CHAPTER 12

The index theorem and formula

Using the earlier results on K-theory and cohomology the families index theorem of Atiyah and Singer is proved using a variant of their 'embedding' proof. The index formula in cohomology (including of course the formula for the numerical index) is then derived from this.

12.1. Outline

The index theorem of Atiyah and Singer is proved here in K-theory, using the results from Chapter 10 and then the cohomological version is derived from this. Here are the main steps carried out below:-

(1) Fibrations of manifolds, $M \longrightarrow B$, are discussed and shown to be embeddable in a trivial fibration following Whitney's embedding theorem.
(2) The 'semiclassical index' is defined using semiclassical smoothing operators, first for odd K-theory and then for even K-theory; it there is an innovation here, this is it. Both exhibit 'excision'.
(3) The odd and even semiclassical index maps are shown to be related by suspension, using a calculus combining semiclassical smoothing operators and standard pseudodifferential operators.
(4) The odd (and hence the even) semiclassical index is shown to be natural for iterated fibrations.
(5) The group of homotopy classes of sections of the bundle $G^{-\infty}(M/B; E)$ is shown to reduce to $\mathrm{K_c}(B)$ using smooth families of projections approximating the identity.
(6) The notion of an elliptic family of pseudodifferential operators on the fibres of a fibration is introduced and the analytic index $\mathrm{Ind_a} : \mathrm{K_c}(T^*(M/B)) \longleftrightarrow \mathrm{K_c}(B)$ is defined.
(7) The analytic and semiclassical index maps are shown to be equal by defining a combined analytic-semiclassical index which extends both.
(8) The topological index map is defined using embeddings and the Thom isomorphism and is shown to be equal to the analytic and semiclassical index maps.

Subsequently the special case of Dirac operators is treated and the formula for the Chern character of the index bundle is deduced.

Maybe other things will go in here, η forms, determinant bundle etc.

12.2. Fibrations

Instead of just considering families of pseudodifferential operators on a manifold but depending smoothly on parameters in some other manifold we allow 'twisting by the diffeomorphism group' and consider the more general setting of a family of pseudodifferential operators on the fibres of a fibration, so the parameters are the variables in the base of the fibration and the operators act on the fibres, which are diffeomorphic to a fixed manifold. This indeed is the setting for the 'families index theorem' of Atiyah and Singer.

So, first we need a preliminary discussion of fibrations. A map between two manifolds

$$\phi : M \longrightarrow B \tag{12.1}$$

is a fibration, with typical fibre a manifold Z, if it is smooth, surjective and has the 'local product' property:-

(12.2)

$$\text{Each } b \in B \text{ has an open neighbourhood } U \subset B$$

for which there exists a diffeomorphism F_U giving a commutative diagramme

$$\begin{array}{ccc} \phi^{-1}(U) & \xrightarrow{F_U} & Z \times U \\ & {}_{\phi}\searrow \quad \swarrow_{\pi_U} & \\ & U. & \end{array}$$

Here of course, π_U is projection onto the second factor. In particular this means that each fibre $\phi^{-1}(b) = Z_b$ is diffeomorphic to Z, and in such a way that the diffeomorphism can be chosen locally to be smooth in $b \in B$. However there is no *chosen* diffeomorphism and of course in general the diffeomorphism cannot be chosen globally smoothly in b – other wise the fibration is trivial in the sense that there exists a diffeomorphism giving a commutative diagramme

$$\begin{array}{ccc} M & \xrightarrow{F} & Z \times B \\ & {}_{\phi}\searrow \quad \swarrow_{\pi_B} & \\ & B. & \end{array} \tag{12.3}$$

I use the notation

$$\begin{array}{ccc} Z & \text{———} & M \\ & & \downarrow{\scriptstyle \phi} \\ & & B \end{array} \tag{12.4}$$

to denote a fibration, the headless arrow meaning that there is no chosen diffeomorphism onto the fibres; often people put an arrow there.

One standard source of fibrations is the implict function theorem.

Proposition 12.1. [1] *If $\phi : M \longrightarrow B$ is a smooth map between connected smooth compact manifolds which is a submersion, i.e. the differential $\phi_* : T_m M \longrightarrow T_{\phi(m)} B$ is surjective for every $m \in M$, then ϕ is a fibration.*

It is easy to see that this implication can fail if M is not compact.

We will discuss operators on the fibres of a fibration below. First however we consider one of the important steps in the proof of the Atiyah-Singer theorem, namely the embedding of a fibration.

[1] See Problem 12.1

PROPOSITION 12.2. *Any fibration of compact manifolds can be embedded in a trivial fibration to give a commutative diagramme*

$$
\begin{array}{ccc}
M & \xrightarrow{\iota} & \mathbb{R}^M \times B \\
& {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\pi_B} & \\
& B. &
\end{array}
\tag{12.5}
$$

PROOF. Following Whitney, simply embed M in $\mathbb{R}^M$ for some M. This is easy to do, much the same way as vector bundle can be complemented to a trivial bundle.[2] Then let ι be the product of this embedding and ϕ, giving a map into $\mathbb{R}^M \times B$. □

Vector bundles give particular examples of fibrations. There are various standard constructions on fibrations, in particular the fibre product.

LEMMA 12.1. *If $\phi_i : M_i \longrightarrow B$, $i = 1, 2$ are two fibrations with the same base and typical fibres Z_i, then*

$$M_1 \times_B M_2 = \{(m_1, m_2) \in M_1 \times M_2; \phi_1(m_1) = \phi_2(m_2)\} \subset M_1 \times M_2$$

is an embedded submanifold and the restriction of $\phi_1 \times \phi_2$ to it gives a fibration

$$
\begin{array}{ccc}
Z_1 \times Z_2 & \longrightarrow & M_1 \times_B M_2 \\
& & \downarrow{\scriptstyle \phi_1 \times \phi_2} \\
& & B.
\end{array}
\tag{12.6}
$$

PROOF. Just look at local trivializations. □

It has become standard to denote 'relative objects' for a fibration, meaning objects on the fibres, using the formal notation M/B for the fibres. Thus $T(M/B)$ is the *fiber tangent bundle*. It is a bundle over the total space M with fibre at $m \in M$ the tangent space to the fibre through m, $\phi^{-1}(\phi(m))$, at m. To see that it is a bundle, just look at local trivializations of the fibration. Its dual bundle is $T^*(M/B)$, with fibre at m the cotangent space for the fibre. This will play a significant role in what we do below.

12.3. Smoothing families

Philosophically, it is often a good idea to think of a space like $\mathcal{C}^\infty(M)$, the smooth functions (or more generally sections of some vector bundle) on the total space of a fibration as an infinite-dimensional bundle over the base. The fibre at b is just $\mathcal{C}^\infty(Z_b)$, the smooth functions on the fibre, and a local trivialization of the fibration gives a local trivialization of this bundle. To be consistent with the notation above I suppose this bundle should be denoted $\mathcal{C}^\infty(M/B) = \mathcal{C}^\infty(M)$ (or $\mathcal{C}^\infty_c(M/B) = \mathcal{C}^\infty_c(M)$ if M is not compact but B is) thought of as a bundle over B.

Next let us consider smoothing operators on the fibres of a fibration from this point of view. Recall that the densities on a manifold form a trivial, but not canonically trivial, real line bundle over the manifold. If this bundle is trivialized

[2]See Problem 12.2 for more details.

then the smoothing operators on Z are identified with the smooth functions (their Schwartz kernels) on $Z \times Z$. Really this is more invariantly written

$$\Psi^{-\infty}(Z) = \mathcal{C}^{\infty}(Z \times Z; \pi_R^* \Omega(Z)) \tag{12.7}$$

where $\pi_R^*\Omega(Z)$ is the density bundle over Z, pulled back to the product under the projection onto to the right-hand factor.

LEMMA 12.2. *For a fibration* (12.4) *the densities bundles on the fibres form a trivial bundle, denoted* $\Omega(M/B)$, *over the total space and the bundle of (compactly-supported) smoothing operators on the fibres may be identified as*

$$\Psi_c^{-\infty}(M/B) = \mathcal{C}_c^{\infty}(M \times_B M; \pi_R^* \Omega(M/B)) \tag{12.8}$$

where π_R *is the right projection from the total space of the fibre product to the total space of the fibration.*

PROOF. Perhaps this is more a definition than a Lemma. The fibre density bundle is just the density bundle for $T(M/B)$. It is then easy to see that an element on the right in (12.8) defines a smoothing operator on each fibre of the fibration and these operators vary smoothly when identified in a local trivialization of the fibration. This leads to the notation on the left. □

Again $\Psi_{\mathrm{c}}^{-\infty}(M/B)$ can be thought of as a (big) bundle over B.

So, now to something a little less formal. As noted above, one case of a fibration is a vector bundle. If we consider a symplectic (or complex) we have discussed the Thom isomoprhism in K-theory above. In doing this we have used, rather extensively, the projections $\pi_{(N)}$ onto the first N eigenspaces of the harmonic oscillators. Since the index theorem is an geometric extension, to a general fibration, of the Thom isomorphism, we need some replacement for these 'exhausting projections' in the general case. Unfortunately there is nothing[3] to take the place of the harmonic oscillators on the fibres. Of course there are similar objects, such as the Laplacians for some family of fibre metrics, but the eigenvalues of such operators are not constant. As a result the eigenspaces are not even smooth and there is not simple replacement for $\pi_{(N)}$. But we really want these, so we have to construct them a little more crudely. I will do this using the embedding construction above; this is a similar argument to the core of the proof of the Atiyah-Singer theorem but in a much simpler setting.

First we note an extension result using these same $\pi_{(N)}$'s, or just $\pi_{(1)}$, the projection onto the ground state of the harmonic oscllator.

PROPOSITION 12.3. *Let* W *be a symplectic vector bundle over a compact manifold* Z *then there is a natural embedding as a subalgebra*

$$\Psi^{-\infty}(Z) \hookrightarrow \dot{\Psi}^{-\infty}(\bar{W}; \Lambda^*), \tag{12.9}$$

into the algebra of smoothing operator on the total space of the bundle of radial of the fibres of W *which vanish to infinite order at the boundary, acting on sections of the exterior algebra, in which an operator on* Z *is identified with an operator on the ground state of the bundle of harmonic oscillators.*

[3] As far as I know, please correct me if you know better.

