CHAPTER 10

K-theory

This is a brief treatment of K-theory, enough to discuss, and later to prove, the Atiyah-Singer index theorem. I am starting from the smoothing algebra discussed earlier in Chapter 4 in order to give a 'smooth' treatment of K-theory (this approach is in fact closely related to the currently-in-vogue subject of 'smooth K-theory'). In particular the K-groups $\mathrm{K}^1_{\mathrm{c}}(X)$ and $\mathrm{K}^0_{\mathrm{c}}(X)$ of any manifold X, corresponding to compactly-supported classes, are defined. The elementary properties are derived and important isomorphism between them are discussed. There is a plethora of such maps which are listed here to try to help keep them straight:-

(10.1) The clutching construction, Proposition 10.6
$$\mathrm{clu} : \mathrm{K}^1_{\mathrm{c}}(X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(\mathbb{R} \times X).$$

(10.2) The 1-dim isotropic index, Proposition 10.9
$$\mathrm{Ind}_{\mathrm{iso}} : \mathrm{K}^1_{\mathrm{c}}(\mathbb{R} \times X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X).$$

(10.3) The 1-dim Toeplitz index, elliptics on the circle §10.7
$$\mathrm{Ind}_{\mathrm{To}} : \mathrm{K}^1_{\mathrm{c}}(\mathbb{R} \times X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X)$$

(10.4) N-dim isotropic index, quantize elliptic symbols
$$\mathrm{Ind}_{\mathrm{iso}} : \mathrm{K}^1_{\mathrm{c}}(\mathbb{R}^{2N-1} \times X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X).$$

(10.5) N-dim odd semiclassical index, quantize invertible matrices
$$\mathrm{Ind}^{\mathrm{odd}}_{\mathrm{iso,sl}} : \mathrm{K}^1_{\mathrm{c}}(\mathbb{R}^{2N} \times X) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(X).$$

(10.6) N-dim even semiclassical index, quantize projections
$$\mathrm{Ind}^{\mathrm{even}}_{\mathrm{iso,sl}} : \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X).$$

(10.7) N-dim isotropic index, quantize elliptic symbols
$$\mathrm{Ind}_{\mathrm{iso}} : \mathrm{K}^0_{\mathrm{c}}(E) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X).$$

(10.8) Odd semiclassical index quantize invertible matrices
$$\mathrm{Thom}^{\mathrm{odd}} = \mathrm{Ind}^{\mathrm{odd}}_{\mathrm{iso,sl}} : \mathrm{K}^1_{\mathrm{c}}(E) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(X).$$

(10.9) Even semiclassical index quantize projections
$$\mathrm{Thom} = \mathrm{Ind}^{\mathrm{even}}_{\mathrm{iso,sl}} : \mathrm{K}^0_{\mathrm{c}}(E) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X).$$

(10.10) The Bott map, tensor with β_E
$$\mathrm{Bott} = \mathrm{Thom}^{-1} : \mathrm{K}^0_{\mathrm{c}}(X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(E).$$

The three maps before the last are for a real vector bundle E over X with symplectic structures on the fibres – they are the same as the preceeding three in the case of a trivial bundle except that the first of those then involves the inverse of the clutching construction.

10.1. What do I need for the index theorem?

Here is a summary of the parts of this chapter which are used in the proof of the index theorem to be found in Chapter 12.

(1) Odd K-theory ($\mathrm{K}^1_{\mathrm{c}}(X)$) defined as stable homotopy classes of maps into $\mathrm{GL}(N,\mathbb{C})$, or as homotopy classes of maps into $G^{-\infty}$.
(2) Even K-theory ($\mathrm{K}_{\mathrm{c}}(X)$) defined as stable isomorphism classes of $\mathbb{Z}_2$-graded bundles.
(3) The gluing identification of $\mathrm{K}^1_{\mathrm{c}}(X)$ and $\mathrm{K}_{\mathrm{c}}(\mathbb{R}\times X)$.
(4) The isotropic index map $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X)\longrightarrow \mathrm{K}_{\mathrm{c}}(X)$ using the eigenprojections of the harmonic oscillator to stabilize the index.
(5) Bott periodicity – proof that this map is an isomorphism and hence that $\mathrm{K}_{\mathrm{c}}(X)\equiv \mathrm{K}_{\mathrm{c}}(\mathbb{R}^2\times X)$.
(6) Thom isomorphism $\mathrm{K}_{\mathrm{c}}(V)\longrightarrow \mathrm{K}_{\mathrm{c}}(X)$ for a complex (or symplectic) vector bundle over X. In particular the identification of the 'Bott element' $\beta_V\in \mathrm{K}_{\mathrm{c}}(V)$ which generates $\mathrm{K}_{\mathrm{c}}(V)$ as a module over $\mathrm{K}_{\mathrm{c}}(X)$.

With this in hand you should be able to proceed to the proof of the index theorem in K-theory in Chapter 12. If you want the 'index formula' which is a special case of the index theorem in cohomology you need a bit more, namely the discussion of the Chern character and Todd class below.

10.2. Odd K-theory

First recall the 'smoothing group'

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)=\left\{A\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n);\ \exists\ B\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ \mathrm{Id}+B=(\mathrm{Id}+A)^{-1}\right\}. \tag{10.11}$$

Note that the notation is potentially confusing here. Namely, I am thinking of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ as the subset consisting of those $A\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ such that $\mathrm{Id}+A$ is invertible. The group product is then not the usual product on $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ since

$$(\mathrm{Id}+A_1)(\mathrm{Id}+A_2)=\mathrm{Id}+A_1+A_2+A_1A_2.$$

Just think of the operator as 'really' being $\mathrm{Id}+A$ but the identity is always there so it is dropped from the notation.

One consequence of the fact that $\mathrm{Id}+A$ is invertible if and only if $\det(\mathrm{Id}+A)\neq 0$ is that[1]

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)\subset\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)=\mathcal{S}(\mathbb{R}^{2n})=\dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^{2n}})\text{ is open and dense.} \tag{10.12}$$

In view of this there is no problem in understanding what a smooth map into $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is. Namely, it is a map into $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ which has range in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and the following statment can be taken as a definition of smoothness, but it is just equivalent to the standard notion of a smooth map with values in a topological vector space. Namely if X is a manifold then

$$\begin{aligned}\mathcal{C}^\infty(X;G^{-\infty})&=\\ &\{a\in\mathcal{C}^\infty(X\times\overline{\mathbb{R}^{2n}});a\equiv 0\text{ at }X\times\mathbb{S}^{2n-1},\ a(x)\in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)\ \forall\ x\in X\},\\ \mathcal{C}^\infty_c(X;G^{-\infty})&=\{a\in\mathcal{C}^\infty(X\times\overline{\mathbb{R}^{2n}});a\equiv 0\text{ at }X\times\mathbb{S}^{2n-1},\\ &\quad a(x)\in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)\ \forall\ x\in X,\exists\ K\Subset X\text{ s.t. }a(x)=0\ \forall\ x\in X\setminus K\}.\end{aligned} \tag{10.13}$$

[1]See Problem 10.8 if you want a proof not using the Fredholm determinant.

Notice that 'compact supports' here means that the actual operator we have in mind, which is $\mathrm{Id} + a$, reduces to the identity outside a compact set.

The two spaces in (10.13) (they are the same if X is compact) are groups. They are in fact examples of gauge groups (with an infinite-dimensional target group), where the composite of a and b is the map $a(x)b(x)$ given by composition in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Two elements $a_0,\ a_1 \in \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}})$ are said to be homotopic (in fact smoothly homotopic, but that is all we will use) if there exists $a \in \mathcal{C}^\infty_c(X \times [0,1]_t; G^{-\infty}_{\mathrm{iso}})$ such that $a_0 = a\big|_{t=0}$ and $a_1 = a\big|_{t=1}$. Clearly if b_0 and b_1 are also homotopic in this sense then $a_0 b_0$ is homotopic to $a_1 b_1$, with the homotopy just being the product of homotopies. This gives the group property in the following definition:-

Definition 10.1. *For any manifold*

$$\mathrm{K}^1_c(X) = \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n))/\sim \tag{10.14}$$

is the group of equivalence classes of elements under homotopy.

Now, we need to check that this is a reasonable definition, and in particular see how is it related to K-theory in the usual sense. To misquote Atiyah, K-theory is the topology of linear algebra. So, the basic idea is that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is just a version of $\mathrm{GL}(N, \mathbb{C})$ where $N = \infty$ (but smoother than the usual topological versions). To make this concrete, recall that finite rank elements are actually dense in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Using the discussion of the harmonic oscillator in Chapter 4 we can make this even more concrete. Let $\pi_{(N)}$ be the projection onto the span of the first N eigenvalues of the harmonic oscillator (so if $n > 1$ it is projecting onto space of dimension a good deal larger than N, but no matter). Thus $\pi_{(N)} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is an operator of finite rank, exactly the sum of the dimensions of these eigenspaces. Then, from the discussion in Chapter 4

$$\begin{gathered} f \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow \pi_{(N)} f \to f \text{ in } \mathcal{S}(\mathbb{R}^n) \text{ as } N \to \infty, \\ A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \Longrightarrow \pi_{(N)} A \pi_{(N)} \to A \text{ in } \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \text{ as } N \to \infty. \end{gathered} \tag{10.15}$$

The range of $\pi_{(N)}$ is just a finite dimensional vector space, so isomorphic to $\mathbb{C}^M$ (where M depends on N and n, the simplest case is $n = 1$ since then $M = N$). We will choose a fixed linear isomorphism to $\mathbb{C}^M$ by choosing a particular basis of eigenfunctions of the harmonic oscillator. If $a \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then $\pi_{(N)} a \pi_{(N)}$ becomes a linear operator on $\mathbb{C}^M$, so an element of the matrix algebra.

Proposition 10.1. *The 'finite rank elements' in $\mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n))$, those for which $\pi_{(N)} A = A \pi_{(N)} = A$ for some N, are dense in $\mathcal{C}^\infty_c(X; G^{-\infty}(\mathbb{R}^n))$.*

These elements are really to be thought of as finite rank perturbations of the identity.

Proof. This just requires a uniform version of the argument above, which in fact follows from the pointwise version, to show that

$$A \in \mathcal{C}^\infty_c(X; \Psi^{-\infty}_{\mathrm{iso}}) \Longrightarrow \pi_{(N)} A \pi_{(N)} \to A \text{ in } \mathcal{C}^\infty_c(X; \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)). \tag{10.16}$$

From this it follows that if $A \in \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n))$ (meaning if you look back, that $\mathrm{Id} + A$ is invertible) then $\mathrm{Id} + \pi_{(N)} A$ is invertible for N large enough (since it vanishes outside a compact set). $\square$

COROLLARY 10.1. *The groups $\mathrm{K}^1_c(X)$ are independent of n, the dimension of the space on which the group acts (as is already indicated by the notation).*

In fact this shows that $\pi_{(N)}a\pi_{(N)}$ and a are homotopic in $\mathcal{C}^\infty_c(X; G^{-\infty}_{\text{iso}}(\mathbb{R}^n))$ provided N is large enough. Thus each element of $\mathrm{K}^1_c(X)$ is represented by a finite rank family in this sense (where the order N may depend on the element and the identity needs to be added). Any two elements can then be represented by finite approximations for the same N. Thus there is a natural isomophism between the groups corresponding to different n's by finite order approximation. In fact this approximation argument has another very important consequence.

PROPOSITION 10.2. *For any manifold $\mathrm{K}^1_c(X)$ is an Abelian group, i.e. the group product is commutative.*

PROOF. In view of the preceeding result it suffices to take $n = 1$ so N and the rank of $\pi_{(N)}$ are the same. As shown above, given two elements $[a]$, $[b] \in \mathrm{K}^1_c(X)$ we can choose representatives a, $b \in \mathcal{C}^\infty_c(X; G^{-\infty}_{\text{iso}}(\mathbb{R}^n))$ such that $\pi_{(N)}a = a\pi_{(N)} = a$ and $\pi_{(N)}b = b\pi_{(N)} = b$. Thus they are represented by elements of $\mathcal{C}^\infty(X; \mathrm{GL}(N, \mathbb{C}))$ for some large N (so the actual element is $\mathrm{Id}_{(N)} + \pi_{(N)}a\pi_{(N)}$). Now, the range of $\pi_{(2N)}$ contains two N dimensional spaces, the ranges of $\pi_{(N)}$ and $\pi_{(2N)} - \pi_{(N)}$. Since we are picking bases in each, we can identify these two N dimensional spaces and then represent an element of the $2N$-dimensional space as a 2-vector of N-vectors. This decomposes $2N \times 2N$ matrices as 2×2 matrices with $N \times N$ matrix elements. In fact this tensor product of the 2×2 and $N \times N$ matrix algebras gives the same product as $2N \times 2N$ matrices (as follows easily from the definitions). Now, consider a rotation in 2 dimensions, represented by the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{10.17}$$

This rotates the standard basis e_1, e_2 to e_2, $-e_1$ as θ varies from 0 to $\pi/2$. If we interpret it as having entries which are multiples of the identity as an $N \times N$ matrix, and then conjugate by it, we get a curve

$$\begin{aligned} a(x,\theta) &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} a\cos^2\theta + \sin^2\theta & (\mathrm{Id} - a)\sin\theta\cos\theta \\ (\mathrm{Id} - a)\sin\theta\cos\theta & \cos^2\theta + a\sin^2\theta \end{pmatrix}. \end{aligned} \tag{10.18}$$

This is therefore an homotopy between a represented as an $N \times N$ matrix and the same element acting on the *second* N dimensional subspace, i.e. it becomes

$$\begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & a \end{pmatrix}. \tag{10.19}$$

This commutes with the second element which acts only in the first N dimensional space, so, because of homotopy equivalence, the product in $\mathrm{K}^1_c(X)$ is commutative. □

So now we see that $\mathrm{K}^1_c(X)$ is an Abelian group associated quite naturally to the space X. I should say that the notation is not quite standard. Namely the standard notation would be $\mathrm{K}^1(X)$, without any indication of the 'compact supports' that are involved in the definition. I prefer to put this in explicitly. Of course if X is compact it is not necessary.

