CHAPTER 2

Lectures 3 and 4, 12th and 14th February: The circle

After the overview last week I will start at the beginning, indeed this week I will talk about analysis (and even a little geometry!) of the circe, U(1). I have moved the general discussion of compact manifolds to an appendix.

1. Functions and Fourier series

Now, we specialize to the circle, U(1) which naturally enough plays a fundamental role in the discussion of loop spaces. The most obvious model for the circle for the discussion of basic regularity is as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ so that $\mathcal{C}^\infty(\mathrm{U}(1))$ is identified with the subspace of $\mathcal{C}^\infty(\mathbb{R})$ consisting of the 2π-periodic functions – and similarly for pretty much every other function space between $\mathcal{C}^{-\infty}(\mathrm{U}(1))$ and $\mathcal{C}^\infty(\mathrm{U}(1))$.

In particular we have a very natural differential operator on functions, $\frac{1}{i}\frac{d}{dt}$ and the spectral theory of this operator corresponds to Fourier series. Thus there are identifications

$$\begin{gathered}\mathcal{C}^\infty(\mathrm{U}(1)) \longrightarrow \mathcal{S}(\mathbb{Z})\\ L^2(\mathrm{U}(1)) \longrightarrow l^2(\mathbb{Z})\\ \mathcal{C}^{-\infty}(\mathrm{U}(1)) \longrightarrow \mathcal{S}'(\mathbb{Z})\\ u(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikt},\ c_k = \frac{1}{2\pi}\int_{\mathrm{U}(1)} u(t)e^{-ikt}|d\theta|\end{gathered} \tag{2.1.1}$$

where the spaces on the right consists of the rapidly decreasing, square summable, and polynomially increasing sequences:-

$$\begin{gathered}\mathcal{S}(\mathbb{Z}) = \{c. : \mathbb{Z} \longrightarrow \mathbb{C}; \sum_k (1+|k|)^l |a_k| < \infty \ \forall\ l \in \mathbb{N}\},\\ l^2(\mathbb{Z}) = \{c. : \mathbb{Z} \longrightarrow \mathbb{C}; \sum_k |a_k|^2 < \infty\},\\ \mathcal{S}'(\mathbb{Z}) = \{c. : \mathbb{Z} \longrightarrow \mathbb{C}; \sum_k (1+|k|)^{-N}|a_k| < \infty \text{ for some } N \in \mathbb{N}\}\end{gathered} \tag{2.1.2}$$

where in the last case N depends on the sequence.

The use of Fourier series allows much of the general discussion above to be made explicit on the circle, but does lead to difficulties when it comes to coordinate-invariance. For example Schwartz kernel theorem in this case becomes the (relatively simply checked) statement that any continuous linear map $\mathcal{C}^\infty(\mathrm{U}(1)) \longrightarrow \mathcal{C}^{-\infty}(\mathrm{U}(1))$ is given by an infinite matrix of polynomial growth on the Fourier transform side

$$(Ac)_k = \sum_j A_{kj} c_j,\ |A_{jk}| \le C(1+|j|+|k|)^N \text{ for some } N. \tag{2.1.3}$$

Smoothing operators correspond to matrices in $\mathcal{S}(\mathbb{Z}^2)$ in the obvious sense. Thus, is $A \in \Psi^{-\infty}(\mathrm{U}(1))$ is a smoothing operator

$$\begin{gathered} A(\sum_k c_k e^{ikt}) = \sum_j d_j e^{ijt},\ d_j = \sum_k A_{jk} c_k, \\ A_{jk} \in \mathcal{S}(\mathbb{Z}^2),\ \text{i.e. } \forall\ N,\ |A_{jk}| \le C_N (1+|j|+|k|)^{-N}. \end{gathered} \tag{2.1.4}$$

Conversely any such rapidly decreasing matrix defines a smoothing operator.

Exercise 2. Write out the relationship of Fourier series on the 2-torus between the matrix A_{jk} and the kernel $A \in \mathcal{C}^\infty(\mathrm{U}(1)^2)$ of a smoothing operator.

2. Hardy space

Of course the circle is extremely special among manifolds! In particular its identification with the unit circle in $\mathbb{C}$ with respect to the Euclidean metric, via the exponential function, means that there is a special (really a whole lot of special) subspaces of $\mathcal{C}^\infty(\mathrm{U}(1))$ of 'half dimension'. The usual choice is the (smooth) Hardy space

$$\mathcal{C}^\infty_H(\mathrm{U}(1)) = \{u \in \mathcal{C}^\infty(\mathrm{U}(1)); c_k = 0 \text{ for } k < 0 \text{ in (2.1.1)}\}. \tag{2.2.1}$$

The elements of $\mathcal{C}^\infty_H(\mathrm{U}(1))$ are precisely those smooth functions which are the restriction to the unit circle of a smooth function on the closed ball, $\{|z| \le 1\}$ in $\mathbb{C}$ which is holomorphic in the interior.