PROOF. The point here is simply that the bundle of ground states of the (bundle of) harmonic oscillators is canonically trivial. Indeed all these functions (and the projections onto them) are positive, so there is a unique choice of unit length basis. A smoothing operator on the manifold is then lifted to the same smoothing operator acting on this line bundle, so as a smoothing operator on the total space it is projection onto this bundle, followed by the action of the smoothing operator. Clearly this forms a subalgebra as claimed, since the Schwartz functions correspond to the functions vanishing at infinity on the radial compactification. □

Now, suppose the total space of W is mapped diffeomorphically to an open subset of a smooth manifold in such a way that $\mathcal{S}(W)$, the space of functions which are Schwartz on the fibres, is identified with the smooth functions with support in the closure of the image set. Then the algebra on the right in (12.9) is identified as the subalgebra of the smoothing operators on this manifold with supports in the closure of the image.

12.4. Semiclassical index maps

As noted above, the index theorem may be thought of as the essential uniqueness of the push-forward map in K-theory. Given a fibration of manifolds as in (12.4) we will first define a 'semiclassical index map'

$$\mathrm{Ind}_{\mathrm{sl}} : \mathrm{K}^1_{\mathrm{c}}(T^*(M/B)) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(B). \tag{12.10}$$

In fact we will do this separately for odd and even K-theory and then compare the results. First we need to discuss the family of fibrewise semiclassical algebras on the fibres of ϕ.

In accordance with the general notation for fibrations the space of semiclassical families of smoothing operators is denoted $\Psi^{-\infty}_{\mathrm{sl}}(M/B; E)$ where E is a vector bundle over M. Repeating again the general principal, this is the space of sections (defined explicitly below) of an infinite dimensional bundle over B whose fibre above $b \in B$ consists of the (space of families of) semiclassical smoothing operators on $Z_b = \phi^{-1}(b)$. There is of course a lot more notation like this below.

Since we have defined the semiclassical algebra on sections of any bundle over any manifold, $\Psi^{-\infty}_{\mathrm{sl}}(Z_b; E_b)$ is well defined. Thus $A \in \Psi^{-\infty}_{\mathrm{sl}}(M/B; E)$ consists of an element of $\Psi^{-\infty}_{\mathrm{sl}}(Z_b; E_b)$ for each $b \in B$, where we only need to specify the meaning of smoothness in $b \in B$. Locally in B the notion of smoothness if obvious enough, since the bundle is trivialized and the meaning of smooth dependence on parameters, which is in any case straightforward, is explained in §6.10. It is therefore only necessary to check that this notion is invariant under diffeomorphisms of the fibres, depending smoothly on the base. I ask you to do this in problems below.****

The results derived earlier for the semiclassical algebra can now be restated for fibrations. The most significant one is the existence and behaviour of the semiclassical symbol map. Here we recall that the semiclassical symbol is 'not quite' a function on the fibrewise cotangent bundle. It is a (Schwartz) function on the slightly different bundle denoted ${}^{\mathrm{sl}}T^*(M/B)$ which is discussed in Section 6.10. In particular this bundle is bundle-isomorphic to $T^*(M/B)$ but *not* equal, i.e. not canonically isomorphic, to it.

PROPOSITION 12.4. *For any fibration the algebra of uniformly properly supported smoothing operators on the fibres, $\Psi_{\rm sl}^{-\infty}(M/B;E)$, gives a short exact, multiplicative, sequence*

(12.11)
$$0 \longrightarrow \epsilon\Psi_{\rm sl}^{-\infty}(M/B;E) \hookrightarrow \Psi_{\rm sl}^{-\infty}(M/B;E) \xrightarrow{\sigma_{\rm sl}} \mathcal{S}({}^{sl}T^*(M/B);\hom(E)) \longrightarrow 0.$$

Recall that $\epsilon\Psi_{\rm sl}^{-\infty}(M/B;E)$ is just this lazy man's notation for sections which are of the form ϵA where A is another semiclassical family.

PROOF. *** Part of this proof will be shifted back to the section on the semiclassical calculus on a single manifold where ${}^{\rm sl}T^*Z$ has already been used but not defined.

In §3.9 there is a rather pedantic definition of the cotangent bundle of a manifold. Namely the fibre at a point $p \in M$ is defined to be the 'linearization' of the space of functions vanishing at p, that is the quotient

(12.12)
$$T_p^*M = \{f \in \mathcal{C}^\infty(M); f(p) = 0\}/\{\sum_{\text{finite}} f_i g_i; f_i, g_i \in \mathcal{C}^\infty(M),\ f_i(p) = g_i(p) = 0\}.$$

Suppose we take the product of M and an interval $[0,1]$. Then

$$\text{(12.13)}\quad \pi^*T^*_{(p,\epsilon)}M = \frac{\{f \in \mathcal{C}^\infty(M\times[0,1]); \partial_\epsilon f = 0,\ f(p,\epsilon) = 0\}}{\{\sum\limits_{\text{finite}} f_i g_i; f_i, g_i \in \mathcal{C}^\infty(M\times[0,1]),\ \partial_\epsilon f_i = \partial_\epsilon g_i = 0,\ f_i(p,\epsilon) = g_i(p,\epsilon) = 0\}}$$

is a rather complicated-looking definition of the pull-back to $M\times[0,1]$ of the cotangent bundle to M, under the projection $\pi : M\times[0,1] \longrightarrow M$ at $(p,0) \in M\times[0,1]$. The latter is just defined to be T_p^*M and the definition (12.13) is obviously isomorphic to T_p^*M since all the functions are independent of ϵ, that is it is simply the same definition as (12.12); i.e. this discussion appears moronic.

Let us just change this slightly by inserting factors of ϵ^{-1}. Namely set

(12.14)
$$\begin{gathered}{}^{\rm sl}T^*M_p = \{f \in \mathcal{C}^\infty(M\times(0,1]); \epsilon f \in \mathcal{C}^\infty(M\times[0,1]),\ \partial_\epsilon(\epsilon f) = 0,\ f(p,0) = 0\}/\mathcal{E},\\ \mathcal{E} = \{h \in \mathcal{C}^\infty((0,1)\times M); \epsilon h = \sum_{\text{finite}} f_i g_i\\ \text{for } f_i, g_i \in \mathcal{C}^\infty(M\times[0,1]),\ \partial_\epsilon f_i = \partial_\epsilon g_i = 0,\ f_i(p) = g_i(p) = 0\}.\end{gathered}$$

Of course this second definition just involves inserting a factor of ϵ. So, given that we know what ϵ is,

$$\text{(12.15)}\qquad {}^{\rm sl}T^*M_p \xrightarrow{\times\epsilon} T^*M_p.$$

On the other hand, suppose that we think of $[0,1]$ as a compact, connected, nonempty, 1-dimensional manifold with boundary. That is, we permit ourselves to make diffeomorphisms in ϵ. The differential condition $\partial_\epsilon f = 0$ is invariant under diffeomorphisms, although ∂_ϵ itself is not. However, ϵ just becomes a defining function for $0 \in [0,1]$, it could as well be 2ϵ or even $\epsilon T(\epsilon)$ with $T > 0$ and smooth. The result of this is that the isomorphism (12.15) is not well-defined. The left side is well-defined for $[0,1]$ as a manifold and it is always isomorphic to T_p^*M, but it is *not canonically isomorphic* to T^*_pM. The result is that ${}^{\mathrm{sl}}T^*M$ is a well-defined vector bundle over M, bundle isomorphic to T^*M but not canonically so.

Now, the claim is that the semiclassical symbol really gives a function on T^*Z, not as one might naively think, on T^*Z – however the error in so thinking will likely never show up! Notice that this is clear from the definition of the semiclassical symbol in local coordinates, i.e. back on $\mathbb{R}^n$. There we took the kernel of the semiclassical family,

$$\epsilon^{-n}B(\epsilon, z, \frac{z-z'}{\epsilon}), \tag{12.16}$$

changed variable to $Z = \frac{z-z'}{\epsilon}$, restricted the result to $\epsilon = 0$ and then took the Fourier transform to get a function $b(z,\zeta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ which is Schwartz in the second variables. Under change of variables we showed before that this transforms as a function on $T^*\mathbb{R}^n$, so in the case of manifolds gives a function on T^*M. However, this depends on knowing precisely what ϵ is. If you think of the variable $\epsilon/2$ instead the resulting function will be $b(z,\zeta/2)$. Note that you might expect a change by an overall factor of 2^n but this does not happen because this is absorbed in the measure when we take the Fourier transform. On the other hand the discussion above shows that after the new identification with ${}^{\mathrm{sl}}T^*M$

$$B \in \Psi^{-\infty}_{\mathrm{sl}}(M) \Longrightarrow \sigma_{\mathrm{sl}}(B) \in \mathcal{S}({}^{\mathrm{sl}}T^*M) \text{ is well-defined.} \tag{12.17}$$

The case of semiclassical families acting on a vector bundle on the total space of a fibration just involves the invariance under diffeomorphisms, and the behaviour under multiplcation by smooth functions, of the semiclassical smoothing algebra. That is, the exact sequence (12.11), including its multiplicativity, just comes from the same result on each fibre. □

One direct way to see why the image space in (12.11) is the right one is to define $\sigma_{\mathrm{sl}}(B)$ by 'oscillatory testing'. This is done in Euclidean space in Problem****. Restating this result more invariantly we get

LEMMA 12.3. *If $A \in \Psi^{-\infty}_{\mathrm{sl}}(M/B; E)$, $u \in \mathcal{C}^\infty(M; E)$ and $f \in \mathcal{C}^\infty(M)$ is real-valued then*

$$\lim_{\epsilon\downarrow 0} e^{-if(z)/\epsilon}B(e^{if(z)/\epsilon}u = \sigma_{\mathrm{sl}}(\frac{df}{\epsilon})u \in \mathcal{C}^\infty(M; E) \tag{12.18}$$

with the limit existing in this space.