LEMMA 10.1. *Any proper smooth map* $f : X \longrightarrow Y$ *induces a homomorphism* $f^* : \mathrm{K}^1_c(Y) \longrightarrow \mathrm{K}^1_c(X)$ *by composition; the map* f^* *only depends on the homotopy class of* f *in proper smooth maps.*

This makes $\mathrm{K}^1_{\mathrm{c}}$ into a contraviant functor on the category of manifolds and proper maps to the category of abelian groups, if you like to think in those terms.

PROOF. If $a \in \mathcal{C}^\infty_c(Y; G^{-\infty}_{\mathrm{iso}})$ then $f^*a = a \circ f \in \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}})$ where the compactness of the support is a consequence of the assumed properness of the map – that $f^{-1}(K) \Subset X$ for any $K \Subset Y$. Homotopies lift to homotopies, so it is straightforward to check that this is a homomorphism at the level of $\mathrm{K}^1_{\mathrm{c}}$ and that it only depends on the homotopy class of f. □

Thus, since it is contravariant, 'pull-back' is the natural operation on K-theory. The index theory that we discuss in Chapter 12 is concerned with the 'wrong-way' map, i.e. push-forward, for K-theory.

LEMMA 10.2 (Excision). *The inclusion of any open set* $i : U \longrightarrow X$ *induces a natural map*

$$i_! : \mathrm{K}^1_c(U) \longrightarrow \mathrm{K}^1_c(X). \tag{10.20}$$

PROOF. Any smooth map with compact support $a \in \mathcal{C}^\infty_c(U; G^{-\infty})$ can be extended as the identity to give a smooth map $\tilde{a} \in \mathcal{C}^\infty_c(X; G^{-\infty})$ so fixed by the properties $\tilde{a} = a$ on U, $\tilde{A} = 0$ on $X \setminus U$. Homotopies also extend in this way so this induces the natural map (10.20). □

A fundamental role is played below by the following result computing the odd K-theory of the product $\mathbb{S} \times X$ of a general manifold and a circle.

PROPOSITION 10.3. *For any manifold the natural projection,* $\pi : X \times \mathbb{S} \longrightarrow X$, *the inclusion* $\iota : X \times \mathbb{R} \longrightarrow X \times \mathbb{S}$ *given by the 1-point compactification of* $\mathbb{R}$ *and the inclusion* $p_1 : X \ni x \longmapsto (x, 1) \in X \times \mathbb{S}$, *lead to a split short exact sequence*

$$0 \longrightarrow \mathrm{K}^1_c(X \times \mathbb{R}) \xrightarrow{\iota_!} \mathrm{K}^1_c(X \times \mathbb{S}) \underset{\pi^*}{\overset{p_1^*}{\rightleftarrows}} \mathrm{K}^1_c(X) \longrightarrow 0 \tag{10.21}$$

and hence an isomorphism

$$\mathrm{K}^1_c(X \times \mathbb{S}) = \mathrm{K}^1_c(X \times \mathbb{R}) \oplus \mathrm{K}^1_c(X) \tag{10.22}$$

PROOF. Certainly $\pi \circ p_1 = \mathrm{Id}_X$ so $p_1^* \circ \pi^* = \mathrm{Id}$ shows that p_1^* must be surjective and π^* injective. Since $1 \in \mathbb{S}$ is not in the image of ι, every class in the image of $\iota_!$ has a representative which is equal to the identity on the image of p_1, so pulls back to zero in $\mathrm{K}^1_{\mathrm{c}}(X)$, so $p_1^* \circ \iota_! = 0$.

Since an element in $\mathcal{C}^\infty_c(X \times \mathbb{S}; G^{-\infty})$ which vanishes at $X \times \{1\}$ is homotopic through such elements to one which vanishes near $X \times \{1\}$ (and with supports uniformly compact) this sequence corresponds to the short exact sequence of groups

$$\begin{aligned}\big\{a \in \mathcal{C}^\infty_c(X \times \mathbb{S}; G^{-\infty}_{\mathrm{iso}}); a(x,1) = 0 \ \forall\ x \in X\big\} \longrightarrow \\ \mathcal{C}^\infty_c(X \times \mathbb{S}; G^{-\infty}_{\mathrm{iso}}) \longrightarrow \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}}).\end{aligned} \tag{10.23}$$

Under homotopy this becomes the direct sum decomposition (10.22). □

Thus there are two Abelian groups $\mathrm{K}^1_{\mathrm{c}}(X)$ and $\mathrm{K}^1_{\mathrm{c}}(X \times \mathbb{R})$ associated to the manifold X with direct sum naturally $\mathrm{K}^1_{\mathrm{c}}(\mathbb{S}\times X)$. As we shall see below it is perfectly natural to *define* the even K-theory of X to be $\mathrm{K}^0_{\mathrm{c}}(X) = \mathrm{K}^1_{\mathrm{c}}(X \times \mathbb{R})$ (although the notation ${\mathrm{K}_{\mathrm{c}}}^{-2}(X)$ would be even better) and to denote the sum of the two terms as

$$\mathrm{K}^*_{\mathrm{c}}(X) = \mathrm{K}^1_{\mathrm{c}}(\mathbb{S} \times X). \tag{10.24}$$

We will not do this now, only because it is potentially confusing and instead will give a more standard definition of $\mathrm{K}^0_{\mathrm{c}}(X)$ and then define a natural index map (the isotropic index)

$$\mathrm{Ind}_{\mathrm{iso}} : \mathrm{K}^1_{\mathrm{c}}(X \times \mathbb{R}) \xrightarrow{\simeq} \mathrm{K}^0_{\mathrm{c}}(X). \tag{10.25}$$

If you know a little topology, you will see that the discussion above starts from the premise that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a *classifying space* for odd K-theory. So this is true by fiat. The corresponding classifying space for even K-theory is then the pointed loop group, the set of maps

$$G^{-\infty}_{\mathrm{iso,sus}}(\mathbb{R}^n) = \{a \in \mathcal{C}^\infty(\mathbb{S}; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n); a(1) = \mathrm{Id}\}. \tag{10.26}$$

10.3. Computations

Let us pause for a moment to compute some simple cases. Namely

Lemma 10.3.

$$\mathrm{K}^1(\{pt\}) = \{0\},\ \mathrm{K}^1_c(\mathbb{R}) = \mathbb{Z},\ \mathrm{K}^1(\mathbb{S}) = \mathbb{Z}. \tag{10.27}$$

Proof. The first two of these statements follow directly from the next two results. The third is a direct consequence of (10.22) and the first two. □

Lemma 10.4. *The group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is connected.*

Proof. If $a \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, the curve $[0,1] \ni t \longmapsto (1-t)a + t\pi_{(N)}a\pi_{(N)}$ lies in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ for N sufficiently large. Thus it suffices to show that $\mathrm{GL}(n,\mathbb{C})$ is connected for large N; of course[2]

$$\mathrm{GL}(N,\mathbb{C}) \text{ is connected for all } N \geq 1. \tag{10.28}$$

□

Proposition 10.4. *A closed loop in $\gamma : S \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is contractible (homotopic through loops to a constant loop) if and only if the composite map*

$$\tilde{\gamma} = \det \circ \gamma : \mathbb{S} \longrightarrow \mathbb{C}^* \tag{10.29}$$

is contractible, so

$$\pi_1(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)) = \mathbb{Z} \tag{10.30}$$

with the identification given by the winding number of the Fredholm determinant.

Proof. Again, as in the previous proof but now a loop can be deformed into $\mathrm{GL}(N,\mathbb{C})$ so it is certainly enough to observe that[3]

$$\pi_1\left(\mathrm{GL}(N,\mathbb{C})\right) = \mathbb{Z} \text{ for all } N \geq 1. \tag{10.31}$$

□

[2]See Problem 10.12
[3]Proof in Problem 10.13

An explicit generator of $\pi_1(G^{-\infty}_{\text{iso}})$ is given by the stabilization of the loop into $\mathrm{GL}(N,\mathbb{C}) = \mathbb{C}\setminus\{0\}$ which is the identity map on the circle embedded in $\mathbb{C}$:

$$\gamma(\theta) = e^{2\pi i\theta}. \tag{10.32}$$

10.4. Vector bundles

The notion of a complex vector bundle was briefly discussed earlier in Section 6.2. Recall from there the notion of a bundle isomorphism and that a bundle is said to be trivial, over some set K, if there is a bundle isomorphism from its restriction to K to $K \times \mathbb{C}^k$. The direct sum of vector bundles and the tensor product are also briefly discussed there.

To see that there is some relationship between K-theory as discussed above and vector bundles consider $\mathrm{K}^1(X)$ for a compact manifold, X. First note that if V is a complex vector bundle over X and $e : V \longrightarrow V$ is a bundle isomorphism, then e defines an element of $\mathrm{K}^1(X)$. To see this we first observe we can always find a complement to V.

Proposition 10.5. *Any vector bundle V which is trivial outside a compact subset of X can be complemented to a trivial bundle, i.e. there exists a vector bundle E, also trivial outside a compact set, and a bundle isomorphism*

$$V \oplus E \longrightarrow X \times \mathbb{C}^N. \tag{10.33}$$

Proof. This follows from the local triviality of V. Choose a finite open cover U_i of X with M elements in which one set is $U_0 = X \setminus K$ for K compact and such that V is trivial over each U_i. Then choose a partition of unity subordinate to U_i – so only the $\phi_0 \in \mathcal{C}^\infty(X)$ with support in U_0 does not have compact support. If $f_i : V\big|_{U_i} \longrightarrow \mathbb{C}^k \times U_i$ is a trivialization over U_i (so the one over U_0 is given by the assumed triviality outside a compact set) consider

$$F : V \longrightarrow X \times \mathbb{C}^{kM},\ u(x) \longmapsto \bigoplus_{i=1}^{M} f_i(\phi_i(u(x))). \tag{10.34}$$

This embeds V as a subbundle of a trivial bundle of dimension $N = kM$ since the map F is smooth, linear on the fibres and injective. Then we can take E to be the orthocomplement of the range of F which is identified with V. □

Thus, a bundle isomorphism e of V can be extended to a bundle isomorphism $e \oplus \mathrm{Id}_E$ of the trivial bundle. This amounts to a map $X \longrightarrow \mathrm{GL}(MN,\mathbb{C})$ which can then be extended to an element of $\mathcal{C}^\infty(X; G^{-\infty}_{\text{iso}}(\mathbb{R}^n))$ and hence gives an element of $\mathrm{K}^1_{\mathrm{c}}(X)$ as anticipated. It is straightforward to check that the element defined in $\mathrm{K}^1(X)$ does not depend on choices made in its construction, only on e (and through it of course on V).

This is one connection between bundles and $\mathrm{K}^1_{\mathrm{c}}$. There is another, similar, connection which is more important. Namely from a class in $\mathrm{K}^1_{\mathrm{c}}(X)$ we can construct a bundle over $\mathbb{S} \times X$. One way to do this is to observe that Proposition 10.5 associates to a bundle V a smooth family of projections $\pi_V \in \mathcal{C}^\infty_{\mathrm{c}}(X; M(N,\mathbb{C}))$ which is trivial outside a compact set, in the sense that it reduces to a fixed projection there. Namely, π_V is just (orthogonal) projection onto the range of V. We will need to think about equivalence relations later, but certainly such a projection defines a bundle as well, namely its range.

For the following construction choose a smooth function $\Theta : \mathbb{R} \longrightarrow [0,\pi]$ which is non-decreasing, constant with the value 0 on some $(-\infty,-T]$, constant with value $\pi/2$ on $[-T/2,T/2]$ and constant with the value π on $[T,\infty)$, for some $T>0$, and strictly increasing otherwise. It may also be convenient to assume that Θ is 'odd' in the sense that

$$\Theta(-t) = \pi - \Theta(t). \tag{10.35}$$

This is just a function which we can used to progressively 'rotate' through angle π but staying constant initially, near the middle and near the end. If $a \in \mathrm{GL}(N,\mathbb{C})$, consider the rotation matrix

$$S(\theta,a) = \begin{pmatrix} \cos(\theta)\,\mathrm{Id}_N & -\sin(\theta)a^{-1} \\ \sin(\theta)a & \cos(\theta)\,\mathrm{Id}_N \end{pmatrix} \in \mathrm{GL}(2N,\mathbb{C}). \tag{10.36}$$

This is invertible, in fact

$$\begin{gathered} S(\theta,a)S(\theta',a) = S(\theta+\theta',a),\ S(0,a)=\mathrm{Id}, \\ \frac{d}{d\theta}S(\theta,a) = S(\theta+\frac{\pi}{2},a) = \begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix} S(\theta,a), \end{gathered} \tag{10.37}$$

Now, for a compact manifold X, consider $a \in \mathcal{C}^\infty(X;\mathbb{C}^N)$ which is everywhere invertible then

$$\mathbb{R}\times X \ni (t,x) \longmapsto R_a(t,x) = S(\Theta(t),a(x)) \tag{10.38}$$

has inverse $R_a(-t,x)$ and is equal to the identity in $|t|>T$. We will use this to construct a bundle on $\mathbb{R}\times X$ which is trivial for $t>0$. The idea is that $R_a(t,x)$ 'rotates by $\pi/2$' as t runs over $(-\infty,0)$. Set

$$\Pi_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \Pi'_a(t,x) = \begin{cases} R_a(t,x)\Pi_\infty R_a(-t,x) & t\le 0 \\ R_{\mathrm{Id}}(t,x)\Pi_\infty R_{\mathrm{Id}}(-t,x) & t \ge 0. \end{cases} \tag{10.39}$$

Clearly, $\Pi'_a(t,x)$ is smooth in $t\le 0$, and in $t\ge 0$, and is constant outside a compact set. In fact Π'_a is globally smooth, since near $t=0$, $\Theta(t)=\pi/2$, by construction, so

$$\Pi'_a(0,x) = \begin{pmatrix} 0 & a^{-1} \\ -a & 0 \end{pmatrix} \Pi_\infty \begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix} \tag{10.40}$$

is independent of a and hence smooth. Thus in fact $\Pi'_a(t,x)$ is constant near $t=\pm\infty$ where it takes the value Π_∞, which is projection onto the first $\mathbb{C}^N$.