Consider the projection onto this subspace, which can be written explicitly in terms of the expansion in Fourier series

$$P_H u = \sum_{k \ge 0} c_k e^{ikt},\ u = \sum_k c_k e^{ikt},\ P_H^2 = P_H. \tag{2.2.2}$$

As such it is clear self-adjoint with respect to the L^2 inner product

$$\langle u, v \rangle = \int_{\mathrm{U}(1)} u(t)\overline{v(t)} dt. \tag{2.2.3}$$

In fact P_H is probably the simplest example of non-differential pseudodifferential operator. To see this of course, you need to know what a pseudodifferential operator is, but for the moment this does not matter.

The most crucial, perhaps non-obvious, property of P_H is that it 'almost commutes' with multiplication. Thus suppose $a \in \mathcal{C}^\infty(\mathrm{U}(1))$ (maybe complex-valued), this defines a multiplication operator which I will also denote a :

$$a : \mathcal{C}^\infty(\mathrm{U}(1)) \ni u \longmapsto au \in \mathcal{C}^\infty(\mathrm{U}(1)). \tag{2.2.4}$$

Lemma 1. *For any $a \in \mathcal{C}^\infty(\mathrm{U}(1))$, the commutator with P_H is a smoothing operator*

$$[a, P_H] \in \Psi^{-\infty}(\mathrm{U}(1)). \tag{2.2.5}$$

PROOF. We only need find expressions for aP_H and P_Ha in terms of Fourier series, which amounts to finding its action on e^{ikt}. First observe that multiplication becomes convolution in the sense that

$$\begin{gathered} a=\frac{1}{2\pi}\sum_l a_l e^{ilt}\in\mathcal{C}^\infty(\mathrm{U}(1))\Longrightarrow \\ au=\frac{1}{2\pi}\sum_j b_j e^{ijt} \text{ where } b_j=\sum_l a_{j-l}c_l \text{ if } u=\frac{1}{2\pi}\sum_k c_k e^{ikt}. \end{gathered} \tag{2.2.6}$$

It follows that

$$\begin{aligned} (aP_H)u&=\frac{1}{2\pi}\sum_j\left(\sum_{l\geq 0}a_{j-l}c_l)\right)e^{ijt} \\ (P_Ha)u&=\frac{1}{2\pi}\sum_{j\geq 0}\left(\sum_l a_{j-l}c_l)\right)e^{ijt}. \end{aligned} \tag{2.2.7}$$

Thus the commutator is given by the difference, which means that

$$\begin{gathered} u=\sum_j\sum_l B_{jl}c_l e^{ijt}, \text{ where} \\ B_{j,l}=\begin{cases} a_{j-l} & \text{if } j<0, l\geq 0 \\ -a_{j-l} & \text{if } j\geq 0, l<0 \\ 0 & \text{otherwise.}\end{cases} \end{gathered} \tag{2.2.8}$$

By assumption, a is smooth, so $|a_{j-l}|\leq C_N(1+|j-l|)^{-N}$ for each N. For the non-zero terms in (2.2.8), $|j-l|=|j|+|l|$ since the signs are always opposite. Thus

$$|B_{j,l}|\leq C_N(1+|j|+|l|)^{-N}$$

is rapidly decreasing in all directions on $\mathbb{Z}^2$. □

Note that this behaviour can be attributed to one oddity of U(1), the circle, that distinguishes it from other connected compact manifolds. Namely its cosphere bundle has two components.

3. Toeplitz operators

This behaviour of commutators allows us to define the Toeplitz algebra. Clearly $\mathcal{C}^\infty(\mathrm{U}(1))$ forms an algebra of multiplication operators. However, if we project this onto $\mathcal{C}^\infty_H(\mathrm{U}(1))$ by defining

$$a_H=P_HaP_H:\mathcal{C}^\infty_H(\mathrm{U}(1))\longrightarrow\mathcal{C}^\infty_H(\mathrm{U}(1)). \tag{2.3.1}$$

This is a Toeplitz operator, the 'projection' of a multiplication operator onto the Hardy space. However, these operators do not form an algebra since the composite is instead

$$P_HaP_HbP_H=P_HabP_H+P_Ha[P_H,b]P_H. \tag{2.3.2}$$

The second term here is a smoothing operator – this is one of the basic properties of pseudodifferential operators, that the composite of a pseudodifferential operator and a smoothing operator is smoothing. In particular this means that the composites with the Hardy projection or multiplication by a smooth function are again smoothing

$$P_H A,\ AP_H,\ aA,\ Aa \in \Psi^{-\infty}(\mathrm{U}(1)) \text{ if } A \in \Psi^{-\infty}(\mathrm{U}(1)),\ a \in \mathcal{C}^\infty(\mathrm{U}(1)). \tag{2.3.3}$$

Exercise 3. Prove (2.3.3).