Notice that we need to interpret $df/\epsilon \in \mathcal{C}^\infty(M; {}^{\mathrm{sl}}T^*M)$ as a section for (12.18) to make sense. Going back to the formal definition (12.14) we can do this by defining its value at $\bar{z} \in M$ to be the class of $(f(z) - f(\bar{z}))/\epsilon$.

Now, having the semiclassical algebra on the fibres at our disposal we can construct the corresponding index.

PROPOSITION 12.5. *If $a \in \mathcal{C}^\infty_c({}^{sl}T^*(M/B); \hom E)$ is such that $\mathrm{Id} + a$ is everywhere invertible and $A \in \Psi^{-\infty}_{\mathrm{sl}}(M/B; E)$ is uniformly properly supported and has $\sigma_{\mathrm{sl}}(A) = a$ then for $\epsilon > 0$ sufficiently small, $\mathrm{Id} + A(\epsilon) \in G^{-\infty}(M/B; E)$ and $[\mathrm{Id} + A] \in \mathrm{K}^1(B)$ depends only on $[\mathrm{Id} + a] \in \mathrm{K}^1_c(T^*(M/B)$ so defining*

$$\mathrm{Ind}_{\mathrm{sl}} : \mathrm{K}^1_c(T^*(M/B)) \longrightarrow \mathrm{K}_c(B). \tag{12.19}$$

PROOF. Note that we are making the effort here not to assume that the fibres are compact – nor does the base need to be compact. The main point is that the quantized family of operators is invertible for small $\epsilon > 0$ with inverse of the same type. Indeed, the discussion in Section **??** shows that there is no problem in constructing a semiclassical family which inverts the quantization to infinite order, so up to an error term which is a standard family of smoothing operators vanishing to infinte order at $\epsilon = 0$. Here all families are uniformly properly supported and so such a perturbation of the identity is invertible with inverse of the same form. Thus it remains only to show that the K-class defined by this invertible section of $G^{-\infty}(M/B; E)$ is independent of choices. Since any two semiclassical quantizations are homotopic for small enough $\epsilon > 0$, independence of choice and homotopy invariance under deformation of a follows from the same construction. Stability is also immediate, so the map(12.19) is well-defined as desired. □

Having defined this 'odd semiclassical index map' we note that there is also an even version, defined using the discussion of projections in Proposition 3.11. Recall that the K-theory with compact supports of a non-compact space X, in this case ${}^{\mathrm{sl}}T^*(M/B)$, is represented by equivalence classes of smooth families of projections $\pi : X \longrightarrow \mathrm{GL}(N, \mathbb{C})$, where $\pi^2 = \pi$ and π is constant outside a compact set. Equivalence of two such projections π_i corresponds to the existence of maps a, $b : X \longrightarrow M(N, \mathbb{C})$ also constant outside a compact set and such that $a\pi_i b = \pi_2$. This just means that $\pi_2 b \pi_2$ is an isomorphism from the range of π_2 to the range of π_1 with inverse $\pi_2 a \pi_1$.

PROPOSITION 12.6. *The semiclassical quantization of projections to projections, in Proposition 3.11, induces a push-forward, or index, map in even K-theory for any fibration with compact fibres*

$$\mathrm{Ind}^0_{\mathrm{sl}} : \mathrm{K}^0_c(T^*(M/B)) \longrightarrow \mathrm{K}^0(B). \tag{12.20}$$

PROOF. Two semiclassical families of projections with the same symbol are homotopic through projections so the map (12.20), in which the index of π is the formal difference of the pair $P \ominus \pi_\infty$ of its quantization and the constant projection 'at infinity' is well-defined up to homotopy. Finite rank approximation shows that it defines an element of the K-theory B and it is straightforward to show independence of choices. □

12.5. Bott periodicity and the semiclassical index

*** Take $E = \mathbb{C}^N$ below, since we know we can do this in constructing the index maps.

In the preceeding section two versions of the index map, as pushforward in K-theory for a fibration, have been defined. Next we show that they are 'equal'.

PROPOSITION 12.7. *For any fibration with compact fibres, the diagramme*

$$\begin{array}{ccc} \mathrm{K}^1_c(\mathbb{R} \times T^*(M/B)) & \xrightarrow{\mathrm{Ind}^1_{\mathrm{sl}}} & \mathrm{K}^1_c(\mathbb{R} \times B) \\ \downarrow{\scriptstyle \mathrm{Ind}_{\mathrm{iso}}} & & \downarrow{\scriptstyle \mathrm{Ind}_{\mathrm{iso}}} \\ \mathrm{K}^0_c(T^*(M/B)) & \xrightarrow{\mathrm{Ind}^0_{\mathrm{sl}}} & \mathrm{K}^0(B) \end{array} \tag{12.21}$$

commutes, where the vertical maps are the realizations of Bott periodicity discussed in (10.53).

The top map is, as indicated, the odd semiclassical index for the fibration $M\times\mathbb{R}\longrightarrow B\times\mathbb{R}$ with an extra factor of $\mathbb{R}$. Clearly the relative cotangent bundle for this fibration is $\mathbb{R}\times T^*(M/B)$.

Of course the problem with proving such a result is that the vertical map are defined by isotropic quantization and the horizontal maps by semiclassical quantization. As usual, the approach adopted here is to construct an algebra of operators which includes both quantizations naturally (i.e. the correspond to the symbol maps). In this case this is relatively straightforward because isotropic quantization is itself rather simple. Thus the algebra $\Psi^0_{\text{iso}}(\mathbb{R}^n)$ arises from a non-commutative product on $\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}})$. Similarly we now the that the algebra of semiclassical operators on the fibres of M can also be identified with a space of smooth functions on a manifold, namely

(12.22)
$$\Psi^{-\infty}_{\text{sl}}(M/B;E)=\epsilon^{-d}\left\{A\in\mathcal{C}^\infty(M^2_{\text{sl}};\operatorname{Hom}(E)\otimes\Omega_R);A\equiv 0\text{ at }\{\epsilon=0\}\setminus\text{ff}\right\}.$$

Here

$$M^2_{\text{sl}}=[[0,1]\times M^2_\phi,\{0\}\times\Delta] \tag{12.23}$$

is obtained by the blow up of the diagonal at $\epsilon=0$ in the fibre product.

What we want is really the completed tensor product of these two algebras. Thus consider

(12.24)
$$\begin{aligned}&\Psi^{0,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)\\&\qquad=\left\{A\in\epsilon^{-d}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}}\times M^2_{\text{sl}};\operatorname{Hom}(E)\otimes\Omega_R);A\equiv 0\text{ at }\{\epsilon=0\}\setminus\text{ff}\right\}.\end{aligned}$$

As spaces of amplitudes (i.e. before quantization) we can define the spaces of other (real or complex) orders by

$$\Psi^{m,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)=\rho^{-m}\Psi^{0,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E) \tag{12.25}$$

where $\rho\in\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}})$ is a boundary defining function, i.e. a non-vanishing real elliptic symobl of order -1 in the isotropic calculus.

Proposition 12.8. *The space in* (12.24) *is an algebra under (the continuous extension of) the isotropic product on* $\mathbb{R}^n$ *for symbols with values in the smoothing operators on the fibres of* $\phi:M\longrightarrow B$. *There are two short exact symbol sequences which are multiplicative*

(12.26)
$$\begin{gathered}\Psi^{-1,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)\hookrightarrow\Psi^{0,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)\xrightarrow{\sigma_{\text{iso}}}\mathcal{C}^\infty(\mathbb{S}^{n-1};\Psi^{-\infty}_{\text{sl}}(M/B;E)\\ \epsilon\Psi^{0,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)\hookrightarrow\Psi^{0,-\infty}_{\text{iso}-\text{sl}}(\mathbb{R}^n\times M/B;E)\xrightarrow{\sigma_{\text{sl}}}\mathcal{S}({}^{sl}T^*(M/B);\Psi^0_{\text{iso}}(\mathbb{R}^n;E))\end{gathered}$$

which have a common double symbol map

(12.27)

$$\begin{array}{ccccc} & & \mathcal{C}^\infty(\mathbb{S}^{n-1};\Psi^{-\infty}_{\mathrm{sl}}(M/B;E)) & & \\ & \overset{\sigma_{\mathrm{iso}}}{\nearrow} & & \overset{\sigma_{\mathrm{sl}}}{\searrow} & \\ \Psi^{0,-\infty}_{\mathrm{iso-sl}}(\mathbb{R}^n\times M/B;E) & & & & \mathcal{S}(\mathbb{S}^{n-1}\times {}^{sl}T^*(M/B);\hom(E)), \\ & \underset{\sigma_{\mathrm{sl}}}{\searrow} & & \underset{\sigma_{\mathrm{iso}}}{\nearrow} & \\ & & \mathcal{S}({}^{sl}T^*(M/B);\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;E)) & & \end{array}$$

and combine to give a joint symbol sequence

$$\begin{aligned}(12.28)\quad \epsilon\Psi^{-1,-\infty}_{\mathrm{iso-sl}}(\mathbb{R}^n\times M/B;E)&\hookrightarrow\Psi^{0,-\infty}_{\mathrm{iso-sl}}(\mathbb{R}^n\times M/B;E)\\ &\xrightarrow{\sigma_{\mathrm{iso}}\oplus\sigma_{\mathrm{sl}}}\mathcal{C}^\infty(\mathbb{S}^{n-1};\Psi^{-\infty}_{\mathrm{sl}}(M/B;E))\oplus\mathcal{S}({}^{sl}T^*(M/B);\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;E))\end{aligned}$$

which is exact in the centre and has range precisely the subspace satisfying the compatibility condition in (12.27)*, that*

$$\sigma_{\mathrm{sl}}\sigma_{\mathrm{iso}}=\sigma_{\mathrm{iso}}\sigma_{\mathrm{sl}}. \tag{12.29}$$

Proof. I will do this ***. The main point is that these are just smooth functions and we can do the quantizations separately in each of the spaces treating the other variables as parameters and then reverse the discussion – really just as though it is a finiter rather than a completed tensor product. Everything should work out pretty well. □