Note, for later reference that

$$\begin{gathered} \Pi'_a(t,x) = \begin{pmatrix} \cos^2(\Theta(t))\,\mathrm{Id}_N & \cos(\Theta(t))\sin(\Theta(t))a^{-1}(x) \\ \sin(\Theta(t))\cos(\Theta(t))a(x) & \sin^2(\Theta(t)) \end{pmatrix},\ t\le 0 \\ \text{has entries linear in } a \text{ and } a^{-1}. \end{gathered} \tag{10.41}$$

Notice that if the conjugating matrix in (10.40) did not jump as it does at $t=0$, but for instance we continued conjugating by $R_a(t,x)$ in $t\ge 0$ instead of switching to $a=\mathrm{Id}$, then the bundle which is the range of the family of projections would be globally isomorphic to the range of Π_∞, with $R_a(t,x)$ being the global isomorphism. In particular if $a=\mathrm{Id}$ this is indeed the case, so that at least $a=\mathrm{Id}$ corresponds to a trivial bundle.

This was all under the assumption that X is compact and the construction will not quite work if it is not, since then since then Π'_a outside a compact set, even when $a=\mathrm{Id}$. To cover the non-compact case we need to 'undo' the twisting at infinity in X which we do with a global isomorphism (not constant at infinity!) and consider instead

$$\Pi_a(t,x) = R_{\mathrm{Id}}(-t,x)\Pi'_a(t,x)R_{\mathrm{Id}}(t,x). \tag{10.42}$$

In case X is compact this is a global isomorphism, constant outside a compact set, and so gives the same bundle up to isomorphism. In the form (10.42) the projection is actually constant in $t \geq 0$.

LEMMA 10.5. *An element $a \in \mathcal{C}^\infty(X; \mathrm{GL}(N,\mathbb{C}))$ equal to the identity outside a compact set defines, through* (10.42)*, a smooth family of matrices with values in the projections, $\Pi_a \in \mathcal{C}^\infty(\mathbb{R} \times X; M(2n,\mathbb{C}))$, which is constant outside a compact subset and so defines a vector bundle over $\mathbb{R} \times X$ which is trivial outside a compact set.*

We will see below that this is one of the important identification maps for K-theory that we need to understand, in fact it leads to (10.1).

So, by now it should not be so surprising that the K-groups introduced above are closely related to the 'Grothendieck group' constructed from vector bundles. The main issue is the equivalence relation.

DEFINITION 10.2. *For a manifold X, $\mathrm{K}_c(X)$ is defined as the set of equivalence classes of pairs of complex vector bundles (V,W), both trivial outside a compact set and with given trivializations a, b there, under the relation $(V_1,W_1;a_1,b_1) \sim (V_2,W_2;a_2,b_2)$ if and only if there is a bundle S and a bundle isomorphism*

$$T : V_1 \oplus W_2 \oplus S \longrightarrow V_2 \oplus W_1 \oplus S \tag{10.43}$$

which is equal to $(a_2 \oplus b_2)^{-1}(a_1 \oplus b_2) \oplus \mathrm{Id}_S$ outside some compact set.

Note that if X is compact then the part about the trivializations is completely void, then we just have pairs of vector bundles (V,W) and the equivalence relation is the existence of a stabilizing bundle S and a bundle isomorphism (10.43).

This is again an Abelian group with the group structure given at the level of pairs of bundles (V_i,W_i), $i = 1, 2$ by[4]

$$[(V_1,W_1)] + [V_2,W_2)] = [(V_1 \oplus V_2, W_1 \oplus W_2)] \tag{10.44}$$

with the trivializations $(a_1 \oplus a_2), (b_1 \oplus b_2)$. In particular $[(V,V)]$ is the zero element for any bundle V (trivial outside a compact set).

The equivalence relation being (stable) bundle isomorphism rather than some sort of homotopy may seen strange, but it is actually more general.

LEMMA 10.6. *If V is a vector bundle over $[0,1]_t \times X$ which is trivial outside a compact set then $V_0 = V\big|_{t=0}$ and $V_1 = V\big|_{t=1}$ are bundle isomorphic over X with an isomorphism which is trivial outside a compact set.*

PROOF. The proof is 'use a connection and integrate'. We can do this explicitly as follows. First we can complement V to a trivial bundle so that it is identified with a constant projection outside a compact set, using Proposition 10.5. Let the family of projections be $\pi_V(t,x)$ in $M \times M$ matrices. We want to differentiate sections of the bundle with respect to t. Since they are M-vectors we can do this, but we may well not get sections this way. However defining the (partial) connection by

$$\nabla_t v(t) = v'(t) - \pi'_V v(t) \Longrightarrow (\mathrm{Id} - \pi_V)\nabla_t v(t) = ((\mathrm{Id} - \pi_V)v(t))' = 0 \tag{10.45}$$

[4] See Problem 10.11 for the details.

if $\pi_V v = v$, i.e. if v is a section. Now, by standard results on the existence, uniquenss and smoothness of solutions to differential equations, the condition $\nabla_t v(t) = 0$ fixes a unique section with $v(0) = v_0 \in V_0$ fixed. Then define $F : V_0 \longrightarrow V_1$ by $Fv_0 = v(1)$. This is a bundle isomorphism. □

PROPOSITION 10.6. *For any manifold X the construction in Lemma 10.5 gives the 'clutching' isomorphism*

$$\text{clu} : \mathrm{K}^1_c(X) \ni [a] \longrightarrow [(\Pi_a, \Pi_a^\infty)] = \mathrm{K}_c(\mathbb{R} \times X) \tag{10.46}$$

where Π_a^∞ is the constant projection to which Π_a restricts outside a compact set.

PROOF. The range of the projection Π_a in Lemma 10.5 fixes an element of $\mathrm{K}_\mathrm{c}(\mathbb{R} \times X)$ but we need to see that it is independent of the choice of a representing $[a] \in \mathrm{K}^1_\mathrm{c}(X)$. A homotopy of a gives a bundle over $[0, 1] \times X$ and then Lemma 10.6 shows that the resulting bundles are isomorphic. Stabilizing a, i.e. enlarging it by an identity matrix adds a constant projection to Π_a and the same projection projection to Π_a^∞. Thus the map in (10.46) is well defined. So we need to show that it is an isomorphism. First we should show that it is additive, this is straightforward – see Problem 10.9.

**** More detail.

If V is a bundle over $\mathbb{R}\times X$ which is trivial outside a compact set, we can embed it as in Proposition 10.5 so it is given by a family of projections π_V (this of course involves a bundle isomorphism). Now, using the connection as in (10.45) we can define an isomorphism of the trivial bundle π_V^∞. Namely, integrating from $t = -T$ to $t = T$ defines an isomorphism a. The claim is that $(\Pi_a, \Pi_a^\infty) = (V, V^\infty)$. I leave the details to you, there is some help in Problem 10.10. Conversely, this construction recovers a from Π_a so shows that (10.46) is injective and surjective. □

10.5. Isotropic index map

Now, (10.46) is part of Bott periodicity. The remaining part is that, for any manifold X there is a natural isomorphism

$$\mathrm{K}^1_\mathrm{c}(\mathbb{R} \times X) \longrightarrow \mathrm{K}_\mathrm{c}(X). \tag{10.47}$$

If we regard this as an identification (and one has to be careful about orientations here) then it means that we have identified

$$\mathrm{K}^0_\mathrm{c}(X) = \mathrm{K}^1_\mathrm{c}(\mathbb{R} \times X) = \mathrm{K}_\mathrm{c}(X) = \mathrm{K}_\mathrm{c}(\mathbb{R}^2 \times X) \tag{10.48}$$

as is discussed more below. For the moment what we will work on is the definition of the map in (10.47). This is the 'isotropic' (or 'Toeplitz'[5]) index map.

So, we get to the start of the connection of this stuff with index theory. An element of $\mathrm{K}^1_\mathrm{c}(\mathbb{R} \times X)$ is represented by a map from $\mathbb{R} \times X$ to $\mathrm{GL}(N, \mathbb{C})$, for some N, and with triviality outside a compact set. In particular this map reduces to the identity near $\pm\infty$ in $\mathbb{R}$ so we can join the ends using the radial compactification of $\mathbb{R} \longmapsto \mathbb{S}$ and get a map

$$\tilde{a} \in \mathcal{C}^\infty(\mathbb{S} \times X; \mathrm{GL}(N, \mathbb{C})),\ \tilde{a} = \mathrm{Id} \text{ near } \{1\} \times X \text{ and outside a compact set.}$$

This indeed is essentially the implied definition of $\mathrm{K}^0_\mathrm{c}(X)$ before (10.25). Now, we can interpret $\tilde{a}$ as the principal symbol of an elliptic family in $\Psi^0_{\mathrm{iso}}(\mathbb{R}; \mathbb{C}^N)$

[5]See Problem **??** for this alternative approach.

depending smoothly on $x \in X$ (and reducing to the identity outside a compact set). Let's start with the case $X = \{\text{pt}\}$ so there are no parameters.

PROPOSITION 10.7. *If $A \in \Psi^0_{\text{iso}}(\mathbb{R}; \mathbb{C}^N)$ is elliptic with principal symbol $a = \sigma_0(A) \in \mathcal{C}^\infty(\mathbb{S}; \text{GL}(N, \mathbb{C}))$ then the index of A is given by the winding number of the determinant of the symbol*

$$\text{Ind}_{\text{iso}}(A) = -\operatorname{wn}(\det(a)) = -\frac{1}{2\pi i}\int_{\mathbb{S}} \operatorname{tr}(a^{-1}\frac{da}{d\theta})d\theta \tag{10.49}$$

and if $a = \text{Id}$ near $\{1\} \in \mathbb{S}$ then $\text{Ind}_{\text{iso}}(A) = 0$ if and only if $[a] = 0 \in \text{K}^1_c(\mathbb{S})$.

PROOF. ****Expand This follows from Proposition 10.4. First, recall what the winding number is. Then check that it defines the identification (10.30). Observe that the index is stable under homotopy and stabilization and that the index of a product is the sum of the indices. Then check one example with index 1, namely for the annihilation operator will suffices. For general A with winding number m, compose with m factors of the creation operator – the adjoint of the annihilation operator. This gives an operator with symbol for which the winding number is trivial. By Proposition 10.4 it can be deformed to the identity after stabilization, so its index vanishes and (10.49) follows. □

Now for the analytic step that allows us to define the full (isotropic) index map.

PROPOSITION 10.8. *If $a \in \mathcal{C}^\infty_c(\mathbb{R} \times X; \text{GL}(N, \mathbb{C}))$ (so it reduces to the identity outside a compact set) then there exists $A \in \mathcal{C}^\infty(X; \Psi^0_{\text{iso}}(\mathbb{R}; \mathbb{C}^N))$ with $\sigma_0(A) = a$, A constant in $X \setminus K$ for some compact K and such that $\text{null}(A)$ is a (constant) vector bundle over X.*

PROOF. We can choose a $B \in \mathcal{C}^\infty(X; \Psi^0_{\text{iso}}(\mathbb{R}; \mathbb{C}^N))$ with $\sigma(B) = a$ by the surjectivity of the symbol map. Moreover, taking a function $\psi \in \mathcal{C}^\infty(X)$ which is equal to 1 outside a compact set in X but which vanishes where $a \neq \text{Id}$, $(1-\psi)B + \psi\,\text{Id}$ has the same principal symbol and reduces to Id outside a compact set.

The problem with this initial choice is that the dimension of the null space may change from point to point. However, we certainly have a parametrix $G_B \in \mathcal{C}^\infty(X; \Psi^0_{\text{iso}}(\mathbb{R}; \mathbb{C}^N))$ which we can take to be equal to the identity outside a compact set, by the same method, and which then satisfies

$$G_B B = \text{Id} + R_1,\ BG_B = \text{Id} + R_2,\ R_i \in \mathcal{C}^\infty_c(X; \Psi^{-\infty}_{\text{iso}}(\mathbb{R}; \mathbb{C}^N)). \tag{10.50}$$

So, recall the finite rank projection $\pi_{(N)}$ onto the span of the first N eigenspaces. We know that $R_1\pi_{(N)} \to R_1$ in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}; \mathbb{C}^N)$ and this is true uniformly on X since the support in X is compact. So, if N is large enough $\sup_{x\in X} \|R_1(x)(\text{Id} - \pi_{(N)})\| < \frac{1}{2}$. Composing the first equation in (10.50) on the right with $\text{Id} - \pi_{(N)}$ we find that

$$G_B B(\text{Id} - \pi_{(N)}) = (\text{Id} + R_1(\text{Id} - \pi_{(N)}))(\text{Id} - \pi_{(N)}) \tag{10.51}$$

where the fact that $\text{Id} - \pi_{(N)}$ is a projection is also used. Now

$$(\text{Id} + R_1(\text{Id} - \pi_{(N)}))^{-1} = \text{Id} + S_1$$

where $S_1 \in \mathcal{C}^\infty_c(X; \Psi^{-\infty}_{\text{iso}}(\mathbb{R}))$ by the openness of $G^{-\infty}_{\text{iso}}(\mathbb{R})$. So if we set $A = B(\text{Id} - \pi_{(N)})$ and $G = (\text{Id} + S_1)G_B$ we see that

$$GA = \text{Id} - \pi_{(N)}. \tag{10.52}$$

In particular the null space of $A(x)$ for each x is exactly the span of $\pi_{(N)}$ – it certainly annihilates this set but can annihilate no more in view of (10.52). Moreover A has the same principal symbol as B and is constant outside a compact set in X. □

Now, once we have chosen A as in Proposition 10.8 it follows from the constancy of the index that family $A(x)^*$ also has null spaces of constant finite dimension, and indeed these define a smooth bundle over X which, if X is not compact, reduces to $\pi_{(N)}$ near infinity – since $A = \mathrm{Id} - \pi_{(N)}$ there. Thus we arrive at the isotropic index map.