Definition 1. The Toeplitz operators (with smooth coefficients) consist of the sum

$$\Psi_{\mathrm{To}}(\mathrm{U}(1)) = P_H \mathcal{C}^\infty(\mathrm{U}(1)) P_H + P_H \Psi^{-\infty}(\mathrm{U}(1)) P_H \tag{2.3.4}$$

as an algebra of operators on $\mathcal{C}^\infty_H(\mathrm{U}(1))$.

Now, the decomposition (2.3.4) of a Toeplitz operator is unique and moreover the multiplier can be recovered from

$$\begin{gathered} P_H a P_H + \Psi^{-\infty}_{\mathrm{To}}(\mathrm{U}(1)),\ a \in \mathcal{C}^\infty(\mathrm{U}(1)). \\ \Psi^{-\infty}_{\mathrm{To}}(\mathrm{U}(1)) = \{P_H A P_H; A \in \Psi^{-\infty}(\mathrm{U}(1))\}. \end{gathered} \tag{2.3.5}$$

To recover a from $a_H = P_H a P_H$ it suffices to look at $a_H e^{ikt}$ for k large. Indeed from (2.2.7)

$$\begin{gathered} a_H e^{ikt} = \sum_{j \ge 0} a_{j-k} e^{ijt},\ k \ge 0 \Longrightarrow \\ \int_{\mathrm{U}(1)} e^{-i(p+k)t} \left(a_H e^{ikt}\right) dt = a_p,\ k+p \ge 0. \end{gathered} \tag{2.3.6}$$

So, by taking k large enough we can recover a_p from $a_H e^{ikt}$ and hence we can recover $a \in \mathcal{C}^\infty(\mathrm{U}(1))$. More generally, if $A \in \Psi^{-\infty}_{\mathrm{To}}(\mathrm{U}(1))$ then

$$\lim_{k \to \infty} \int_{\mathrm{U}(1)} e^{-i(p+k)t} \left(A e^{ikt}\right) dt = 0 \tag{2.3.7}$$

Proposition 1. *The Toeplitz-smoothing operators form an ideal in the Toeplitz operators (with smooth coefficients) and gives a short exact sequence of algebras*

$$\Psi^{-\infty}_{\mathrm{To}}(\mathrm{U}(1)) \longrightarrow \Psi_{\mathrm{To}}(\mathrm{U}(1)) \xrightarrow{\sigma} \mathcal{C}^\infty(\mathrm{U}(1)). \tag{2.3.8}$$

The second homomorphism here is a special case of the 'symbol map' for pseudodifferential operators.

Proof. Basically this is proved above. □

Although this is what I will call the Toeplitz algebra, the name is applied to several closely related algebras (particularly the norm closure of this algebra as bounded operators on the L^2 version of the Hardy space). Even keeping things 'smooth' there is another algebra which is important here, at least as an aid to understanding. Namely, we can simply 'compress' the pseudodifferential operators on the Hardy space and define

$$\Psi^{\mathbb{Z}}_H(\mathrm{U}(1)) = P_H \Psi^{\mathbb{Z}}(\mathrm{U}(1)) P_H. \tag{2.3.9}$$

The usual definition of a Toeplitz algebra in higher dimensions is derived from this. Since multiplication operators and smoothing operators are pseudodifferential operators,

$$\Psi_{\mathrm{To}}(\mathrm{U}(1)) \subset \Psi^0_H(\mathrm{U}(1)). \tag{2.3.10}$$

The question then is:- What else is in the space on the right – we could call it the 'extended Toeplitz algebra'.

The main extra terms are the compressions of pseudodifferential operators of negative integral order. As matix operator on the Fourier series side these are of the form, for $m \in \mathbb{N}$

(2.3.11)
$$Ae^{ikt} = \sum_{j=0}^{\infty} A_{jk} e^{ijt}, A_{jk} = a_{j-k}(k+1)^{-m},\ j,k \geq 0,\ \sup_j |a_j|(1+|p|)^N < \infty\ \forall\ N.$$

This is in fact just the composite of the compression of a multiplication operator, by $a \in \mathcal{C}^\infty(\mathrm{U}(1))$ with Fourier coefficients a_j and the convolution operator given by multiplication by $(1+k)^{-m}$ on the Fourier series side. Note that the $k+1$ is just to avoid problems at $k=0$. Beyond this the extended Toeplitz algebra has a completeness property.