Proof of Proposition 12.7. Now – and really this is essentially the same argument as recurs below in the proof of multiplicativity – we consider the quantization procedure determined by this algebra. Starting with a 'double symbol'

$$\tilde a\in\mathcal{S}(\mathbb{R}\times{}^{\mathrm{sl}}T^*(M/B)\text{ s.t. }(\mathrm{Id}+\tilde a)^{-1}=\mathrm{Id}+\tilde b,\ \tilde b\in\in\mathcal{S}(\mathbb{R}\times{}^{\mathrm{sl}}T^*(M/B), \tag{12.30}$$

we take the radial compactification of the line into $\mathbb{S}$ and so realize $\tilde a$ as an element a of the image space in (12.27),

(12.31)

$$a\in\mathcal{S}(\mathbb{S}\times{}^{\mathrm{sl}}T^*(M/B);\hom(E))\text{ s.t. }(\mathrm{Id}+a)^{-1}=\mathrm{Id}+b,\ b\in\mathcal{S}(\mathbb{S}\times{}^{\mathrm{sl}}T^*(M/B);\hom(E))$$

we proceed to 'quantize' a in two ways. First, we can use semiclassical quantization to choose a family

$$\alpha'\in\mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}_{\mathrm{sl}}(M/B;E))\text{ s.t. }\sigma_{\mathrm{sl}}(\alpha')=a. \tag{12.32}$$

Here the circle just consists of parameters which should be added to both the base and the fibre and so do not contribute at all to the fibre quantization. It follows that $\mathrm{Id}+\alpha'$ is invertible for small $\epsilon>0$ and that, by definition of semiclassical quantization,

$$\mathrm{Ind}_{\mathrm{sl}}([\mathrm{Id}+\tilde a])=[\mathrm{Id}+\alpha']\in\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times B). \tag{12.33}$$

Secondly we can proceed in the opposite way and construct a family

$$\alpha''\in\mathcal{S}({}^{\mathrm{sl}}T^*(M/B);\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;E))\text{ s.t. }\sigma_{\mathrm{iso}}(\alpha'')=a. \tag{12.34}$$

From the discussion of isotropic quantization, the invertibility of $\operatorname{Id}+a$ means that $\operatorname{Id}+\alpha''$ is a Fredholm family. In fact we know that for N sufficiently large,

$$(\operatorname{Id}+\alpha'')(\operatorname{Id}-\pi_{(N)}) \text{ has null space precisley the range of } \pi_{(N)}, \tag{12.35}$$

where $\pi_{(N)}$ is the projection onto the span of the first N eigenspaces of the harmonic oscillator (extended to act on sections of E ****). Then we can choose a generalized inverse $\operatorname{Id}+\beta''$ of $\operatorname{Id}+\alpha''$ with

$$\begin{gathered}\beta'' \in \mathcal{S}({}^{\mathrm{sl}}T^*(M/B);\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;E)) \text{ and}\\ (\operatorname{Id}+\beta'')(\operatorname{Id}-\pi')(\operatorname{Id}+\alpha'')(\operatorname{Id}-\pi_{(N)}) = (\operatorname{Id}-\pi_{(N)}),\\ (\operatorname{Id}+\alpha'')(\operatorname{Id}-\pi_{(N)})(\operatorname{Id}+\beta'')(\operatorname{Id}-\pi') = (\operatorname{Id}-\pi')\\ \pi'\pi' = \pi',\ \pi' - \pi_{(N)} \in \mathcal{S}({}^{\mathrm{sl}}T^*(M/B);\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;E)).\end{gathered} \tag{12.36}$$

Note that I have written things out this way to avoid having to allow the main families to remain trivial at infinity on ${}^{\mathrm{sl}}T^*(M/B)$ – although there has to be non-triviality there in order to get the errors to be represented by projections in this way. Then, from the definition of the Bott periodicity map by isotropic quantization,

$$\operatorname{Ind}_{\mathrm{iso}}([\operatorname{Id}+\tilde{a}]) = [\pi_{(N)} \ominus \pi'] \in \mathrm{K}^0_{\mathrm{c}}({}^{\mathrm{sl}}T^*(M/B)) \tag{12.37}$$

is the left vertical map in (12.21).

So, as it should be, the semiclassical quantization is easier.

Using the properties of the algebra, we can find a common element $A \in \Psi^{0,-\infty}_{\mathrm{iso-sl}}(\mathbb{R}^n \times M/B;E)$ such that $\sigma_{\mathrm{sl}}(A) = \alpha''$ and $\sigma_{\mathrm{iso}}(A) = \alpha'$. Moreover, $\operatorname{Id}+A$ has a 'two-sided parameterix' $\operatorname{Id}+B$, $B \in \Psi^{0,-\infty}_{\mathrm{iso-sl}}(\mathbb{R}^n \times M/B;E)$ which can be constructed so that, as elements of the semiclassical-isotropic algebra

$$\begin{gathered}(\operatorname{Id}+B)(\operatorname{Id}+A) = \operatorname{Id}-\pi_1,\ (\operatorname{Id}+A)(\operatorname{Id}+B) = \operatorname{Id}-\pi_2,\\ \pi_i \in \Psi^{-\infty,-\infty}_{\mathrm{iso-sl}}(M/B;E),\ \pi_i^2 = \pi_i,\ i = 1,2.\end{gathered} \tag{12.38}$$

Now, of necessity

$$\sigma_{\mathrm{sl}}(\pi_1) = \pi_{(N)},\ \sigma_{\mathrm{sl}}(\pi_2) = \pi'. \tag{12.39}$$

Now, we claim that for $\epsilon > 0$ small,

$$\begin{gathered}\operatorname{Ind}(\operatorname{Id}+A) = [\pi_1 \ominus \pi_2] = \operatorname{Ind}^0_{\mathrm{sl}}(\pi_{(N)},\pi') = \operatorname{Ind}_{\mathrm{sl}}\operatorname{Ind}^0_{\mathrm{iso}}(\operatorname{Id}+\tilde{a}) \text{ and}\\ \operatorname{Ind}(\operatorname{Id}+A) = \operatorname{Ind}_{\mathrm{iso}}(\operatorname{Id}+\alpha') = \operatorname{Ind}_{\mathrm{iso}}\operatorname{Ind}_{\mathrm{sl}}(\operatorname{Id}+\tilde{a}).\end{gathered} \tag{12.40}$$

The first equality on the top line in (12.40) is the essentially definition of the index in the isotropic algebra (here extended a bit because of the values in the smoothing algebra) because of (12.38). The second equality follows from the definition of the even semiclassical index map and (12.39) and the third equality is the combination of this and (12.37). Similarly the second line in (12.40) follows from the choice of A as a semiclassical quantization of α'. □

12.6. Hilbert bundles and projections

** This section, or parts of it, may need to be moved back into the K-theory chapter.

As is well-known, all infinite-dimensional separable Hilbert spaces (here always over $\mathbb{C}$) are isomorphic. Namely, if one takes as model ℓ_2, the space of square-summable complex sequences, then an isomorphism to a separable infinite-dimensional Hilber space, $\mathcal{H}$, corresponds exactly to a choice of complete orthonormal basis in $\mathcal{H}$ and such can be constructed by the application of the Gram-Schmidt orthonormalization procedure to any countable dense subset. The group $\mathrm{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$ is then an infinite-dimensional analogue of $\mathrm{U}(N)$, the group of $N \times N$ unitary matrices. However, in contrast the the finite dimensional case, $\mathrm{U}(\mathcal{H})$ is contractible. In the infinite-dimensional setting it is necessary to specify the topology in which this contractibility is to take place. The 'serious' theorem here is Kuiper's theorem that the unitary group is contractible in the norm topology. We will only use the weaker result,

PROPOSITION 12.9. *For any infinite-dimensional separable Hilbert space,* $\mathrm{U}(\mathcal{H})$ *is contractible in the strong topology, meaning there is a map*

$$\mathrm{U}(\mathcal{H}) \times [0,1] \ni (U,t) \Longrightarrow U_t \in \mathrm{U}(\mathcal{H}) \ s.t.\ U_t v \longrightarrow v \ in\ \mathcal{H}\ \forall\ v \in \mathcal{H},\ U \in \mathrm{U}(\mathcal{H}). \tag{12.41}$$

PROOF. One example of an infinite-dimensional separable Hilbert space is $L^2([0,1])$ with an orthonormal basis given by the exponentials of period 1. Thus it suffices to prove strong contractibility for this example. For given $v \in L^2([0,1])$, $U \in \mathrm{U}(L^2([0,1]))$ and $t \in [0,1]$ set

$$v_t(x) = v(tx),\ x \in [0,1] \text{ and } U_t v(x) = \begin{cases} (Uv_t)(x/t) & 0 \le x \le t \\ v(x) & x > t. \end{cases} \tag{12.42}$$

Thus $U_t v(x)$ is given by the identity operator on $[t,1]$ and by a rescaled version of U on $[0,1]$. Clearly U_t is linear and

$$\int_0^1 |U_t v(x)|^2 = \int_0^t |Uv_t(x/t)|^2 dx + \int_t^1 |v(x)|^2 dx = t\|v_t\|_{L^2}^2 + \int_t^1 |v(x)|^2 dx = \|v\|_{L^2}^2 \tag{12.43}$$

so U_t is unitary. Similarly

$$\|U_t v - v\|_{L^2([0,1])} = \|U_t v - v\|_{L^2([0,t])} \le \|U_t v\|_{L^2([0,t])} + \|u\|_{L^2([0,t])} \le 2\|v\|_{L^2([0,t]} \to 0 \text{ as } t \to 0 \tag{12.44}$$

shows that $U_t \to \mathrm{Id}$ strongly.

So, it suffices to check that the continuity of the map

$$\mathrm{U}(\mathcal{H}) \times \mathcal{H} \times [0,1] \ni (U,v) \longmapsto U_t v \in \mathcal{H} \tag{12.45}$$

with respect to the strong topology on $\mathrm{U}(\mathcal{H})$. □

Note that the contraction constructed in (12.42) is multiplicative, namely

$$(UV)_t = U_t V_t \tag{12.46}$$

as follows directly from the definition. This should be useful somewhere!

The main application we need of this contractibility is the existence of finite rank approximations to the identity for fibre bundles. One way to do this is to note the following general topological result.