Proposition 10.9. *If A is as in Proposition 10.8 the the null spaces of $A^*(x)$ form a smooth vector bundle $\mathrm{Nul}(A^*)$ over X defining a class $[(\pi_{(N)}, \mathrm{Nul}(A^*))] \in \mathrm{K}_c(X)$ which depends only on $[a] \in \mathrm{K}^1_c(\mathbb{R}\times X)$ and so defines an additive map*

$$\mathrm{Ind}_{\mathrm{iso}} : \mathrm{K}^1_c(\mathbb{R}\times X) \longrightarrow \mathrm{K}_c(X). \tag{10.53}$$

Proof. In the earlier discussion of isotropic operators it was shown that an elliptic operator has a generalized inverse. So near any particular point $\bar{x} \in X$ we can add an element $E(\bar{x}) \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N)$ to $G(\bar{x})$ so that $H(\bar{x}) = G(\bar{x}) + E(\bar{x})$ is a generalized inverse, $H(\bar{x})A(\bar{x}) = \mathrm{Id} - \pi_{(N)}$, $A(\bar{x}_H(\bar{x}) = \mathrm{Id} - \pi'(\bar{x})$ where $\pi'(\bar{x})$ is a finite rank projection onto a subspace of $\mathcal{S}(\mathbb{R})$. Then $H(x) = G(x) + E(\bar{x})$ is still a parametrix nearby and

$$H(x)A(x) = \mathrm{Id} - \pi_{(N)},\ A(x)H(x) = \mathrm{Id} - p(x) \text{ near } \bar{x} \tag{10.54}$$

where $p(x)$ must have constant rank. Indeed, it follows that $p(x)\pi'(\bar{x})$ is a smooth bundle isomorphism, near $\bar{x}$, from the range of $\pi'(\bar{x})$ to the null space of A^*. This shows that the null spaces of the $A^*(x)$ form a bundle, which certainly reduces to $\pi_{(N)}$ outside a compact set. Thus

$$[(\pi_{(N)}, \mathrm{null}(A^*))] \in \mathrm{K_c}(X). \tag{10.55}$$

Next note the independence of this element of the choice of N. It suffices to show that increasing N does not change the class. In fact increasing N to $N+1$ replaces A by $A(\mathrm{Id} - \pi_{(N+1)})$ which has null bundle increased by the trivial line bundle $(\mathrm{Id}_{(N+1)} - \pi_{(N)})$. The range of A then decreases by the removal of the trivial bundle $A(x)(\mathrm{Id}_{(N+1)} - \pi_{(N)})$ and $\mathrm{null}(A^*)$ increases correspondingly. So the class in (10.55) does not change.

To see that the class is independent of the choice of A, for fixed a, consider two such choices. Initially the choice was of an operator with a as principal symbol, two choices are smoothly homotopic, since $tA + (1-t)A'$ is a smooth family with constant symbol. The same construction as above now gives a pair of bundles over $[0,1]\times X$, trivialized outside a compact set, and it follows from Lemma 10.6 that the class is constant. A similar discussion shows that homotopy of a is just a family over $[0,1]\times X$ so the discussion above applies to it and shows that the bundles can be chosen smoothly, again from Lemma 10.6 the class is constant. □

It is important to understand what the index tell us.

Proposition 10.10. *If $a \in \mathcal{C}^\infty_c(\mathbb{R}\times X; \mathrm{GL}(N,\mathbb{C}))$ then $\mathrm{Ind}_{\mathrm{iso}}(a) = 0$ if and only if there is a family $A \in \mathcal{C}^\infty(X; \Psi^0_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N))$ with $\sigma_0(A) = a$ which is constant outside a compact set in X and everywhere invertible.*

PROOF. The definition of the index class above shows that a may be quantized to an operator with smooth null bundle and range bundle such with $\operatorname{Ind}_{\mathrm{iso}}(a)$ represented by $(\pi_{(N)}, p')$ where p' is the null bundle of the adjoint. If A can be chosen invertible this class is certainly zero. Conversely, if the class vanishes then after stabilizing with a trivial bundle $\pi_{(N)}$ and p' become bundle isomorphic. This just means that they are isomorphic for sufficiently large N with the isomorphism being the trivial one near infinity. However this isomorphism is itself an element of $\mathcal{C}^\infty(X; \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}; \mathbb{C}^N))$ which is trivial near infinity. Adding it to A gives an invertible realization of the symbol, proving the Proposition. $\square$

10.6. Bott periodicity

Now to the proof of Bott periodicity. Choose a 'Bott' element, which in this case is a smooth function

$$\beta(t) = e^{i\Theta(t)} \Longrightarrow \begin{cases} \beta : \mathbb{R} \longrightarrow \mathbb{C}^*,\ \beta(t) = 1 \text{ for } |t| > T, \\ \arg \beta(t) \text{ increasing over } (0, 2\pi) \text{ for } t \in (-T, T) \end{cases} \tag{10.56}$$

where Θ satisfies (10.35) and the preceeding conditions. Thus β has winding number one but is constant near infinity.

We first show

PROPOSITION 10.11. *The map* (10.53) *is surjective with explicit left inverse generated by mapping a smooth projection (constant near infinity) to*

$$(\pi_V, \pi_V^\infty) \longmapsto \beta(t)^{-1}\pi_V + (\mathrm{Id} - \pi_V) \in \mathcal{C}^\infty_c(\mathbb{R} \times X; \mathrm{GL}(N, \mathbb{C})). \tag{10.57}$$

PROOF. The surjectivity follows from the existence of a left inverse, so we need to investigate (10.57). Observe that $\beta(t)^{-1}$, when moved to the circle, is a symbol with winding number 1. By Proposition 10.7 we may choose an elliptic operator $b \in \Psi^0_{\mathrm{iso}}(\mathbb{R})$ which has a one-dimensional null space and has symbol in the same class in $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R})$ as β^{-1}. In fact we could take the annihilation operator, normalized to have order 0. Then we construct an elliptic family $B_V \in \Psi^0_{\mathrm{iso}}(\mathbb{R}; \mathbb{C}^N)$ by setting

$$B_V = \pi_V(x) b + (\mathrm{Id} - \pi_V(x)),\ x \in X. \tag{10.58}$$

The null space of this family is clearly $\pi_V \otimes N$, where N is the fixed one-dimensional vector space $\mathrm{null}(b)$. Thus indeed

$$\operatorname{Ind}_{\mathrm{iso}}(B_V) = [(\pi_V, \pi_V^\infty)] \in \mathrm{K}_{\mathrm{c}}(X). \tag{10.59}$$

This proves the surjectivity of $\operatorname{Ind}_{\mathrm{iso}}$, the index map in this isotropic setting. $\square$

With some danger of repeating myself, if X is compact the 'normalizing term' at infinity π_V^∞ is dropped. You will now see why we have been dragging this non-compact case along, it is rather handy even if interest is in the compact case.

This following proof that $\operatorname{Ind}_{\mathrm{iso}}$ is injective is a variant of the 'clever' argument of Atiyah (maybe it is *very* clever – look at the original proof by Bott or the much more computational, but actually rather enlightening, argument in [**1**]).

PROPOSITION 10.12. *For any manifold* X, *the isotropic index map in* (10.47), (10.53) *is an isomorphism*

$$\operatorname{Ind}_{\mathrm{iso}} : \mathrm{K}^1_c(\mathbb{R} \times X) \xrightarrow{\simeq} \mathrm{K}_c(X). \tag{10.60}$$

Proof. Following Proposition 10.11 only the injectivity of the map remains to be shown. Rather than try to do this directly we use another carefully chosen homotopy.

So, we need to show that if $a \in \mathcal{C}^\infty_c(\mathbb{R} \times X; \mathrm{GL}(N, \mathbb{C}))$ has $\mathrm{Ind}_{\mathrm{iso}}(a) = 0$ then $0 = [a] \in \mathrm{K}^1_{\mathrm{c}}(\mathbb{R}_s \times X)$. As a first step we use the construction of Proposition 10.6 and Lemma 10.5 to construct the image of $[a]$ in $\mathrm{K}_{\mathrm{c}}(\mathbb{R}^2 \times X)$. It is represented by the projection-valued matrix

$$\Pi_a(t, s, x) \in \mathcal{C}^\infty_c(\mathbb{R}^2 \times X; M(2N, \mathbb{C})) \tag{10.61}$$

which is constant near infinity. Then we use the surjectivity of the index map in the case

$$\mathrm{Ind}_{\mathrm{iso}} : \mathrm{K}_{\mathrm{c}}(\mathbb{R} \times (\mathbb{R}^2 \times X)) \longrightarrow \mathrm{K}_{\mathrm{c}}(\mathbb{R}^2 \times X) \tag{10.62}$$

and the explicit lift (10.58) to construct

(10.63)
$$e \in \mathcal{C}^\infty_c(\mathbb{R}^2 \times X; \mathrm{GL}(2N, \mathbb{C})),\ e(r, t, s, x) = \beta(r)\Pi_a(t, s, x) + (\mathrm{Id} - \Pi_a(t, s, x)),$$
$$\mathrm{Ind}_{\mathrm{iso}}(e) = [\Pi_a, \Pi^\infty_a] \in \mathrm{K}_{\mathrm{c}}(\mathbb{R}^2 \times X).$$

Here the 'r' variable is the one which is interpreted as the variable in the circle at infinity on $\mathbb{R}^2$ to turn e into a symbol and hence a family of elliptic operators with the given index. However we can rotate between the variables r and s, which is an homotopy replacing $e(r, t, s, x)$ by $e(-s, t, r, x)$. Since the index map is homotopy invariant, this symbol must give the same index class. Now, the third variable here is the argument of a, the original symbol. So the quantization map just turns a and a^{-1} which appears in the formula for Π_a – see (10.41) – into any operator with these symbols. By Proposition 10.10 a (maybe after a little homotopy) is the symbol of an invertible family. Inserting this in place of a and its inverse for a^{-1} gives an invertible family of operators with symbol $e(-s, t, r, x)$[6]. Thus $\mathrm{Ind}_{\mathrm{iso}}(e) = 0$, but this means that

$$0 = [(\Pi_a, \Pi^\infty_a)] \in \mathrm{K}_{\mathrm{c}}(\mathbb{R}^2 \times X) \Longrightarrow 0 = [a] \in \mathrm{K}^1_{\mathrm{c}}(\mathbb{R} \times X). \tag{10.64}$$

This shows the injectivity of the isotropic index map and so that (10.60) is an isomorphism. □

What does this tell us? Well, as it turns out, lots of things! For one thing the normalization conditions extend to all Euclidean space:-

$$\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}^k) = \begin{cases} \{0\} & k \text{ even} \\ \mathbb{Z} & k \text{ odd}, \end{cases} \quad \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^k) = \begin{cases} \mathbb{Z} & k \text{ even} \\ \{0\} & k \text{ odd}. \end{cases} \tag{10.65}$$

This in turn means that we understand a good deal more about $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

Theorem 10.1 (Bott periodicity). *The homotopy groups $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ are*

$$\pi_j(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)) = \begin{cases} \{0\} & k \textit{ even} \\ \mathbb{Z} & k \textit{ odd}. \end{cases} \tag{10.66}$$

Indeed Bott proved this rather directly using Morse theory.

[6] See Problem 10.1 for some more details

10.7. Toeplitz index map

Although the map from $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X)$ to $\mathrm{K}_{\mathrm{c}}(X)$ has been discussed here in terms of the quantization of symbols to isotropic pseudodifferential operators it could equally, and more conventionally, be done by quantization to 'Toeplitz operators'. The advantage of the isotropic quantization is that it extends directly to higher dimensions. The Toeplitz algebra is the 'compression' of the pseudodifferential algebra on the circle to the positive Fourier components, some form of the Hardy space. This is discussed in § 6.9. In the Toeplitz context, $\pi_{(N)}$ is projection onto the span of the first N exponentials $\exp(il\theta)$, $1\le l\le N$.

Proposition 10.13. *If $A\in\mathcal{C}^\infty(X;\Psi^0_{\mathrm{To}}(\mathbb{S};\mathbb{C}^k))$ is an elliptic family of Toeplitz operators, which is constant outside a compact subset of X and has $\sigma(A)(1,x)\equiv\mathrm{Id}_{k\times k}$ then for N sufficiently large $A(\mathrm{Id}-\pi_{(N)})$ and its adjoint have null spaces forming a smooth vector bundle over X, the class $[(\mathrm{Nul}(A),\mathrm{Nul}(A^*))]\in\mathrm{K}^0_c(X)$ depends only on the class in of the symbol in $\mathrm{K}^1_c(\mathbb{R}\times\mathbb{S})$ and the map so defined*

$$\mathrm{Ind}_{\mathrm{To}}:\mathrm{K}^1_c(\mathbb{R}\times X)\longrightarrow\mathrm{K}^0_c(X)\tag{10.67}$$

is equal to the isotropic index map discussed above.

Notice that the assumption that the symbol of A is equal to the identity at $\theta=1$ on the circe, for all $x\in X$, means that it can be interpreted (after a little deformation) as defining an element in the compactly supported K-group on the left in (10.67), where $\mathbb{R}$ is identified with $\mathbb{S}\setminus\{1\}$ by the map

$$\mathbb{R}\ni t\longmapsto\exp(i\Theta(t)),\ \Theta\in\mathcal{C}^\infty(\mathbb{R}),\ \Theta'(t)\ge0,\ \Theta(t)=0,\ t<<0,\ \Theta(t)=2\pi,\ t>>0.\tag{10.68}$$

where the orientation is important.

Problem 10.1. Go through the argument for the stability of the null bundle and the independence of choices, it is essentially the same as for the isotropic case but using the properties of the Toeplitz algebra, and smoothing operators on the circle instead.

Proof. The proof of the stability of the index etc, leading to the map (10.67) is essentially the same as in the isotropic case so is omitted. It remains to show that quantization by Toeplitz operators gives the same index map as quantization by istropic operators.