4. Toeplitz index

How is this related to loops? One way to see a relationship is to observe that

$$\mathcal{L}\,\mathrm{U}(1) \subset \mathcal{C}^\infty(\mathrm{U}(1)) \tag{2.4.1}$$

consisting of the maps with values in $\mathrm{U}(1) \subset \mathbb{C}$. These form a group and so we can look for the invertible elements in $\Psi_{\mathrm{To}}(\mathrm{U}(1))$ which map into this group. The answer is at first a bit disappointing!

Lemma 2. *The elements of $\mathcal{L}\,\mathrm{U}(1) \subset \mathcal{C}^\infty(\mathrm{U}(1))$ which lift to invertible elements of $\Psi_{\mathrm{To}}(\mathrm{U}(1))$ are precisely those with winding number* 0, *i.e. are contractible to constant in $\mathcal{L}\,\mathrm{U}(1)$.*

This you might say is the simplest form of the Atiyah-Singer index theorem, long predating it of course since it was known to Toeplitz (in the 1920s I think). If we do a little more we get a true index theorem:-

Theorem 2. *If $a \in \mathcal{C}^\infty(\mathrm{U}(1);\mathbb{C}^*)$ (i.e. is non-zero) then, as an operator on $\mathcal{C}^\infty_H(\mathrm{U}(1))$, any element $B \in \Psi_{\mathrm{To}}(\mathrm{U}(1))$ with $\sigma(B) = a$ is Fredholm – has finite dimensional null space and closed range of finite codimension – and*

$$\mathrm{ind}(B) = \dim \mathrm{null}(B) - \dim\left(\mathcal{C}^\infty_H(\mathrm{U}(1))/B\mathcal{C}^\infty_H(\mathrm{U}(1))\right) = -\,\mathrm{wn}(a) \tag{2.4.2}$$

is determined by the winding number of a.

5. Diffeomorphisms and increasing surjections

6. Toeplitz central extension

Now let me get a little closer to the core topic and show one construction of the central extension of the loop group on any connected and simply connected Lie group. There are quite a few other constructions (see for instance that of Mickelsson).

We can embed $\mathrm{Spin}(n)$ in the (real) Clifford algebra $\mathrm{Cl}(n)$ and so think of it concretely as a group in an algebra. This means that $\mathcal{L}\,\mathrm{Spin}(n)$ sits inside the algebra $\mathcal{C}^\infty(\mathrm{U}(1);\mathrm{Cl}(n))$. There is a 'Hardy' subalgebra of $H \subset \mathcal{C}^\infty(\mathrm{U}(1);\mathrm{Cl}(n)$ consisting of the functions with vanishing negative Fourier coefficients and a natural projection Π_H onto it. The Toeplitz operators $\Pi_H\mathcal{C}^\infty(\mathrm{U}(1);\mathrm{Cl}(n))\Pi_H$, do not form an algebra, but do so when extended by the Toeplitz smoothing operators with values in $\mathrm{Cl}(n)$ – because $[H, \mathcal{C}^\infty(\mathrm{U}(1); \mathrm{Cl}(n)]$ consists of such smoothing operators. We can call this the Clifford-Toeplitz algebra.

Invertible elements in $\mathcal{C}^\infty(\mathrm{U}(1); \mathrm{Cl}(n))$ give Fredholm operators and the fact that $\mathrm{Spin}(n)$ is simply connected means that the index vanishes and so there is a group of invertible operators in the Clifford-Toeplitz algebra and a subgroup of unitary extension. This gives a short exact sequence of groups

$$G_H^{-\infty}(\mathrm{U}(1); \mathrm{Cl}(n)) \longrightarrow \mathcal{G}_H \longrightarrow \mathcal{L}\,\mathrm{Spin}(n)$$

with kernel the unitary smoothing perturbations of Π_H. The Fredholm determinant is defined on $G_H^{-\infty}(\mathrm{U}(1); \mathrm{Cl}(n))$ so we may take subgroup of determinannt one; it is also a normal subgroup of $\mathcal{G}_H$. Finally then the quotient group

$$E\mathcal{L}\,\mathrm{Spin}(n) = \mathcal{G}/\{A \in G_H^{-\infty}(\mathrm{U}(1); \mathrm{Cl}(n)); \det(A) = 1\}$$

is the desired (basic) central extension of $\mathcal{L}\,\mathrm{Spin}(n)$. This construction can be modified to work even for non-simply connected groups such as $\mathrm{U}(n)$.