PROPOSITION 12.10. *If* $F : \mathcal{M} \longrightarrow B$ *is a topological fibre bundle over a compact manifold and the typical fibre,* $\mathcal{Z}$ *is contractible, then* F *has a right inverse, i.e. the bundle has a section.*

PROOF. Maybe I will put a detailed proof in somewhere. This is pretty easy using a triangulation but it might be better to have a proof using small geodesic balls. □

Thus, for a fibration $M \longrightarrow B$ consider the Hilbert spaces of square-integrable sections of a vector bundle E over M these combine to give a bundle $L^2(M/B;E)$ over B. Each of the fibres is unitarily equivalent to the fixed Hilbert space $L^2(Z;E\big|_Z)$ and we can consider the bundle $\mathcal{P} \longrightarrow B$ with fibre at b

$$\mathcal{P}_b = \{G : L^2(Z_b;E\big|_{Z_b}) \longrightarrow L^2(Z;E\big|_Z) \text{ unitary} \tag{12.47}$$

consisting of all such unitary equivalences – this is a principal bundle for the action of $\mathrm{U}(L^2(Z;E\big|_Z))$ by composition. In fact the bundle is locally trivial for the norm topology but we have only checked the contractibility of the fibre for the strong topology. Applying Proposition 12.10 we conclude that there is a section of $B \longrightarrow \mathcal{P}$ which is strongly continuous – of course if we used Kuiper's theorem we could show that there is a norm-continuous section.

PROPOSITION 12.11. *For any fibration $M \longrightarrow B$, Hermitian vector bundle E over M and choice of smooth positive fibre density, there are sections $e_i \in \mathcal{C}^\infty(M;E)$, $i \in \mathbb{N}$, which form an orthonormal basis in each fibre.*

PROOF. As discussed before the statement of the Proposition, the bundle $\mathcal{P}$ has a strongly continuous section G. Then for any orthonormal basis f_i of $L^2(Z;E\big|_Z)$ the sections $e_i' = G^{-1}f_i \in \mathcal{C}^0(B;L^2(M/B;E))$ are continuous and form an orthonormal basis at each point. Now, we can approximate such continuous-L^2 sections by smooth sections $e_i'' \in \mathcal{C}^\infty(M;E)$ as closely as we wish. In particular we can choose these smooth sections so that

$$\sup_{b\in B} \|e_i' - e_i''\| \le 2^{-i-4},\ i \ge 1. \tag{12.48}$$

Now, we claim that these new sections can in turn be modified to a smooth orthonormal basis. Certainly from (12.48), the operator

$$T_b u = \sum_i \langle u, e_i'(b)\rangle e_i''(b) \text{ has } \|T_b - \operatorname{Id}\|_{L^2} \le \left(\sum_i \|e_i' - e_i''\|^2\right)^{\frac12} \le 1/4 \tag{12.49}$$

so is invertible. Thus the finite span of the e_i'' is certainly dense and they are independent. Gram-Schmidt orthonormalization therefore gives an orthonormal basis all elements of which are smooth. □

The main use we put such a smooth orthonormal basis to is the construction of approximate identities which give uniform, i.e. norm convergent, finite rank approximations to smoothing operators.

PROPOSITION 12.12. *Having chosen a smooth orthonormal basis $e_i \in \mathcal{C}^\infty(M;E)$ for a fibration $M \longrightarrow B$ (corresponding to a choice of Hermitian structure and smooth fibre densities) the orthogonal projections $\pi_{(N)}$ onto the span of the first N elements are such that*

$$\|\pi_{(N)}A\pi_{(N)} - A\|_{L^2(M/B;E)} \to 0 \textit{ as } N \to \infty\ \forall\ A \in \Psi^{-\infty}(M/B;E). \tag{12.50}$$

PROOF. This follows from the compactness of smoothing operators on L^2. □

12.7. Adiabatic limit

The main content of the K-theory version of the families index theorem of Atiyah and Singer is that there really is only one way to define an index map, essentially because this is a push-forward map in K-theory. We start the proof by showing this in one particular case. Namely we have shown above **** that if M is a compact manifold which is fibred over B then the K-theory of B can be realized as the homotopy classes of sections of the bundle of groups $G^{-\infty}(M/B;E)$ for any bundle E over M. That is, instead of maps from B into $G^{-\infty}(Z)$ it is fine to consider the twisted case where Z is the varying fibre of $\phi : M \longrightarrow B$. The proof above is by deformation to finite rank, i.e. in both cases sections can be replaced by maps into $\mathrm{GL}(N;\mathbb{C})$ for some appropriately large N depending on the section.

Now, suppose that the total space M of a fibration is itself the base of another fibration

$$\begin{array}{ccc} \tilde{Z} & \text{------} & \tilde{M} \\ & & \downarrow{\scriptstyle\tilde{\phi}} \\ Z & \text{------} & M \\ & & \downarrow{\scriptstyle\phi} \\ & & B. \end{array} \qquad \psi : \tilde{M} \longrightarrow B \tag{12.51}$$

In this setting we will show that the 'adiabatic calculus' of smoothing operators gives a quantization map

$$K^1(T^*(M/B)) \longrightarrow K^1(B) \tag{12.52}$$

which is defined in terms of smoothing operators on the fibres of $\tilde{\phi} : \tilde{M} \longrightarrow M$.

To define this we need to investigate the adiabatic algebra of smoothing operators for a fibration and then for an iterated fibration. First we start with the case that the overall base, B is a point. Thus M may be replaced by Z and we consider a fibration, with compact fibres

$$\begin{array}{ccc} \tilde{Z} & \text{------} & Y \\ & & \downarrow{\scriptstyle\tilde{\phi}} \\ & & Z \end{array} \tag{12.53}$$

Definition 12.1. *A smooth family of smoothing operators,* $A \in \mathcal{C}^\infty((0,1]_\delta;\Psi^{-\infty}(Y;E))$ *(for a vector bundle over* Y*) is an* adiabatic family *for the fibration* $\tilde{\phi}$ *in* (12.53) *if and only if its Schwartz kernel (also denoted* A*) has the following properties as* $\delta \downarrow 0$ *:*

(1) *If* $\tilde{\chi} \in \mathcal{C}^\infty(Y^2)$ *has support disjoint from the fibre diagonal* $\{(p,p') \in Y^2; \tilde{\phi}(p) = \tilde{\phi}(p')\}$ *then*

$$\tilde{\chi} A \in \delta^\infty \mathcal{C}^\infty([0,1];\Psi^{-\infty}(Y;E)) \tag{12.54}$$

i.e. is smooth down to $\delta = 0$ *where it vanishes to infinite order.*

(2) *If* $\chi \in \mathcal{C}^\infty(Z)$ *has support in a coordinate patch over which* $\tilde{\phi}$ *is trivial (and* E *reduces to the pull-back of a bundle* $\tilde{E}$ *over* $\tilde{Z}$*) with coordinate* z

then

(12.55)
$$\chi(z)A\chi(z') = \delta^{-m}A'(z,\frac{z-z'}{\delta};\Psi^{-\infty}(\tilde{Z};\tilde{E})),$$
$$A \in \mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R}^m \times \tilde{Z} \times \tilde{Z}; \operatorname{Hom}(\tilde{E}) \otimes \Omega(\tilde{Z})_R)$$

having compact support in the first variable and being Schwartz in the second.

The set space of such operators will be denoted $\Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(Y;E)$.

*** This is just a quick redefintion of the semiclassical cotangent bundle.

We may define a bundle $\pi_{\mathrm{sl}} : {}^{\mathrm{sl}}T^*Z \longrightarrow Z$ as the restriction to $\delta = 0$ of the bundle over $[0,1]_\delta \times Z$ which has global sections of the form $\alpha(\delta,z)/\delta$ where $\alpha \in \mathcal{C}^\infty([0,1];T^*Z)$ is a smooth 1-form on Z (depending smoothly on δ.) This bundle is bundle isomorphic to T^*Z (since it is so over $(0,1)\times Z$) but not naturally so, however there is a well-defined homotopy class of bundle isomorphisms between ${}^{\mathrm{sl}}T^*Z$ and T^*Z. We may then pull the fibration $Y \longrightarrow Z$ back to a fibration $\pi^*Y \longrightarrow Z$ which has the same fibre, $\tilde{Z}$, but now has base ${}^{\mathrm{sl}}T^*Z$. This allows us to define the smoothing operators on the fibres and also to see that Schwartz sections are well-defined, giving the algebra $\mathcal{S}({}^{\mathrm{sl}}T^*Z;\Psi^{-\infty}(\pi^*_{\mathrm{sl}}Y/{}^{\mathrm{sl}}T^*Z;E))$.

Proposition 12.13. *The adiabatic smoothing operators for a fibration form an algebra of operators on* $\mathcal{C}^\infty([0,1]\times Y;E)$ *with a multiplicative short exact symbol sequence*

(12.56)
$$0\longrightarrow \delta\Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(Y;E) \hookrightarrow \Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(Y;E) \xrightarrow{\sigma_{\mathrm{ad}}} \mathcal{S}({}^{sl}T^*Z;\Psi^{-\infty}(\pi^*_{\mathrm{sl}}Y/{}^{sl}T^*Z;E)) \longrightarrow 0.$$

Proof. The usual. □

Now, we can extend this construction to the iterated fibration (12.51), to define a similar algebra $\Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(\tilde{M}/B;E)$ for any bundle E over $\tilde{M}$. These are just smooth families with respect to the variables in B with each operator being an adiabatic family on the fibre above $b \in B$ – so for the fibration of $\tilde{\phi}^{-1}(Z_b) \subset \tilde{M}$ over $Z_b = \phi^{-1}(b)$. Thus when $\delta \downarrow 0$ additional commutative variables appear in the fibre ${}^{\mathrm{sl}}T^*(Z_b)$; combined with the variables in B this means that the adiabatic symbol has parameters in ${}^{\mathrm{sl}}T^*(M/B)$ as a bundle over M. So the multiplicative short exact sequence (12.56) becomes

(12.57)
$$\delta\Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(\tilde{M}/B;E) \hookrightarrow \Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(\tilde{M}/B;E) \xrightarrow{\sigma_{\mathrm{ad}}} \mathcal{S}({}^{\mathrm{sl}}T^*(M/B);\Psi^{-\infty}(\pi^*_{\mathrm{sl}}\tilde{M}/{}^{\mathrm{sl}}T^*(M/B);E))$$

where I dropped off the zeros to save space.