The shift operator, which is multiplication by $e^{-i\theta}$ followed by projection back onto the Hardy projection, is elliptic and has index 1 as a Toeplitz operator. Its symbol is homotopic, after the identification (10.68), with the symbol of the annihilation operator in the isotropic algebra (after change of order using the square root of the harmonic oscillator), which also has index 1 and is the Bott element. Thus the two indexes agree on this element, with X a point. The argument of sujectivity for the isotropic index above, which involves twisting the annihilation operator with a bundle on X applies equally well in the Toeplitz setting. Thus both maps are surjective and the injectivity of the isotropic index shows that these element span $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X)$, so the two maps give the same isomorphism. □

10.8. Odd isotropic index

Especially since the geometric version of the odd index plays a considerable role in the proof of the index theorem of Atiyah and Singer below, we next discuss the ‘odd’ version of the isotropic index theorem which arises from the semiclassical limit for isotropic operators. This is used in the next section to obtain the Thom isomorphism in K-theory.

We shall show that for any even dimensional Euclidean space the symbol map for isotropic smoothing operators leads to an isomorphism

$$\mathrm{Ind}^{\mathrm{odd}}_{\mathrm{iso}} : \mathrm{K}^1_{\mathrm{c}}(\mathbb{R}^{2N} \times X) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(X) \tag{10.69}$$

for any manifold X. This is consistent with the other Bott periodicity constructions, as is shown below.

Proposition 10.14. *If $a \in \mathcal{C}^\infty(\mathbb{R}^{2n} \times X; \mathrm{GL}(N, \mathbb{C}))$ has compact support, in the sense that $a = \mathrm{Id}$ outside a compact set, the there exists $A' \in \mathcal{C}^\infty_c(X; \Psi^{-\infty}_{\mathrm{sl\,iso}}(\mathbb{R}^n); \mathbb{C}^N)$ such that $\sigma_{\mathrm{sl}(A')} = a - \mathrm{Id}_N$ and then for small $\epsilon > 0$ $[\mathrm{Id}_N + A_\epsilon] \in \mathrm{K}^1_c(X)$ depends only on $[a] \in \mathrm{K}^1_c(\mathbb{R}^{2n} \times X)$ and gives the isomorphism* (10.69).

Proof. The main step is the existence of the semiclassical family, reducing to the identity outside a compact set, but this is shown in Chapter 4.

The fact that $[A_\epsilon]$, for $\epsilon > 0$ so small that A_δ is invertible for all $0 < \delta < \epsilon$, only depends on a follows from the homotopy equivalence of all possible semiclassical quantizations. The independence of choices follows from similar arguments to those above (and maybe should be moved back from the index chapter ****) □

It is important that this odd index map is consistent under iteration and with Bott periodicity, as discussed in the preceeding section.

Lemma 10.7. *For any manifold X and any N, $M \in \mathbb{N}$, the diagrammes*

$$\begin{array}{ccccc} & & \mathrm{K}^1_c(\mathbb{R}^{2M} \times X) & & \\ & \overset{\mathrm{Ind}_{\mathrm{iso}(N)}}{\nearrow} & & \overset{\mathrm{Ind}_{\mathrm{iso}(M)}}{\searrow} & \\ \mathrm{K}^1_c(\mathbb{R}^{2N+2M} \times X) & & & & \mathrm{K}^1_c(X) \\ & \underset{\mathrm{Ind}_{\mathrm{iso}(M)}}{\searrow} & & \underset{\mathrm{Ind}_{\mathrm{iso}(N)}}{\nearrow} & \\ & & \mathrm{K}^1_c(\mathbb{R}^{2N} \times X) & & \end{array} \tag{10.70}$$

and

$$\begin{array}{ccc} \mathrm{K}^1_c(\mathbb{R}^{2N+1} \times X) & \xrightarrow{\mathrm{Ind}^{\mathrm{odd}}_{\mathrm{iso}(N),\mathrm{sl}}} & \mathrm{K}^1_c(\mathbb{R} \times X) \\ \Big\downarrow \mathrm{Ind}_{\mathrm{iso}} & & \Big\downarrow \mathrm{Ind}_{\mathrm{iso}} \\ \mathrm{K}^0_c(\mathbb{R}^{2N} \times X) & \xrightarrow[(\mathrm{clu} \circ \mathrm{Ind}_{\mathrm{iso}})^N]{} & \mathrm{K}^0_c(X) \end{array} \tag{10.71}$$

commute.

10.9. Complex and symplectic bundles

In Chapter 4 the algebra of istropic pseudodifferential on a symplectic vector space F is discussed. For example the algebra of operators of order 0 is just a non-commutative product on the space of smooth functions on the radial compactification of F, $\mathcal{C}^\infty(\overline{F})$. This product varies smoothly with the symplectic form used to define it. Now suppose that $E \longrightarrow X$ is a real vector bundle over a manifold X which has a symplectic structure, that is a section

$$\begin{gathered}\omega \in \mathcal{C}^\infty(X; \Lambda^2 F'),\\ v \in F_x,\ \omega_x(v,w) = 0\ \forall\ w \in F_x \Longrightarrow v = 0\end{gathered} \tag{10.72}$$

is given. Then the isotropic algebras on the fibre combine to a smooth bundle of algebras. It is this bundle of algebras which we will use to discuss the Thom isomorphism. Since the Thom isomorphism in K-theory is usually thought of in terms of complex bundles, not realy symplectic bundles, we recall the relationship between them here.

Recall that any complex vector space F has an underlying real vector space, $F_{\mathbb{R}}$, which is the same set with only real multiplication allowed. Then multiplication by i on F becomes a real isomorphism $J : F_{\mathbb{R}} \longrightarrow \mathbb{R}_{\mathbb{R}}$ with the property that $J^2 = -\operatorname{Id}$. Conversely, on a real vector space with such an isomorphism, complex multiplication, defined with multiplication by $z = \alpha + i\beta$ being $\alpha + \beta J$, is a complex vector space with the original real vector space underlying it.

Lemma 10.8. *A real vector bundle of even rank admits a complex structure if and only if it admits a symplectic structure and the homotopy classes of these structures are in 1-1 correspondence.*

If X is not compact, this correspondence of complex or symplectic structures extends to those which are trivialized outside a compact set.

Proof. This is really just the corresponding construction in linear algebra. Any complex vector space F admits an Hermitian structure, a sequilinear positive definite form:

$$\begin{gathered}h : F \times F \longrightarrow \mathbb{C},\\ h(z_1v_1 + z_2v_2, w) = z_1h(v_1,w) + z_2h(v_2,w),\ h(v,w) = \overline{h(w,v)},\ h(v,v) \geq 0,\ h(v,v) = 0 \Longrightarrow v = 0.\end{gathered} \tag{10.73}$$

To see this, just take the Euclidean inner product with respect to a basis. The imaginary part of h,

$$\omega_h(v,w) = \Im h(v,w) \tag{10.74}$$

is a symplectic form on $F_{\mathbb{R}}$. Moreover $h(v,w) = \omega_h(v,Jw) + i\omega(v,w)$ so the Hermitian structure can be recovered from the symplectic strucure and J. Conversely, if V is a real vector space with symplectic form ω_V then choosing a real Euclidean structure g on V defines a linear map $J' : V \longrightarrow V$ by

$$\begin{gathered}\omega_V(v, J'w) = g(v,w) \Longrightarrow\\ \omega_V(v,J'w) = g(v,w) = g(w,v) = \omega_V(w,J'v) = -\omega_V(Jv,w).\end{gathered} \tag{10.75}$$

Thus $g(J'v,w) = \omega_V(J'v,J'w) = -\omega_V(J'w,J'v) = -g(J'w,v) = -g(v,J'w)$ shows that J' is skew-adjoint with respect to g and $g((J')^2v,w) = -g(J'v,J'w)$ shows that its square is negative definite and self-adoint. Thus $-(J')^2 = A^2$ where A is a positive definite real self-adjoint matrix, with respect to g, which commutes with J' (since iJ' is self-adjoint and its eigenvectors are eigenvectors of A^2 and hence A. Thus $J = A^{-1}J'$ is a complex structure, $J^2 = -\operatorname{Id}$.

For a symplectic vector bundle, this construction can be carried out smoothly, simply by choosing a smooth family of real metrics on the fibres. The construction of J from J' is determined and hence is easily seen to yield a smooth homorphism

J of the real bundle, and hence a smooth complex structure. Moreover both the construction of a complex structure from the symplectic and of the symplectic structure from the complex can lift to homotopies, since they can be carried out smoothly in parameters. □

10.10. Thom isomorphism

The even semiclassical isotropic index map is shown above to generate an isomorphism

$$\mathrm{Ind}_{\mathrm{iso,sl}} : \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X) \tag{10.76}$$

for any manifold X. Here the product can be interpreted as a trivial even-rank bundle over X. The Thom isomorphism extends this to the bundles discussed in the previous section.

Proposition 10.15. *If $F \longrightarrow X$ is an even-rank real vector bundle over X, trivial outside a compact set and with a symplectic structure constant outside a compact set then semiclassical isotropic quantization gives an isomorphism*

$$\begin{gathered}\mathrm{Thom} : \mathrm{K}^0_c(F) \longrightarrow \mathrm{K}^0_c(X) \ \textit{with inverse} \\ \mathrm{K}^0_c(X) \ni [\mathbb{V}] \longmapsto [\mathbb{V} \otimes b_E] \in \mathrm{K}^0_c(E)\end{gathered} \tag{10.77}$$

where $b_E \in \mathrm{K}^0_c(E)$ is the Bott element corresponding to the harmonic oscillator.

Proof. We first show that isotropic quantization of projections on the fibres descends to an index map in the bundle case (10.77) in the bundle case. Certainly an element of $\mathrm{K}^0_{\mathrm{c}}(F)$ is represented by a smooth map $F \longrightarrow M(N, \mathbb{C})$ for some N with values in the projections and constant outside a compact subset of F (which of course projects to a compact subset of X). Mainly we just need to show that the previous discussion extends smoothly to this case and also that there is a smooth 'Bott element' $\beta_F \in \mathrm{K}^0_{\mathrm{c}}(F)$, so represented by a family of projections, such that $\mathrm{Ind}(F) = [\mathbb{C}]$ is a trivial one-dimensional bundle. Then the second line in (10.77) gives a left inverse of the index,

$$\mathrm{Ind}(\pi^*[\mathbb{V}] \otimes \beta_E) = [\mathbb{V}] \in \mathrm{K}^0_{\mathrm{c}}(X). \tag{10.78}$$

As for the original isotropic index, this proves that the index map is surjective for any symplectic bundle F as in the statement above. So only the injectivity remains to be shown.

If F is a real vector bundle with symplectic structure then it is shown above that it can be realized as the underlying real vector bundle for a complex vector bundle with the symplectic structure being the imaginary part of an Hermitian structure on E. If F is trivial, with constant symplectic structure outside a compact set, then E can be taken to be trivial with complex Hermitian structure outside a compact set. Then E can be embedded as a subbundle of a trivial complex bundle $\mathbb{C}^N$ with constant inclusion outside a compact set. Extending the Hermitian structure to the whole bundle, as a direct sum, and constant outside a compact set, shows that F can be complemented to a trivial bundle with direct symplectic forms constant outside a compactum. Let the complementary bundle be G so $F \oplus G = \mathbb{R}^{2k}$ for some k. Now we have maps

$$\begin{array}{ccc} \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2k}\times X) & \xrightarrow{\ \mathrm{Ind}\ } & \mathrm{K}^0_{\mathrm{c}}(X). \\ & \searrow^{\mathrm{Ind}_G} \quad \nearrow_{\mathrm{Ind}_F} & \\ & \mathrm{K}^0_{\mathrm{c}}(F) & \end{array} \tag{10.79}$$

We claim that this diagramme commutes. *** This is supposed to be done back in the isotropic chapter, namely that when quantizing a projection on the product of two vector spaces one can first quantize in one subspace and then the other. For the moment the more complicated case of the adiabatic limit has already been done so this should be clear enough.

From the commutativity of (10.79) it follows that Ind_F is an isomorphism. Indeed, the bottom two are injective and top is known to be an isomorphism from the preceeding discussion. Thus Ind_F must also be surjective and hence is an isomorphism and Ind_G is Ind_F for a different bundle and base. □

10.11. Chern forms

I would not take this section seriously yet, I am going to change it.

Let's just think about the finite-dimensional groups $\mathrm{GL}(N,\mathbb{C})$ for a little while. Really these can be replaced by $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, as I will do below, but it may be a strain to do differential analysis and differential topology on such an infinite dimensional manifold, so I will hold off for a while.

Recall that for a Lie group G the tangent space at the identity (thought of as given by an equivalence to second order on curves through Id), $\mathfrak{g}$, has the structure of a Lie algebra. In the case of most interest here, $\mathrm{GL}(n,\mathbb{C})\subset M(N,\mathbb{C})$ is an open subset of the algebra of $N\times N$ matrices, namely the complement of the hypersurface where $\det=0$. Thus the tangent space at Id is just $M(N,\mathbb{C})$ and the Lie algebra structure is given by the commutator

$$[a,b]=ab-ba,\ \ a,b\in\mathfrak{gl}(N,\mathbb{C})=M(N,\mathbb{C}). \tag{10.80}$$

At any other point, g, of the group the tangent space may be naturally identified with $\mathfrak{g}$ by observing that if $c(t)$ is a curve through g then $g^{-1}c(t)$ is a curve through Id with the equivalence relation carrying over. This linear map from T_gG to $\mathfrak{g}$ is herlpfully denoted

$$g^{-1}dg : T_gG \longrightarrow \mathfrak{g}. \tag{10.81}$$

In this notation 'dg' is the differential of the identity map of G at g. This 'Maurier-Cartan' form as a well-defined 1-form on G with values in $\}$ – which is a fixed vector space.

The fundamental property of this form is that

$$d(g^{-1}dg)=-\frac{1}{2}[g^{-1}dg,g^{-1}dg]. \tag{10.82}$$

In the case of $\mathrm{GL}(N,\mathbb{C})$ this can be checked directly, and written slightly differently. Namely in this case as a 'function' 'g' is the identity on G but thought of as the canonical embedding $\mathrm{GL}(N,\mathbb{C})\subset M(N,\mathbb{C})$. Thus it takes values in $M(N,\mathbb{C})$, a vector space, and we may differentiate directly to find that

$$d(g^{-1}dg)=-dgg^{-1}dg\wedge dg \tag{10.83}$$

where the product is that in the matrix algebra. Here we are just using the fact that $dg^{-1} = -g^{-1}dgg^{-1}$ which comes from differentiating the defining identity $g^{-1}g = \mathrm{Id}$. Of course the right side of (10.83) is antisymmetric as a function on the tangent space $T_gG \times T_gG$ and so does reduce to (10.82) when the product is repalced by the Lie product, i.e. the commutator.