Proposition 12.14. *If* $a \in \mathcal{C}^\infty_c({}^{sl}T^*(M/B);G^{-\infty}(\pi^*_{\mathrm{sl}}\tilde{M}/{}^{sl}T^*(M/B);E))$, *i.e. is the sum of the identity and a family in the image of the symbol map in* (12.57) *which has compact support such that the result is always invertible, then any adiabatic family* $A \in \Psi^{-\infty}_{\mathrm{ad}(\tilde{\phi})}(\tilde{M}/B;E)$ *with* $\sigma_{\mathrm{ad}}(A) = a$ *is such that* $\operatorname{Id}+A(\delta) \in G^{-\infty}(\tilde{M}/B;E)$ *for* $0 : \delta < \delta_0$ *for* $\delta_0 > 0$ *small enough, and this defines unambiguously an index map*

$$\operatorname{Ind}_{\mathrm{sl}} : K^1(T^*(M/B)) \longrightarrow K^1(B) \qquad (12.58)$$

which (as the notation indicates) is equal to the semiclassical index map as previously defined.

PROOF. The first step is to show homotopy invariance and stability as before. Then, use the (*** currently non-existent) result above showing that smoothing operators on a fibration can be uniformly approximated by finite rank families to deform the symbol a to a finite rank operator i.e. acting on a trivial finite dimensional bundle of $\mathcal{C}^\infty(\tilde{M}/M;E)$ as a bundle over M, and then just observe that a semiclassical quantization of this gives an adiabatic quantization. Hence the maps are the same – the adiabatic quantization is just a more general construction which is 'retractible' to the semiclassical case. □

12.8. Multiplicativity

One of the crucial properties of the semiclassical index, defined in (12.19) is that it gives a commutative diagramme under iteration of fibrations. Thus suppose we are again in the set-up of (12.51). The composite map is then a fibration and we wish to prove that the same (semiclassical) index map arises by quantization in 'one step' and in 'two steps'.

PROPOSITION 12.15. *For an iterated fibration as in* (12.51)*, the semiclassical index map for* ψ*, the overall fibration, is the composite*

$$\operatorname{Ind}_{\mathrm{sl}(\psi)} = \operatorname{Ind}_{\mathrm{sl}(\psi)} \circ \operatorname{Ind}_{\mathrm{sl}(\tilde{\phi}^*)} . \tag{12.59}$$

Notice that the map on the rightmost here is *not* quite the usual semiclassical index map for $\tilde{\phi}$ as a fibration from $\tilde{M}$ to M (with fibre $\tilde{Z}$), rather it is the semiclassical index for the pull-back of this fibration to ${}^{\mathrm{sl}}T^*(M/B)$ as a bundle over M to give a fibration

$$\begin{array}{ccc} \tilde{Z} & \text{———} & {}^{\mathrm{sl}}T^*(M/B) \times_M \tilde{M} \\ & & \downarrow \tilde{\phi}^* \\ & & {}^{\mathrm{sl}}T^*(M/B). \end{array} \tag{12.60}$$

It is precisely for this purpose that the adiabatic-semiclassical algebra was discussed earlier. Unfortunately, at the time of writing, the notation in the earlier discussion has not been reversed – it is here but this is potentially confusing.

So consider the very special case where the iterated fibration (12.60) reverts to a single fibration, namely $B = \{\mathrm{pt}\}$. We can then declare $M = Z$ and so write the single fibration as

$$\begin{array}{ccc} \tilde{Z} & \text{———} & M \\ & & \downarrow \tilde{\phi} \\ & & Z. \end{array} \tag{12.61}$$

So, we proceed to construct the algebra $\Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M};E)$ of adiabatic-semiclassical smoothing operators associated to this fibration. Going back to Section 2.20 we consider 2-parameter families, where the parameters are $\epsilon > 0$ and $\delta > 0$ of smoothing operators on $\tilde{M}$ (acting on sections of E). Thus

$$\Psi^{-\infty}_{\mathrm{sl\,ad}(\tilde{\phi})}(\tilde{M};E) \subset \mathcal{C}^\infty((0,1)_\epsilon \times (0,1)_\delta; \Psi^{-\infty}(\tilde{M};E)) \tag{12.62}$$

and we only need describe exactly the admissible behaviour of the kernels as $\epsilon \downarrow 0$ and $\delta \downarrow 0$, separately and jointly. Of course we do this in terms of the local families discussed in Section 2.20.

Let z_i, $\tilde{z}_j$ be local coordinates in $\tilde{M}$ with the z_i coordinates in the base and $\tilde{z}_j$ coordinates in the fibre. In M^2 we take two copies of such local coordinates with primed versions in the right factors. The fibre diagonal is the globally well-defined manifold given locally by $z_i = z_i'$. The assumptions we make on the kernels are

- As $\delta \downarrow 0$ the kernels vanish rapidly with all derivatives in $z \neq z'$
- As $\epsilon \downarrow 0$ the kernels vanish rapidly with all derivatives in $z \neq z'$ or $\tilde{z} \neq \tilde{z}'$.
- As $\delta \downarrow 0$ but $\epsilon \geq \epsilon_0 > 0$, the kernels a of the form

$$\delta^{-k} A(\delta, \epsilon, \tilde{z}, \tilde{z}', z, \frac{z - z'}{\delta}),\ A \in \mathcal{C}^\infty([0,1] \times [\epsilon_0, 1] \times \tilde{U}, \tilde{U}', U; \mathbb{S}(\mathbb{R}^k)) \tag{12.63}$$

near $z = z'$ (for possibly different coordinate patches U, U' in the fibres.)

- In $\epsilon < \epsilon_0$ for $\epsilon_0 > 0$ sufficiently small, the kernels are uniformly of the form

$$\delta^{-n-k} A(\delta, \epsilon, \tilde{z}, \frac{\tilde{z} - \tilde{z}'}{\epsilon}, z, \frac{z - z'}{\epsilon\delta}),\ A \in \mathcal{C}^\infty([\delta_0, 1] \times [0,1] \times \tilde{U}, U; \mathbb{S}(\mathbb{R}^n)) \tag{12.64}$$

near $z = z'$, $\tilde{z} = \tilde{z}'$.

*** Describe the two tangent and cotangent bundles, adiabatic and semiclassical-adiabatic.

The adiabatic-semiclassical calculus corresponds to a modified cotangent bundle ${}^{\rm sl\,ad}T^*\tilde{M}$, over $\tilde{M} \times [0,1]_\epsilon \times [0,1]_\delta$. Namely the fibre at any point is the quotient of the space of smooth linear combinations

$$\frac{df}{\epsilon},\ \frac{dg}{\epsilon\delta},\ f \in \mathcal{C}^\infty(\tilde{M}),\ g \in \mathcal{C}^\infty(Z) \tag{12.65}$$

by the corresponding product with the ideal of functions vanishing at that point. There are canonical isomorphisms

$$\begin{aligned} {}^{\rm sl\,ad}T^*\tilde{M}\big|_{\delta>0} &\equiv {}^{\rm sl}T^*\tilde{M}, \\ {}^{\rm sl\,ad}T^*\tilde{M}\big|_{\epsilon>0} &\equiv {}^{\rm ad}T^*\tilde{M} \text{ and} \\ {}^{\rm sl\,ad}T^*\tilde{M}\big|_{\epsilon>0,\delta>0} &\equiv T^*\tilde{M}. \end{aligned} \tag{12.66}$$

Proposition 12.16. *For a fibration* (12.61) *the space of operators* $\Psi^{-\infty}_{\rm sl\,ad(\tilde{\phi})}(\tilde{M}; E)$ *forms an alegbra under composition with two multiplicative exact symbol sequences*
(12.67)

$$\delta\Psi^{-\infty}_{\rm sl\,ad(\tilde{\phi})}(\tilde{M}; E) \longrightarrow \Psi^{-\infty}_{\rm sl\,ad(\tilde{\phi})}(\tilde{M}; E) \xrightarrow{\sigma_{\rm ad}} \Psi^{-\infty}_{\rm sl}({}^{sl}T^*(M/B) \times_M \tilde{M}/{}^{sl}T^*(M/B))$$

$$\epsilon\Psi^{-\infty}_{\rm sl\,ad(\tilde{\phi})}(\tilde{M}; E) \longrightarrow \Psi^{-\infty}_{\rm sl\,ad(\tilde{\phi})}(\tilde{M}; E) \xrightarrow{\sigma_{\rm sl}} \mathcal{C}^\infty\left([0,1]_\delta; \mathcal{S}({}^{\rm sl\,ad}T^*\tilde{M})\right).$$

Furthermore the joint symbol map gives a short exact sequence

$$\epsilon\delta\Psi^{-\infty}_{\rm sl\,ad(\tilde{\phi})}(\tilde{M}; E) \longrightarrow \Psi^{-\infty}_{\rm sl\,ad(\tilde{\phi})}(\tilde{M}; E) \xrightarrow{\sigma_{\rm sl} \oplus \sigma_{\rm ad}} \oplus \tag{12.68}$$

12.9. Analytic index

The analytic index map of Atiyah and Singer is defined for any fibration with compact fibres

$$\mathrm{Ind}_{\mathrm{a}} : \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) \longrightarrow \mathrm{K}^0(B). \tag{12.69}$$

It starts with the realization of the K-theory as equivalence classes of pairs $(\mathbb{E}, a)$ of a superbundle, i.e. pair or complex vector bundles $\mathbb{E} = (E^+, E^-)$, over M, the total space of the fibration, and an isomorphism $a \in \mathcal{C}^\infty(S^*(M/B); \mathbb{E})$ between the pull-backs to $S^*(M/B)$, the boundary of the radial compactification of $T^*(M/B)$. From the surjectivity of the symbol map for the algebra of pseudodifferential operators on the fibres of $\phi : M \longrightarrow B$,

$$\Psi^0(M/B; \mathbb{E}) \longrightarrow \mathcal{C}^\infty(S^*(M/B); \mathbb{E}) \tag{12.70}$$

we know there exists a family $A \in \Psi^0(M/B; \mathbb{E})$ with $\sigma_0(A) = a$.