Since we are dealing with matrix, or infinite matrix, groups throughout, I will use the 'non-intrinsic' form (10.83) in which the product is the matrix product, rather than the truly intrinsic (and general) form (10.82).

Proposition 10.16 (Chern forms). *If* tr *is the trace functional on* $N \times N$ *matrices then on* $\mathrm{GL}(N, \mathbb{C})$,

$$\begin{gathered}\operatorname{tr}((g^{-1}dg)^{2k}) = 0 \ \forall\ k \in \mathbb{N},\\ \beta_{2k-1} = \operatorname{tr}((g^{-1}dg)^{2k-1}) \textit{ is closed} \ \forall\ k \in \mathbb{N}.\end{gathered} \tag{10.84}$$

Proof. This is the effect of the antisymmetry. The trace idenitity, $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ means precisely that tr vanishes on commutators. In the case of an even number of factors, for clarity evaluation on $2k$ copies of $T_g \mathrm{GL}(N, \mathbb{C})$, given for $a_i \in M(N, \mathbb{C})$, $i = 1, \dots, 2k$, by the sum over

$$\begin{gathered}\operatorname{tr}((g^{-1}dg)^{2k})(a_1, a_2, \dots, a_{2k}) = \sum_e \operatorname{sgn}(e) \operatorname{tr}(g^{-1}a_{e(1)}g^{-1}a_{e(2)} \dots g^{-1}a_{e(2k)}) = \\ -\sum_e \operatorname{sgn}(e) \operatorname{tr}(g^{-1}a_{e(2k)}g^{-1}a_{e(1)} \dots g^{-1}a_{e(2k-1)}) = -\operatorname{tr}((g^{-1}dg)^{2k})(a_1, a_2, \dots, a_{2k}).\end{gathered} \tag{10.85}$$

In the case of an odd number of factors the same manipulation products a trivial identity. However, notice that

$$g^{-1}dgg^{-1} = -d(g^{-1}) \tag{10.86}$$

is closed, as is dg. So in differentiating the odd number of wedge products each pair $g^{-1}dgg^{-1}dg$ is closed, so (tr being a fixed functional)

$$d\beta_{2k-1} = \operatorname{tr}(dg^{-1})(g^{-1}dgg^{-1}dg)^{2k-2}) = -\operatorname{tr}((g^{-1}dg)^{2k}) = 0 \tag{10.87}$$

by the previous discussion. □

Now, time to do this in the infinite dimensional case. First we have to make sure we know that we are talking about.

Definition 10.3 (Fréchet differentiability). *A function on an open set of a Fréchet space,* $O \subset F$, $f : O \longrightarrow V$, *where* V *is a locally convex topological space (here it will also be Fréchet, and might be Banach) differentiable at a point* $u \in O$ *if there exists a continuous linear map* $D : F \longrightarrow V$ *such that for each continuous seminorm* $\|\cdot\|_\alpha$ *on* V *there is a continuous norm* $\|\cdot\|_i$ *on* F *such that for each* $\epsilon > 0$ *there exists* $\delta > 0$ *for which*

$$\|v\|_i < \delta \Longrightarrow \|f(u + v) - f(u) - Tv\|_\alpha \le \epsilon \|v\|_i. \tag{10.88}$$

This is a rather strong definition of differentiability, stronger than the Gâteaux definition – which would actually be enough for most of what we want, but why not use the stronger condition when it holds?

PROPOSITION 10.17. *The composition of smoothing operators defines a bilinear smooth map*

$$\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \times \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n), \|ab\|_k \le C_k \|a\|_{k+N} \|b\|_{k+N} \tag{10.89}$$

(where the kth norm on u is for instance the $\mathcal{C}^k$ norm on $\langle z\rangle^k u$ and inversion is a smooth map

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n). \tag{10.90}$$

PROOF. I did not define smoothness above, but it is iterated differentiability, as usual. In fact linear maps are always differentiable, as follows immediately from the definition. The same is true of jointly continuous bilinear maps, so the norm estimates in (10.89) actually prove the regularity statement. The point is that the derivative of a bilinear map P at $(\bar{a}, \bar{b})$ is the linear map

$$Q_{\bar{a},\bar{b}}(a, b) = P(a, \bar{b}) + P(\bar{a}, b), P(\bar{a}+a, \bar{b}+b) - P(\bar{a}, \bar{b}) - Q_{\bar{a},\bar{b}}(a, b) = P(a, b). \tag{10.91}$$

The bilinear estimates themselves follow directly by differentiating and estimating the integral composition formula

$$a \circ b(z, z') = \int a(z, z'')b(z'', z')dz''. \tag{10.92}$$

The shift in norm on the right compared to the left is to get a negative factor of $\langle z''\rangle$ to ensure integrability.

Smoothness of the inverse map is a little more delicate. Of course we do know what the derivative at the point g, evaluated on the tangent vector a is, namely $g^{-1}ag^{-1}$. So to get differentiability we need to estimate

$$(g + a)^{-1} - g^{-1} + g^{-1}ag^{-1} = g^{-1}a \left(\sum_{k \ge 0} (-1)^{k+1} g^{-1}(ag^{-1})^k \right) ag^{-1}. \tag{10.93}$$

This is the Neumann series for the inverse. If a is close to 0 in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then we know that $\|a\|_{L^2}$ is small, i.e. it is bounded by some norm on $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus the series on the right converges in bounded operators on $L^2(\mathbb{R}^n)$. However the smoothing terms on both sides render the whole of the right side smoothing and with all norms small in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ when a is small.

This proves differentiability, but in fact infinite differentiability follows, since the differentiability of g^{-1} and the smoothness of composition, discussed above, shows that $g^{-1}ag^{-1}$ is differentiable, and allows one to proceed on inductively. □

We also know that the trace functional extends to $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ as a trace functional, i.e. vanishing on commutators. This means that the construction above of Chern classes on $\mathrm{GL}(N, \mathbb{C})$ extends to $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

PROPOSITION 10.18. *(Universal Chern forms) The statements* (10.84) *extend to the infinite-dimensional group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ to define deRham classes $[\beta_{2k-1}]$ in each odd dimension.*

In fact these classes generate (not span, you need to take cup products as well) the cohomology, over $\mathbb{R}$, of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

PROOF. We have now done enough to justify the earlier computations in this setting. □

Proposition 10.19. *If X is a manifold and $a \in \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then the forms $a^*\beta_{2k-1}$ define deRham classes in $\mathrm{H}_c^{2k+1}(X;\mathbb{R})$ which are independent of the homotopy class and so are determined by $[a] \in \mathrm{K}^1_c(X)$. Combining them gives the (odd) Chern character*

$$\mathrm{Ch}_o([a]) = \sum_k c_{2k-1} a^*\beta_{2k-1}. \tag{10.94}$$

the particular constants chosen in (10.94) corresponding to multiplicativity under tensor products, which will be discussed below.

Proof. The independence of the (smooth) homotopy class follows from the computation above. Namely if $a_t \in \mathcal{C}^\infty_c(X\times[0,1]; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^b n)$ then $B_{2k-1} = a_t^*\beta_{2k-1}$ is a closed $(2k-1)$-form on $X \times [0,1]$. If we split it into the two terms

$$B_{2k-1} = b_{2k-1}(t) + \gamma_{2k-1}(t) \wedge dt \tag{10.95}$$

where $b_{2k-1}(t)$ and $\gamma_{2k-1}(t)$ are respectively a t-dependent $2k-1$ and $2k-2$ form, then

$$\begin{gathered} dB_{2k-1} = 0 \Longleftrightarrow \frac{\partial}{\partial t} b_{2k-1}(t) = d_X \gamma_{2k-2}(t) \text{ and hence} \\ b(1)_{2k-1} - b(0)_{2k-1} = d\mu_{2k-2},\ \mu_{2k-2} = \int_0^1 dt \gamma_{2k-2}(t) \end{gathered} \tag{10.96}$$

shows that $b(1)_{2k-1}$ and $b(0)_{2k-1}$, the Chern forms of a_1 and a_0 are cohomologous. □

The even case is very similar. Note above that we have defined even K-classes on X as equivalence classes under homotopy of elements $a \in \mathcal{C}^\infty_c(X; G^{-\infty}_{iso,\mathrm{sus}}(\mathbb{R}^n)$. The latter group consists of smooth loops in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ starting and ending at Id. This means there is a natural (smooth) map

$$T : G^{-\infty}_{\mathrm{iso,sus}}(\mathbb{R}^n) \times \mathbb{S} \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ (a,\theta) \longmapsto a(\theta). \tag{10.97}$$

This map may be used to pull back the Chern forms discussed above to the product and integrate over $\mathbb{S}$ to get forms in even dimensions:-

$$\beta_{2k} = \int_0^{2\pi} \mathrm{tr}(g^{-1}dg)^{2k+1},\ k = 0,1,\dots. \tag{10.98}$$

Proposition 10.20. *The group $G^{-\infty}_{\mathrm{iso,sus}}(\mathbb{R}^n)$ has an infinite number of components, labelled by the 'index' β_0 in* (10.98)*, the other Chern forms define cohomology classes such that for any map*

$$\mathrm{Ch}([a]) = \sum_{k=0}^\infty c_{2k} a^*\beta_{2k} \tag{10.99}$$

defines a map $\mathrm{K}^0_c(X) \longrightarrow \mathrm{H}^{\mathrm{even}}(X)$.

The range of this map spans the even cohomology, this is a form of a theorem of Atiyah-Hurzebruch.

If $f : X \longrightarrow Y$ is a smooth map then it induces a pull-back operation on vector bundles (see Problem 10.2) and this in turn induces an operation

Problem 10.2.

$$f^* : \mathrm{K}(Y) \longrightarrow \mathrm{K}(X). \tag{10.100}$$

Now we can interpret Proposition **??** in a more K-theoretic form.

10.12. Chern character

We have seen above that the 'unnormalized' Chern forms $\mathrm{Tr}((a^{-1}da)^{2k+1})$ are well-defined closed forms on the group $G^{-\infty}$ and allow manipulation in the uisual way. In particular they each pull back to give cohomology classes associated to a given odd K-class on a manifold. It is important for us to understand the behaviour of these forms under the basic maps in K-theory that we have defined above. The most important is the isotropic/Toeplitz index map (10.47). For the moment, we will take X to be compact even though this is not necessary.

The inverse of (10.47) we know explicitly, that is we know how to represent a bundle as the index bundle of a family of isotropic (or Toeplitz) operators. Namely if E is a vector bundle over X then it can be embedded as a subbundle of a trivial bundle so there is a smooth family of projections $\Pi \in \mathcal{C}^\infty(X; M(N, \mathbb{C}))$ such that we may identify $E = \mathrm{Ran}(\Pi)$. Then E (as an element of $K(X)$) is the index bundle for any isotropic family with symbol

$$a(x, \theta) = e^{-i\theta}\Pi(x) + (\mathrm{Id}_N - \Pi(x)),\ \Pi(x)^2 = \Pi(x). \tag{10.101}$$

So, it is naturally of interest to compute the (odd) Chern forms of a on $\mathbb{S} \times X$.

Computing away,

$$\begin{aligned} a^{-1}da &= \left(e^{i\theta}\Pi + \mathrm{Id} - \Pi\right)\left(-ie^{-i\theta}d\theta\Pi + (e^{-i\theta} - 1)d_X\Pi\right) \\ &= e^{i\theta}(e^{-i\theta} - 1)\Pi d_X\Pi + (e^{-i\theta} - 1)(\mathrm{Id} - \Pi)d_X\Pi - id\theta\Pi. \end{aligned} \tag{10.102}$$

As a form on a product manifold we may decompose

$$\mathrm{Tr}((a^{-1}da)^{2k+1}) = d\theta \wedge \alpha + \beta \tag{10.103}$$

where α and β are forms on X depending smoothly on θ. Since we know it is closed it follows that

$$d\theta(\partial_\theta\beta - d_X\alpha) + d_X\beta = 0 \Longrightarrow d_X\beta = 0,\ d_X\alpha = \partial_\theta\beta. \tag{10.104}$$

Expanding α and β in Fourier series

$$\alpha = \sum_{k\in\mathbb{Z}} e^{ik\theta}\alpha_k,\ \beta = \sum_{k\in\mathbb{Z}} e^{ik\theta}\beta_k \tag{10.105}$$

it follows from (10.104) that all the β_k with $k \neq 0$ are exact. In fact all the terms in (10.103) corresponding to $k \neq 0$ are exact since

$$d\theta \wedge e^{ik\theta}\alpha_k + e^{ik\theta}\beta_k = d(\frac{1}{ik}e^{ik\theta}\alpha_k). \tag{10.106}$$

So the only cohomology which can arise comes from the terms α_0 and β_0 since separately $d\alpha_0 = 0$ and $d\beta_0 = 0$.

Problem 10.3. Show that β_0 arising from the Chern form in (10.103) is cohomologous to a constant (i.e. is exact except in form degree 0. What is the constant?

So, we most want to compute α_0. By definition α is the contraction of $\operatorname{Tr}((a^{-1}da)^{2k+1})$ with ∂_θ. Watching out for normalizations it follows from antisymmetry and (10.102) that

$$\alpha = -i \operatorname{Tr}\left(\Pi(e^{i\theta}(e^{-i\theta}-1)\Pi(d_X\Pi) + (e^{-i\theta}-1)(\operatorname{Id}-\Pi)(d_X\Pi))^{2k}\right) \tag{10.107}$$

where you should note that for a projection $\Pi(d_X\Pi) = \Pi(d_X\Pi)(\operatorname{Id}-\Pi)$ (meaning that the differential is completely off-diagonal with respect to the projection at that point). So in fact

$$\alpha = -i(e^{ik\theta}(e^{-i\theta}-1)^{2k} \operatorname{Tr}\left(\Pi d_X\Pi(\operatorname{Id}-\Pi)d_X\Pi)^k\right). \tag{10.108}$$

Thus

$$\alpha_0 = -\frac{i}{2\pi}\int_0^{2\pi} e^{ik\theta}(e^{-i\theta}-1)^{2k} d\theta \operatorname{Tr}(\omega^k),\ \omega = \Pi(d_X\Pi)(\operatorname{Id}-\Pi)(d_X\Pi). \tag{10.109}$$

The constant here can be readily evaluated and is (perhaps)

$$-\frac{i}{2\pi}\int_0^{2\pi} e^{ik\theta}(e^{-i\theta}-1)^{2k} d\theta = -i\frac{(2k)!}{k!}. \tag{10.110}$$

Now, ω is in fact the curvature of a connection on the bundle $E = \operatorname{Ran}(\Pi)$. Namely, d can be applied to sections of $\operatorname{Ran}(\Pi)$ but will not give a new section of the bundle (with values in 1-forms as well), however $\nabla^\Pi s = \Pi ds = \Pi d\Pi s$ is a connection since it distributes over functions

$$\nabla^\Pi(fs) = dfs + f\nabla^\Pi s.$$

The curvature of this connection is easily computed, especially if one uses extension of the distribution law to all forms

$$\nabla(s\alpha) = d\alpha s + (-1)^k\alpha\nabla s,\ \alpha \in \mathcal{C}^\infty(X;\Lambda^k),$$

since then

$$(\nabla^\Pi)^2 s = \Pi d(\Pi ds) = \Pi(d\Pi)(\operatorname{Id}-\Pi)(d\Pi) = \omega. \tag{10.111}$$

Thus for this one connection we see that α_0 is a multiple of $\operatorname{Tr}(\omega^k)$. The basic observation of Chern-Weil theory is

LEMMA 10.9. *For any connection ∇ on a complex vector bundle E the forms*

$$\operatorname{tr}(\omega^k) \in \mathcal{C}^\infty(X;\Lambda^{2k}),\ \omega = \nabla^2,$$

are closed and represent a fixed deRham cohomology class.

PROOF. The crucial point is that (10.9) is always a closed form. The connection ∇ acts on sections of E but also defines a connection on the bundle $\hom(E)$ of homomorphisms. Namely if $b \in \mathcal{C}^\infty(X;\hom(E))$ then

$$(\nabla b)s = \nabla(bs) - b\nabla s = [\nabla, b]s$$

is a connection. As before it extends to homomorphisms with values in forms and in this sense Bianchi's identity holds

$$\nabla\omega = 0 \Longrightarrow \nabla\omega^k = 0. \tag{10.112}$$

Indeed, (10.112) just comes from the associativity of operators, that $\nabla(\nabla)^2 = (\nabla)^2\nabla$.

Locally on a coordinate patch in X over which the bundle E is trivial, i.e. can be identified with $\mathbb{C}^N$, any connection takes the form $d+\gamma$ where γ is a homomorphism of $\mathbb{C}^N$ with values in 1-forms on X in the open set. Then the connection acting on homomorphisms becomes $\nabla b = db + [\gamma, b]$ and so

$$d\,\mathrm{tr}(\omega^k) = \mathrm{tr}(d\omega^k) = \mathrm{tr}(d\omega^k + [\gamma, \omega^k]) = \mathrm{tr}(\nabla\omega^k) = 0 \tag{10.113}$$

using the trace identity.

Thus, $\mathrm{tr}(\omega^k)$ is a closed form for the curvature of any connection on E. To see that its cohomology class does not depend on which connection is used, observe that any two connections ∇_i $i = 0, 1$ are connected by a smooth path of connections, $\nabla_t = (1-t)\nabla_0 + t\nabla_1$, $t \in [0, 1]$. This 1-parameter family of connections is also a connection on E pulled back from X to $X \times [0, 1]$ in the sense that it defines

$$\nabla s(t, x) = \nabla_t s(t, x) + dt\partial_t s(t, x). \tag{10.114}$$

The Chern form $\mathrm{tr}(\nabla^2)$ is therefore closed as a form on $X \times [0, 1]$ from which it follows that $\mathrm{tr}(\nabla_0^2)$ and $\mathrm{tr}(\nabla_1^2)$, which are its pull-backs to $t = 0$ and $t = 1$, are cohomologous by the analogue of (10.104). □

This means that the cohomology classes

$$\mathrm{Ch}(E, \nabla) = \mathrm{tr}(\omega^k),\ \mathrm{Ch}(E) = [\mathrm{Ch}(E, \nabla)] \in H^{2k}(X) \tag{10.115}$$

are well-defined.

Lemma 10.10. *The Chern forms in* (10.115) *define maps*

$$K(X) \longrightarrow H^{2k}(X),\ k \in \mathbb{N}_0. \tag{10.116}$$

Proof. For the formal difference (E_+, E_-) of two bundles the Chern classes are just the differences. To see that this gives a well-defined map (10.116) we need to check that it respect equivalence classes. Invariance under bundle isomorphisms is obvious enough ****. To see invariance under stability, that $(E_+ \oplus F, E_- \oplus F)$ defines the same class as (E_+, E_-) it suffices to consider the Chern classes of sums of bundles. In fact the Chern classes are additive, since we can always take as connection on a sum the direct sum of connections on the summands. Then the curvature is the direct sum of the curvatures and it follows that

$$\mathrm{Ch}(E \oplus F, \nabla^E \oplus \nabla^F) = \mathrm{Ch}(E, \nabla^E) + \mathrm{Ch}(F, \nabla^F) \tag{10.117}$$

at the level of forms, and hence certainly at the level of cohomolgy. □

It is also straightforward to see what happens to these Chern forms for the tensor product of two bundles. Again on $E \otimes F$ on can take as connection the tensor product of connections on the bundles. Then

$$(\nabla^E \otimes \nabla^F)^2 = (\nabla^E)^2 \otimes \mathrm{Id}_E + \mathrm{Id}_E \otimes (\nabla^F)^2 \tag{10.118}$$

and it follows that the Chern forms decompose (for this connection)

$$\mathrm{tr}_{E\otimes F}(\omega_{E\otimes F})^k = \sum_{j=0}^{k} \binom{k}{j} \mathrm{tr}_E((\omega_E)^j) \wedge \mathrm{tr}((\omega_F)^{k-j}). \tag{10.119}$$

From the properties of the exponential and binomial coefficients it follows that the Chern character, formally a sum of all the Chern forms,

$$\mathrm{Ch}(E) = \mathrm{tr}(\exp(\omega)) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{tr}(\omega^k) \in \mathcal{C}^\infty(X; \Lambda^*) \tag{10.120}$$

defines a map which is both additive and multiplicative

(10.121)

$$\mathrm{Ch} : \mathrm{K}^0(X) \longrightarrow H^{\mathrm{even}}(X),\ \mathrm{Ch}(e+f) = \mathrm{Ch}(e) + \mathrm{Ch}(f),\ \mathrm{Ch}(ef) = \mathrm{Ch}(e) \wedge \mathrm{Ch}(f)$$

in cohomolgy (where you might prefer to think of wedge as the cup product). The basic normalization ensures that the constant terms is the (effective) rank of the bundle. A second normalization is possible, multiplying the curvature by a constant. This is frequently chosen so that the term of degree 2 is integral, i.e. is in the image of the integral cohomology.

Now, having normalized the even Chern character, consider the second map involved in Bott periodicity. Namely the injection

$$\mathrm{K}^1(X) \longrightarrow \mathrm{K}^0(\mathbb{S} \times X). \tag{10.122}$$

Here we use an element $a \in \mathcal{C}^\infty(X; \mathrm{GL}(N, \mathbb{C}))$ to define a vector bundle over $\mathbb{S} \times X$ by 'clutching'. The bundle can be defined in terms of its global section, so set, for $\epsilon > 0$ small,

(10.123)

$$\mathcal{C}^\infty(\mathbb{S} \times X; E_a) = \left\{ s \in \mathcal{C}^\infty([0, 2\pi + \epsilon); \mathbb{C}^N); s(t + 2\pi, x) = a(x)s(t, x),\ t \in [0, \epsilon) \right\}.$$

Problem 10.4. Go through the proof that there is a smooth vector bundle over $\mathbb{S} \times X$ such that $\mathcal{C}^\infty(\mathbb{S} \times X; E_a)$, as defined in (10.123), is the space of global sections. Hint:- Define the fibre as a quotient of the putative space of sections.

We wish to consider the Chern character of the bundle E_a and related it to a sum of forms on X. To do so we need to choose a connection on E_a; this can be thought of as a differential operator on sections. Namely if $\rho \in \mathcal{C}^\infty(\mathbb{R})$ has $\rho(t) = 1$ in $t < 1$ and $\rho(t) = 0$ in $t > \pi$ then

$$\nabla s(t) = d_X s + dt\partial_t s + \rho(t) a^{-1} das \tag{10.124}$$

is a well-defined operator

$$\nabla : \mathcal{C}^\infty(\mathbb{S} \times X; E_a) \longrightarrow \mathcal{C}^\infty(\mathbb{S} \times X; E_a \otimes \Lambda^1). \tag{10.125}$$

Indeed, if $\epsilon > 0$ is small enough, $\rho(t) = 1$ in $t < \epsilon$ and

$$\nabla s(2\pi + t, x) = ds(2\pi + t, x) = das(t, x) = a(\nabla s(t, x)). \tag{10.126}$$

It is convenient to choose ρ so that $\rho' \leq 0$.

Lemma 10.11. *The bundle E_a is isomorphic to the range of Π_a in Lemma 10.5.*

Proof. We proceed to show that E_a can be embedded as a subbundle of $\mathbb{C}^{2N}$ as a bundle over $\mathbb{S} \times X$. Consider $E_a \oplus E_{a^{-1}}$. This is defined by the same construction as E_a with a replaced by

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tag{10.127}$$

acting on $\mathbb{C}^{2N}$. It was shown above that this matrix is trivial as an odd K-class, i.e. can be connected to the identity. This can be done explicitly, for instance the family B(r)=

$$\begin{pmatrix} a\cos(\theta(r)) & \mathrm{Id}_N \sin(\theta(r)) \\ -\mathrm{Id}_N \sin(\theta(r)) & a^{-1}\cos(\theta(r)) \end{pmatrix} \tag{10.128}$$

connects it to

$$\begin{pmatrix} 0 & \mathrm{Id}_N \\ -\mathrm{Id}_N & 0 \end{pmatrix} \tag{10.129}$$

if $\theta : [0,1] \longrightarrow [0, 2\pi]$ is weakly increasing and constant at 0 and 2π near the end points. Reversing the curve with a replaced by the identity connects (10.127) to the identity.

Now, to embed E_a as a subbundle of $\mathbb{C}^{2N}$ it suffices to consider the bundle $E_{B(t)}$ over $\mathbb{S} \times X \times I$ where $B(r)$, for $r \in I$, is the curve connecting (10.127) to the identity. Thus $E_{B(r)}$ is $E_a \oplus E_{a^{-1}}$ at one end of the interval and $\mathbb{C}^{2N}$ at the other. Choosing a connection on $E_{B(r)}$ and integrating from E_a and integrating from one side to the other embeds E_a as a subbundle of $\mathbb{C}^{2N}$.

It remains to show that this subbundle is isomorphic to the range if Π_a as defined in before Lemma 10.5; to do so consider in more detail the connection on $E_{B(r)}$. From (10.124) the ∂_r component of the connection is

(10.130)

$$\nabla_{\partial_r} s = \partial_r s + \rho(t) B(r)^{-1}(\partial_r B(r)) s = \begin{cases} \partial_r s + \rho(t)\Theta'(r) A_1(x) s & r \in [0, \frac{\pi}{2}] \\ \partial_r s + \rho(t)\Theta'(-r) A_2 s & r \in [\frac{\pi}{2}, \pi] \end{cases}$$

$$A_1(x) = \begin{pmatrix} 0 & a^{-1}(x) \\ -a(x) & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & \mathrm{Id}(x) \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

The induced connection on homomorphisms acts by conjugation, so the projection in $\mathbb{C}^{2N}$ which gives the embedding is the solution of

$$\partial_r \Pi(r) + \rho(t)\Theta'(r)[A(x), \Pi(r)] = 0, \ \Pi(0) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}. \tag{10.131}$$

We will do this in two stages, corresponding to the two subintervals for $B(r)$. It is natural to look for $\Pi(r) = Q(r)\Pi(0)Q(r)^{-1}$ with Q invertible. Then the differential condition (10.131) can be replaced by

(10.132)

$$\partial_r Q(r) + \rho(t)\Theta'(r) A_1(x) Q(r) = 0 \Longrightarrow \partial_r (Q(r)^{-1}) + \rho(t)\Theta'(r) Q(r)^{-1} A_1(x) = 0,$$

where $Q(0) = \mathrm{Id}$. This is satisfied by

$$Q(r) = S(\rho(t)\Theta(r), -a^{-1}) \tag{10.133}$$

where $S(\theta, a)$ is defined in (10.38).