Since a is assumed to be invertible, A is, by definition, elliptic. Again from the properties of the calculus we can choose $B \in \Psi^0(M/B; \mathbb{E}^-)$, $\mathbb{E}^- = (E^-, E^+)$, which is a two-sided parameterix for A, so

$$BA = \mathrm{Id} - R_+,\ AB = \mathrm{Id} - R^-,\ R^\pm \in \Psi^{-\infty}(M/B; E^\pm). \tag{12.71}$$

The existence of finite rank exhaustions $\pi_N^\pm \in \mathcal{C}^\infty(M/B; E^\pm)$, for which $\pi_N^\pm R^\pm \longmapsto R$ in $\Psi^{-\infty}(M/B; E^\pm)$ for any element R of this space, allows A to be stabilized to have finite rank. Namely, for N large enough, $\mathrm{Id} - R^+(\mathrm{Id} - \pi_N^+)$ is invertible, with inverse necessarily of the form $\mathrm{Id} - S$, $S \in \Psi^{-\infty}(M/B; E^+)$ and then

$$(\mathrm{Id} - R^+)(\mathrm{Id} - \pi_N^+) = (\mathrm{Id} - R^+(\mathrm{Id} - \pi_N^+))(\mathrm{Id} - \pi_N^+) \Longrightarrow (\mathrm{Id} - S)BA(\mathrm{Id} - \pi_N) = \mathrm{Id} - \pi_N. \tag{12.72}$$

From this it follows that $A(\mathrm{Id} - \pi_N)$ has null space precisely the range of π_N on each fibre. In particular its null spaces form a smooth bundle over B and since it has the same symbol we can replace A with $A(\mathrm{Id} - \pi_N)$. Since the numerical index of a Fredholm family, such as A, is constant, the range of this new choice of A has a finite dimensional complement of constant rank, which can be identified with the null space of A^* for choices of smooth inner products and a smooth family of fibre densities. Let $\pi^- \in \Psi^{-\infty}(M/B; E^-)$ be the family of projections onto this finite dimensional bundle. Then

$$(\mathrm{Id} - \pi^-)A = A \tag{12.73}$$

and B can be replaced by the generalized inverse, which is the inverse of A as a map from the range of $\mathrm{Id} - \pi^-$ to the range of $\mathrm{Id} - \pi_N$ extended as zero to the range of π^-. With this choice (12.71) is replaced by

$$BA = \mathrm{Id} - \pi_N^+,\ AB = \mathrm{Id} - \pi^-. \tag{12.74}$$

Proposition 12.17 (Analytic index). *The class $[\pi_N^+, \pi^-] \in \mathrm{K}^0(B)$ constructed above for N large enough depends only on $[a] \in \mathrm{K}^0_c(T^*(M/B))$ and not on the choices made and gives a well-defined homomorphism* (12.69)*.*

Proof. Choices:

(1) a as representative of $[a]$.
(2) A with symbol a
(3) N
(4) Adjoints, densities

Increasing N to N' adds the bundle $\pi^+_{N'} - \pi^+_N$ to the null space of the quantization of the symbol, changing $A(\mathrm{Id}^+_N)$ to $A(\mathrm{Id}^+_{N'})$ and adds the image of this bundle under A to the complement of the range – i.e. subtracts it from the range. Thus it leaves the index class unchanged. Stabilizing a by adding the identity on some bundle F which is added to both E^+ and E^- also does not change the index and bundle isomorphisms of $E^\pm$ do not change the index either. All the other equivalences can be done by smooth homotopies and this corresponds to adding an interval (or if you want to be very careful, a circle – by reversing the homotopy on the other side) to the base. Then the null and conull bundles are defined over $B \times [0, 1]$ and it follows that their restrictions to the ends are bundle isomorphic. Thus the analytic index is well-defined. □

The existence of an invertible family of pseudodifferential operators of any real order shows that the definition of the index can be extended to elliptic families of any (fixed) order.

12.10. Analytic and semiclassical index

We have now defined the analytic index in the form given by Atiyah and Singer and also a similar map using semiclassical quantization of projections. The latter has also been reduced to the odd semiclassical index by suspension. So the main remaining step in the proof of the index theorem, in K-theory, of Atiyah and Singer is the equality of the analytic and semiclassical index maps.

Theorem 12.1 (Analytic=Semiclassical index). *The maps* (12.69) *and* (12.20) *are equal.*

Obviously we need to 'put the semiclassical and standard quantizations together.' Once again we do this by developing (yet) another calculus of operators! Fortunately in this case it is the semiclassical calculus for symbols of finite order, rather than the smoothing operators used in the semiclassical calculus, and we have been carrying this along for some time.

The main difference between the two index maps is the realizations of the compactly supported K-theory of the fibrewise cotangent bundle on which they are based. To remove irrelevancies, consider the more general case of a real vector bundle V over a compact manifold B. The index may of Atiyah and Singer is based on the identification of the K-theory with compact supports of V as the K-theory of the radial compactification $\bar{V}$, relative to its boundary $\mathbb{S}V = (V \setminus 0/\mathbb{R}^+$, together with the fact that any vector bundle over $\bar{V}$ is isomorphic to the pull-back of a bundle over B.

Forgetting the latter fact we consider the more general 'chain space' for K-theory consisting of triples (π^+, π^-, a) where $\pi^\pm \in \mathcal{C}^\infty(\bar{V}; M(N, \mathbb{C}))$ are smooth families of projections, $(\pi^\pm)^2 = \pi^\pm$, over the radial compactification and $a \in \mathcal{C}^\infty(\mathbb{S}V; M(N, \mathbb{C}))$ is an identificaiton of their ranges over $\mathbb{S}V$, namely we demand that

$$a\pi^+ = a,\ \pi^- a = a,\ \mathrm{Nul}(a) = (\pi^+),\ \mathrm{Ran}(a) = \mathrm{Ran}(\pi^-). \tag{12.75}$$

That is, a is precisely an isomorphism of the range of π^+ to the range of π^- over the boundary of the radial compactication.

This could be said better!

PROPOSITION 12.18. *The triples* (π^+,π^-,a) *above give* $\mathrm{K}^1_c(V)$ *under the equivalence relations of bundle isomorphism over* $\bar V$ *for* $\pi^\pm$*, homotopy for* a *and stability, in the sense of taking direct sum with* (p,p,Id_p) *where* $p\in\mathcal{C}^\infty(\bar V;M(N',\mathbb{C}))$ *is some family of projections. Moreover the inclusion of* $(\pi,\pi_\infty,\pi_\infty)$*, where* $\pi\in\mathcal{C}^\infty(\bar V,M(N,\mathbb{C}))$ *is a family of projections constant near infinity (with constant value* π_∞*) and of* $(\mathbb{E},a)$ *by complementing* E^- *to a trivial bundle, induce retractions of the chain spaces.*

PROOF. Deformation. □

Now the idea is that if we can define a 'big' index map from these general triples (π^+,π^-,a) which reduces to the semiclassical and the analytic index maps under the inclusions of these chain spaces, then we prove the desired equality. In fact we will simplify the 'big' chain space by arranging that

$$\pi^+\in M(N,\mathbb{C}) \tag{12.76}$$

is actually constant. We can do this by stabilizing to a trivial bundle. **** Do more

To 'quantize' a general triple subject to (12.76), we first use Proposition **** to choose a semiclassical family of projections $\Pi^-\in\Psi^0_{\mathrm{sl}}(M/B;\mathbb{C}^N)$ with

$$\sigma_{\mathrm{sl}}(\Pi^-)=\pi^-\text{ and }\sigma_0(\Pi^-)=\pi^-\big|_{S^*(M/B)}\ \forall\ \epsilon>0. \tag{12.77}$$

Then we choose a standard quantization of the family of matrices a, namely $A'\in\Psi^0(M/B;\mathbb{C}^N)$ with $\sigma_0(A')=a$. For $\epsilon>0$ but sufficiently small consider the family of 'Toeplitz' operators

$$A=\Pi^-A'\Pi^+\in\Psi^0(M/B;\mathbb{C}^N),\ \Pi^+=\pi^+\in M(N,\mathbb{C}). \tag{12.78}$$

Now, let π_N be our usualy family of finite rank smooth projecitons approximation the identity on the fibres of M/B. As in the standard case, we shall check that

$$A(\mathrm{Id}-\pi_N)\text{ has null space }\mathrm{Ran}(\pi_N\pi^+) \tag{12.79}$$

where by arrangement, $\pi_N\pi^+$ is a family of projections, so defines a smooth bundle over B. Now, (12.79) is just the usual parametrix argument. Let b be the inverse of a as a map from $\mathrm{Ran}(\pi^-)$ to $\mathrm{Ran}(\pi^+)$. Thus it has the same properties as a in (12.75) but with the signs reversed. Then quantize it to $B'\in\Psi^0(M/B;\mathbb{C}^N)$ and replace this by $B=\Pi^+B'\Pi^-$. From the symbol calculus it follows that

$$BA=\Pi^+(\mathrm{Id}+R^+)\Pi^+,\ \Pi^+R^+\Pi^+=R^+ \tag{12.80}$$

where initially $R^+\in\Psi^{-1}(M/B;\mathbb{C}^N)$. Then taking an asymptotic sum $(\mathrm{Id}+R)$, with $R\in\Psi^{-1}(M/B;\mathbb{C}^N)$ and $\Pi^+R\Pi^+=R$ of the Neumann series for $(\mathrm{Id}+R)$ and composing on the left with $\Pi^+(\mathrm{Id}+R)\Pi^+$ gives (12.80) with error $R^+\in\Psi^{-\infty}(M/B;\mathbb{C}^N)$. Then (12.77) follows since $R^+(\mathrm{Id}-\Pi_N)\to 0$ in $\Psi^{-\infty}(M/B;\mathbb{C}^N)$.