Thus after the first interval of integration the projection is

$$S(2\pi\rho(t), -a^{-1})\Pi(0)S(-2\pi\rho(t), -a^{-1}). \tag{10.134}$$

In the second interval of the homotopy, a is replaced by the identity so E_a is embedded in $\mathbb{C}^{2N}$ through the projection

$$S(-2\pi\rho(t), -\mathrm{Id})S(2\pi\rho(t), -a^{-1})\Pi(0)S(-2\pi\rho(t), -a^{-1})S(2\pi\rho(t), -\mathrm{Id}). \tag{10.135}$$

This is the same as $\Pi_a(t, x)$ in (10.39) except that all the signs are wrong at once!**** Better try to get the orientations right! □

Now the curvature of this connection over $\mathbb{S} \times X$ is

$$\nabla^2 = (d + \rho(t) a^{-1} da)^2 = \rho'(t) dt a^{-1} da + (\rho^2(t) - \rho(t))(a^{-1} da)^2. \tag{10.136}$$

The Chern character form of E_a with respect to this connection is

$$\mathrm{tr}\sum_k \frac{1}{k!}(\rho'(t)dta^{-1}da + (\rho^2(t)-\rho(t))(a^{-1}da)^2)^k. \tag{10.137}$$

From this even-degree sum of closed forms on $\mathbb{S}\times X$ we can extract an odd-degree sum of forms on X by integration over $\mathbb{S}$. Changing variable from t to $\rho(t)$ gives

$$\begin{aligned}\mathrm{Ch}^{\mathrm{odd}}(a) &= \sum_k \frac{1}{(k-1)!}\int_0^1 a^{-1}da((\rho^2(t)-\rho(t))(a^{-1}da)^2))^{k-1} \\ &= \int_0^1 \mathrm{tr}(a^{-1}da\exp(w(s)))ds,\ w(s)=s(s-1)(a^{-1}da)^2.\end{aligned} \tag{10.138}$$

Proposition 10.21. *The odd Chern character, defined by* (10.138)*, gives an additive map*

$$\mathrm{K}^1(X)\longrightarrow \mathrm{H}^{\mathrm{odd}}(X) \tag{10.139}$$

which has the multiplicative property

$$\mathrm{Ch}^{\mathrm{odd}}(a\otimes \mathrm{Id}_E) = \mathrm{Ch}^{\mathrm{odd}}(a)\wedge \mathrm{Ch}(E) \tag{10.140}$$

for any vector bundle E over X.

Proof. This follows directly from the discussion above. The multiplicativity in (10.140) is a consequence of the fact that if $a\otimes \mathrm{Id}_E$ is used to define a bundle over $\mathbb{S}\times X$ following the clutching construction above then the resulting bundle is $E_a\otimes E$. Then (10.140) is a consequence of the multiplicativity of the even Chern character under tensor products. □

In fact it is rather useful to generalize the formula in (10.138) by allowing a to be an isomorphism of a general bundle F over X, rather than a trivial bundle. Then a defines a class by stabilization, meaning that if F is complemented to a trivial bundle then a is extended by the identity on the complement. Proceeding directly the space of global sections of the new bundle over $\mathbb{S}\times X$ is defined by the obvious replacement of (10.123):

$$\mathcal{C}^\infty(\mathbb{S}\times X;E_a) = \{s\in\mathcal{C}^\infty([0,2\pi+\epsilon);F); s(t+2\pi,x)=a(x)s(t,x),\ t\in[0,\epsilon)\}. \tag{10.141}$$

The trivial connection d in (10.124) can then be replaced by a connection ∇^F on F and used in the same way to define a connection

$$\nabla s = (\nabla^F + \rho(t)a^{-1}\nabla a)s. \tag{10.142}$$

The formula for the odd Chern character in this more general setting is due (I believe) to Fedosov (beware of possible sign errors below, to say the least)

$$\begin{aligned}\mathrm{Ch}^{\mathrm{odd}}(a) &= \int_0^1 \mathrm{tr}(a^{-1}\nabla a\exp(w(s)))ds, \\ &\qquad w(s) = (1-s)\omega_F + sa^{-1}\omega_F a + s(s-1)(a^{-1}da)^2.\end{aligned} \tag{10.143}$$

Problem 10.5. Go through the derivation of (10.143) and correct it as necessary!

Problem 10.6. Formula (10.138) normalizes the constants in (10.94); what are they?

Going back to the discussion at the beginning of this section we can now deduce the 'Toeplitz index in cohomology'.

PROPOSITION 10.22. *Under the isotropic/Toeplitz index map* (10.47),

$$\mathrm{Ch}(\mathrm{Ind}(a)) = -\frac{1}{2\pi i}\int_{\mathbb{S}} \mathrm{Ch}^{\mathrm{odd}}(a). \tag{10.144}$$

Of course this is consistent with (10.140) since we know that if E is a bundle over X then $\mathrm{Ind}(a\otimes \mathrm{Id}_E) = \mathrm{Ind}(a)\otimes E$, where this should really be thought of as products in K-theory.

PROOF. Check the constants, I haven't. *** □

I also should discuss here the extension to non-compact manifolds. This is quite straightforward.

10.13. Todd class

Now, we need to go on and see the effect on the Chern character, i.e. in cohomology, of the Thom isomorphism; whoops it isn't there yet ***. Thus, if E is a complex (or symplectic) vector bundle over X then there is an isomorphism

$$\mathrm{Thom} : \mathrm{K}^0_{\mathrm{c}}(E) \longrightarrow \mathrm{K}^0(X) \tag{10.145}$$

which is given by the isotropic index map.

PROPOSITION 10.23. *If E is a complex vector bundle over X then there is a cohomology class* $\mathrm{Td}(E) \in \mathrm{H}^{\mathrm{even}}(E)$ *such that under the Thom isomorphism*

$$\mathrm{Ch}(\mathrm{Thom}(f)) = \int_{E/X} \mathrm{Ch}(f)\wedge \mathrm{Td}(E). \tag{10.146}$$

Note that this Todd class $\mathrm{Td}(E)$ represents a 'twisting' in the behaviour of K-theory as opposed to cohomology under push-forward.

PROOF. We are supposed to know by now that the inverse of (10.145) is given by 'twisting with the Bott element'. That is, we know there is an element $\beta_E \in \mathrm{K}^0_{\mathrm{c}}(E)$, the Bott element, represented by a family of harmonic oscillators, which has index class, $\mathrm{Thom}(\beta_E)$, a trivial 1-dimensional line bundle.

Consider first the case that E is a trivial line bundle, hence a trivial bundle with fibre $\mathbb{R}^2$ as a real space. Thus we know about Bott periodicity and in fact we get a commutative diagramme

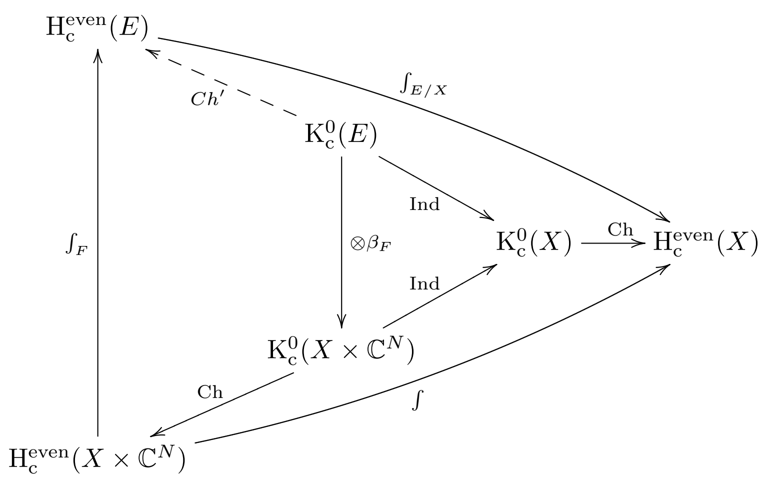
$$\begin{array}{ccccc} \mathrm{K}^0_{\mathrm{c}}(X\times\mathbb{R}^2) & \xrightarrow{\simeq} & \mathrm{K}^1_{\mathrm{c}}(X\times\mathbb{R}) & \xrightarrow{\simeq} & \mathrm{K}^0_{\mathrm{c}}(X) \\ \downarrow{\scriptstyle \mathrm{Ch}} & & \downarrow{\scriptstyle \mathrm{Ch}^{\mathrm{odd}}} & & \downarrow{\scriptstyle \mathrm{Ch}} \\ \mathrm{H}^{\mathrm{even}}_{\mathrm{c}}(X\times\mathbb{R}^2) & \xrightarrow{\simeq} & \mathrm{H}^{\mathrm{odd}}_{\mathrm{c}}(X\times\mathbb{R}) & \xrightarrow{\simeq} & \mathrm{H}^{\mathrm{even}}_{\mathrm{c}}(X). \end{array} \tag{10.147}$$

The top row we know to be isomorphisms and the two bottom maps are also isomorphisms, given by integration. We have defined the odd Chern character so that the left square commutes. We also know that the Bott element, the symbol $e^{-i\theta}$ on the circle, induces an element of $\mathrm{K}^1_{\mathrm{c}}(\mathbb{R}\times X)$ which is mapped to the trivial line by the index map, the second map on the top, and has Chern character equal to 1. The commutativity of the right square then follows from the multiplicativity of the Chern character in (10.140). This proves (10.146) in the case that E is a trivial line bundle.

Since we have not assumed that X is compact here the case of a general rank n trivial complex or rank $2n$ real bundle follows by iteration of (10.147); again the Todd class is 1.

Now, as with the Thom isomorphism for K-theory, we pass to the general case by complementing a complex bundle E to a trivial bundle $E \subset \mathbb{C}^N$ with complementary bundle F. Then we know we have isomorphisms in K-theory and cohomology leading to a commutative diagramme

(10.148)



Here all three inner maps and all three outer maps are isomorphisms. The inner triangle commutes and the outer triangle also commutes, being fibre integration. The quadrangle towards the lower right commutes, this being the case of a trivial bundle just discussed. Thus the diagramme without the dotted arrow is commutative. Moreover there is only one way to get the left quadrangle to commute, namely by defining

$$\mathrm{Ch}'(e) = \int_F \mathrm{Ch}(e \otimes \beta_F) \tag{10.149}$$

where the integral is over the fibres of F. Then the whole diagramme commutes and gives us the formula in cohomology that we want. On the other hand, $\mathrm{Ch}(e \otimes \beta_F) = \pi^* \mathrm{Ch}(e) \otimes \mathrm{Ch}(\beta_F)$ where π is the projection from $\mathbb{C}^N \times X$ to E along the fibres of F. Since

$$\int_f \pi^* a \wedge b = a \wedge \int_F b \tag{10.150}$$

for any form b on $\mathbb{C}^N \times X$ with compact support relative to the fibres of F, the integral being fibre integration, we conclude that

$$\mathrm{Ch}'(e) = \mathrm{Ch}(e) \wedge \mathrm{Td}(E),\ \mathrm{Td}(E) = \int_F \mathrm{Ch}(\beta_F) \tag{10.151}$$

with the Todd class being, by definition, a form on the total space of E, but not with compact support. □

10.14. Stabilization

In which operators with values in $\Psi^{-\infty}_{\text{iso}}$ are discussed.

10.15. Delooping sequence

The standard connection between even and odd classifying groups.

10.16. Looping sequence

The quantized connection between classifying groups.

10.17. $\mathcal{C}^*$ algebras

10.18. K-theory of an algeba

10.19. The norm closure of $\Psi^0(X)$

10.20. Problems

Problem 10.7. There is a natural adjoint map on $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ so we could also look at the unitary subgroup

$$\mathrm{U}^{-\infty}_{\text{iso}}(\mathbb{R}^n) = \left\{A \in G^{-\infty}_{\text{iso}}(\mathbb{R}^n); (\mathrm{Id}+A)^{-1} = \mathrm{Id}+A^*\right\}. \tag{10.152}$$

Show that the natural inclusion induces an homotopy equivalence, so there is a natural identification

$$\mathrm{K}^1_{\mathrm{c}}(X) \simeq \mathcal{C}^\infty_c(X; U^{-\infty}_{\text{iso}})/\sim \tag{10.153}$$

where the equivalence relation is again homotopy.

Problem 10.8. Remind yourself of the proof that $G^{-\infty}_{\text{iso}}(\mathbb{R}^n) \subset \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is open. Since $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is a group, it suffices to show that a neighbourhood of $0 \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is a neighbourhood of the identity. Show that the set $\|A\|_{\mathcal{B}(L^2)} < \frac{1}{2}$, given by the operator norm, fixes an open neighbourhood of $0 \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ (this is the L^2 continuity estimate). The inverse of $\mathrm{Id}+A$ for A in this set is given by the Neumann series and the identity (which follows from the Neumann series)

$$(\mathrm{Id}+A)^{-1} = \mathrm{Id}+B = \mathrm{Id}-A + A^2 - ABA \tag{10.154}$$

in which *a prioiri* $B \in \mathcal{B}(L^2)$ shows that $B \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ by the 'corner' property of smoothing operators (meaning $ABA' \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ if $A, A' \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ and $B \in \mathcal{B}(L^2)$.

Problem 10.9. Additivity of the map (10.46).

Problem 10.10. Details that (10.46) is an isomorphism.

Problem 10.11. Check that (10.44) is well-defined, meaning that if (V_1, W_2) is replaced by an equivalent pair then the result is the same. Similarly check that the operation is commutative and that it make $\mathrm{K}(X)$ into a group.

Problem 10.12. Check that you do know how to prove (10.28). One way is to use induction over N, since it is certainly true for $N = 1$, $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$. Proceeding by induction, note that an element $a \in \mathrm{GL}(N, \mathbb{C})$ is fixed by its effect on the standard basis, e_i. Choose $N-1$ elements ae_j which form a basis together with e_1. The inductive hypothesis allows these elements to be deformed, keeping their e_1 components fixed, to e_k, $k > 1$. Now it is easy to see how to deform the resulting basis back to the standard one.

Problem 10.13. Prove (10.31). Hint:- The result is very standard for $N = 1$. So proceed by induction over N. Given a smooth curve in $\mathrm{GL}(N, \mathbb{C})$, by truncating its Fourier series at high frequencies one gets, by the openness of $\mathrm{GL}(N, \mathbb{C})$, a homotopic curve which is real-analytic, denote it $a(\theta)$. Now there can only be a finite number of points at which $e_1 \cdot a(\theta)e_1 = 0$. Moreover, by deforming into the complex near these points they can be avoided, since the zeros of an analytic function are isolated. Thus after homotopy we can assume that $g(\theta) = e_1 \cdot a(\theta)e)1 \neq 0$. Composing with a loop in which e_1 is roatated in the complex by $1/g(\theta)$, and e_2 in the opposite direction, one reduces to the case that $e_1 \cdot a(\theta)e)1 = 0$ and then easily to the case $a(\theta)e_1 = e_1$, then induction takes over (with the determinant condition still holding). Thus it is enough to do the two-dimensional case, which is pretty easy, namely e_1 rotated in one direction and e_2 by the inverse factor.