Once the null space of A has been stabilized to a bundle, i.e. it is replaced by $A(\mathrm{Id}-\pi_N)$ for N sufficiently large, it follows that its range inside the range of the family Π^- has finite dimensional complement, given by a smooth family of projections $\tilde\pi\in\Psi^{-\infty}(M/B;\mathbb{C}^N$ with $\Pi^-\tilde\pi\Pi^-=\tilde\pi$. Then the index in this more general setting is

$$\mathrm{Ind}(\pi^+,\pi^-,A)=[\Pi^+\pi_N\ominus\tilde\pi]\in\mathrm{K}^0(B). \tag{12.81}$$

So, it remains to check that this is independent of the choices made in its definition and that it reduces to the semiclassical index and the analytic index in the corresponding special cases. ****

12.11. Atiyah-Singer index theorem in K-theory

In Theorem 12.1 the two variants of the index map introduced above, have been shown to be equal. The index theorem of Atiyah and Singer therefore reduces to the equality of either of these and the topological index

$$\operatorname{Ind}_{\mathrm{t}} : \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) \longrightarrow \mathrm{K}^0(B) \tag{12.82}$$

which we proceed to define. As the name indicates this map does not involve any 'analytic constructions', except that Bott periodicity is involved which we proved analytically. This third map (12.82) for a fibration (12.61) is defined using an embedding into a trivial fibration as in Proposition 12.1. The Collar Neighbourhood Theorem shows that for each point in the base $b \in B$ the corresponding fibre has a neighbourhood $\Omega_b \subset \mathbb{R}^M$ which is a bundle over Z_b which is diffeomorphic (with Z_b mapped to the zero section) to the normal bundle of $Z_b \subset \mathbb{R}^M$. Moverover this is all smooth in b so that in

$$\begin{array}{ccccc} N & \longleftrightarrow & \Omega & \hookrightarrow & B \times \mathbb{S}^M \\ & \searrow & \uparrow & & \swarrow \\ & & M & \swarrow & \\ & & \downarrow & & \\ & & B. & & \end{array} \tag{12.83}$$

the bundle maps are consistent.

Since Ω is smoothly (although by no means naturally) identified with a bundle over M it follows that the relative cotangent bundle of Ω as a bundle over B is smoothly identified as

$$T^*(\Omega/B) \simeq T^*(M/B) \oplus N \oplus N^*. \tag{12.84}$$

Since $N \oplus N^*$ is a symplectic bundle over M we know from the Thom isomorphism that

$$\mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/B)) \simeq \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)). \tag{12.85}$$

On the other hand $\Omega \hookrightarrow B \times \mathbb{R}^M$ is an open embedding of fibrations, so there is a pull-back map for compactly supported K-theory:

$$\mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/B)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\mathbb{R}^M) \times B) \tag{12.86}$$

where on the right the relative cotangent bundle of $\mathbb{R}^M \times B$ as a bundle over B is written out. Finally $T^*(\mathbb{R}^M) = \mathbb{R}^{2M}$ so using Bott periodicity combined with the previous maps we define the topological index as the composite

$$\operatorname{Ind}_{\mathrm{t}} : \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/B)) \xrightarrow{\iota^*} \mathrm{K}^0_{\mathrm{c}}(T^*(\mathbb{R}^M) \times B)) \equiv \mathrm{K}^0(B). \tag{12.87}$$

It is not immediately clear that this map is independent of the embedding of the fibration which is used to define it. This is not difficult to show directly but we will instead show that it is equal to something which we already know to be independent of choices.

THEOREM 12.2 (Index theorem in K-theory). *The topological index map* (12.87) *is equal to the analytic index map.*

PROOF. We will follow the proof in [**2**] at least in outline. That is, we follow the semiclassical index through the diagramme (12.83) and check that the analytic index factors through each step. In view of Theorem 12.1 we can use either the analytic or the semiclassical index map at each stage.

The first stage is to consider the iterated fibration

$$N \longrightarrow M \longrightarrow B. \tag{12.88}$$

Here, N is the normal bundle to the embedding of M. Theorem 12.15 applies to this iterated fibration and gives the commutativity of the three maps on the right, corresponding to (12.59)

$$\begin{array}{c} \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)\oplus(N\oplus N^*)) \\ \mathrm{Thom}\uparrow\downarrow\mathrm{Ind_a} \qquad \searrow \mathrm{Ind_{sl}} \\ \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) \\ \downarrow \mathrm{Ind_a} \\ \mathrm{K}^0(B). \end{array} \tag{12.89}$$

Since we know that the analytic index is the inverse of the Thom isomorphism we conclude that

$$\mathrm{Ind_a} = \mathrm{Ind_{sl}} \circ \mathrm{Thom}\,. \tag{12.90}$$

The space on the top in (12.89) is diffeomorphic to Ω as a bundle over B so the same identity (12.90) holds with the semiclassical index map for Ω. Thus we have passed through the first map in (12.87) to start the commutative digramme

$$\begin{array}{ccccc} \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) & \longrightarrow & \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/B)) & \xrightarrow{\iota^*} & \mathrm{K}^0_{\mathrm{c}}(T^*(\mathbb{R}^M)\times B)) \\ & \searrow_{\mathrm{Ind_a}} & \downarrow \mathrm{Ind_{sl}} & \swarrow_{\mathrm{Ind_{sl}}} & \\ & & \mathrm{K}^0(B). & \xleftarrow{\mathrm{Ind_t}} & \end{array} \tag{12.91}$$

Since we already know excision for the semiclassical index, the right triangle in (12.91) also commutes. Then again Bott perioditicity is the same as the index map (now thought of as 'analytic') so in fact the (12.91) shows that the topological index in (12.87) also defines the analytic, or semiclassical, index. □

12.12. Chern character of the index bundle

In the case of the isotropic or Toeplitz index, which is to say the Thom isomorphism, we have already obtained a formula for the Chern character of the index in Proposition 10.23. Starting form this and following the proof above of the index theorem in K-theory, the computation of the Chern character of the index bundle is fairly straightforward. The main complication is that we have a plethora of index maps and we have to keep them a little separated (even though the are all the same). The simplest, and from the current perspective the most fundamental, is the semiclassical odd index.

THEOREM 12.3. *For any fibration the Chern character of the semiclassical odd index map is given by*

$$\mathrm{Ch}^{\mathrm{odd}}(\mathrm{Ind}_{\mathrm{sl}}(a)) = \int_{T^*(M/B)} \mathrm{Ch}^{\mathrm{odd}}(a) \wedge \mathrm{Td}(\phi), \tag{12.92}$$

where the Todd class of the fibration is fibre integral of the Chern class of the Bott element on the normal bundle to an embedding of the fibration in a trivial bundle over the base.

PROOF. The index map for a trivial bundle has been shown in §10.13 to be given by the integration of the Chern character over the fibres, in either the odd or even cases. The index map itself is shown above to factor through this Thom case by embedding

$$\mathrm{Ind}(a) = \mathrm{Ind}(\iota(a \otimes \beta_N)) \tag{12.93}$$

where β_N is the Bott element on the fibres of the normal bundle to the embedding and ι represents the inclusion ('excision') map for K-theory with compact supports in a neighbourhood of the embedding of the total space of the fibration. Thus, applying the bundle case and then the consistency properties of the Chern character,

$$\begin{gathered}\mathrm{Ch}^{\mathrm{odd}}(\mathrm{Ind}(a)) = \mathrm{Ch}^{\mathrm{odd}}(\mathrm{Ind}(\iota(a \otimes \beta_N)) \\ = \int_{\mathbb{R}^{2N}} \mathrm{Ch}^{\mathrm{odd}}(\iota(a \otimes \beta_N)) = \int_{N \oplus N*} \mathrm{Ch}^{\mathrm{odd}}(a) \wedge \mathrm{Ch}(\beta_N) \\ = \int_{T^*(M/B)} \mathrm{Ch}^{\mathrm{odd}}(a) \wedge \mathrm{Td}(\phi).\end{gathered} \tag{12.94}$$

Here the Todd class of the bundle is the integral over the fibres of the Bott element on normal bundle to the embedding. □

Since $\mathrm{Td}(\phi)$ is an absolute cohomology class on $T^*(M/B)$ it can also be identified, via the 'easy' Thom isomorphism, with the pull-back of a cohomology class on M; this is the usual interpretation. Of course one would like to know that $\mathrm{Td}(\phi)$ is determined by ϕ and not by the chosen embedding of M in a trivial bundle. However, the Todd class, being the Chern character of the Bott element, or harmonic oscillator, is stable under the addition of trivial bundles – this again follows from the discussion in §10.13. Under duality it switches sign, since this is just reversing the order of N and N^*. Since the embedding is into a trivial space we see that the normal bundle is a summand of the tangent bundle to a trivial bundle. It follows that its the Todd class is independent of choices. It can of course be identified with a characteristic class but I will not do this here.

Other cases, even semiclassical and Atiyah-Singer now follow from the previous identifications.

12.13. Dirac families

The most commonly encountered families of non-self-adjoint elliptic differential operators, at least in a geometric setting, are Dirac operators. So we discuss these briefly and derive the index formula in cohomology in that case. Indeed, computations based on the special properties of Dirac operators can be used to derive the index formula in general.

12.14. Spectral sections

Problems

Problem 12.1. Proof of Proposition 12.1.

Problem 12.2. Embedding manifolds.

APPENDIX A

Bounded operators on Hilbert space

Some of the main properties of bounded operators on a complex Hilbert space, H, are recalled here; they are assumed at various points in the text.

(1) Boundedness equals continuity, $\mathcal{B}(H)$.
(2) $\|AB\| \le \|A\|\|B\|$
(3) $(A - \lambda)^{-1} \in \mathcal{B}(H)$ if $|\lambda| \ge \|A\|$.
(4) $\|A^*A\| = \|AA^*\| = \|A\|^2$.
(5) Compact operators, defined by requiring the closure of the image of the unit ball to be compact, form the norm closure of the operators of finite rank.
(6) Fredholm operators have parametrices up to compact errors.
(7) Fredholm operators have generalized inverses.
(8) Fredholm operators for an open subalgebra.
(9) Hilbert-Schmidt operators?
(10) Operators of trace class?
(11) General Schatten class?

[1]

[1] Known as Gerard, my PhD advisor

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