CHAPTER 1

Lectures 1 and 2: An overview, February 5 and 7, 2013

1. Manifolds, maybe big

There is general agreement about what a finite dimensional manifold is, less about the infinite dimensional case. Let me try to indicate what the issues are.

Let's agree from the beginning that a manifold X is a topological space, that it is Hausdorff and that it is paracompact. You might wonder about this last condition but although the infinite-dimensional spaces we consider are big, they are not so big. One significant thing that is lost in passing to the infinite dimensional case is *local compactness*. This makes it very difficult to integrate functions.

In addition to these basic properties, a manifold is supposed to have a 'local regularity structure' given by local coordinate systems.

The local coordinates correspond to the choice of a model space V and X is supposed to have a covering by open sets U_a with homeomorphisms $F_a : U_a \longrightarrow V_a$ where $V_a \subset V$ is open and

$$F_{ab} : V_{ab} = F_a(U_a \cap U_b) \longrightarrow F_b(U_a \cap U_b) = V_{ba}$$

is required to lie in a specified groupoid $\underline{\mathrm{Dff}}(V)$ consisting of homeomorphisms between open subsets of V.

2. Transformation groupoid

This is where the real issues start,

What is $\underline{\mathrm{Dff}}(V)$?

We will want $\underline{\mathrm{Dff}}(V)$ to preserve some space of 'smooth functions' $\underline{\mathcal{C}^\infty}(V)$ on open subsets of V – certainly they are supposed to be continuous. Usually V will be some linear space; in the cases of interest here a Fréchet space – a complete countably normed space. This includes Banach and so Hilbert spaces.

I will usually denote a finite dimensional manifold by M. In this familiar case $V = \mathbb{R}^n$ and the transition groupoid $\underline{\mathrm{Dff}}(\mathbb{R}^n)$ consists of the smooth maps with smooth inverses between open subsets, $F_{ab} : V_{ab} \longrightarrow V_{ba}$ such that $F_{ab}^*(\mathcal{C}^\infty(V_{ba})) = \mathcal{C}^\infty(V_{ab})$. There are restricted choices (for instance symplectic diffeomorphisms) but pretty much everyone agrees that this is the definition of a finite-dimensional smooth manifold. Of course one can vary the local structure by considering $\mathcal{C}^k$ or real-analytic manifolds.

Note however that the transition groupoid, or even the global group of diffeomorphisms $\mathrm{Dff}(\mathbb{R}^n)$ is not so easy to deal with. For any (say compact) finite-dimensional manifold, $\mathrm{Dff}(M)$ is a Fréchet Lie group, with Lie algebra $\mathcal{V}(M)$, the space of real smooth vector fields on M. So $\mathrm{Dff}(M)$ *is* in some sense a manifold modelled on $\mathcal{V}(M)$ as model space. This is the sort of infinite dimensional manifold we will consider, one modelled on a space like the smooth functions on a finite dimensional manifold. I will review the properties of $\mathrm{Dff}(M)$ later, but let me warn you that it is not quite like a finite-dimensional Lie group. For instance, the exponential map (which does exist in the compact case) is not surjective from a neighbourhood of 0. Fortunately the structure of $\underline{\mathrm{Dff}}(\mathbb{R}^n)$ does not arise directly in dealing with finite-dimensional manifolds since we really only consider a finite number of elements at any one time.

3. Riemann manifolds

While I am at it, let me remind you about Riemann manifolds.

With any finite-dimensional smooth manifold we can associate may others, one particularly important one being the tangent bundle which comes with a smooth map $TM \longrightarrow M$. Because it is constructed from M, the groupoid for TM is also $\underline{\mathrm{Dff}}(\mathbb{R}^n)$, but we can expand it to $\underline{\mathcal{C}^\infty}(\mathbb{R}^n; \mathrm{GL}(n))$ (sending a diffeomorphism to its differentials). On the tangent bundle we can choose a Riemann metric – a smooth family of fibre metrics and so shrink the groupoid again to $\underline{\mathcal{C}^\infty}(\mathbb{R}^n; \mathrm{O}(n))$ which we can think of in terms of the principal $\mathrm{O}(n)$-bundle of orthonormal frames in TM. The choice of a Riemann metric also gives a 'nice' covering of M by coordinate systems corresponding to small balls and the exponential map with all (non-trivial) intersections contractible. I will come back to all of this later.

4. Examples

Now, what about really infinite-dimensional manifolds? When the model V is infinite-dimensional, there are many possible choices for $\underline{\mathrm{Dff}}(V)$, many of them quite ugly! I will not try to discuss the zoo of choices, but just mention some important examples. One important Banach manifold is $\mathrm{GL}(H)$ (or $\mathrm{U}(H)$) the space of invertible (or unitary) operators on a separable Hilbert space. The model here is $B(H)$ (or $A(H)$) the bounded (and self-adjoint) operators. An important (and reasonably simple) result is Kuiper's theorem, that $\mathrm{GL}(H)$ is contractible (in the uniform topology). A similar, but 'smoother' manifold is $G^{-\infty}(M)$, the group of invertible smoothing perturbations of the identity acting on functions on a compact finite-dimensional manifold M. The model here is $\mathcal{C}^\infty(M^2)$. This is a Fréchet Lie group, but one where the exponential map is a local diffeomorphism – it is also a classifying group for (odd) K-theory.

The fundamental problem with defining an infinite-dimensional manifold is: What is the regularity we require of the functions on open subsets of the model, V, and what do we require of the transition groupoid which is supposed to preserve this regularity between open sets – and so transfer it to our manifold. The 'obvious' definitions or infinite differentiability of functions and taking the 'maximal groupoid' – which is the direct extension of the finite-dimensional case – do not work at all well. This is the problem. For instance we know almost nothing about the space of homeomorphisms between open subsets of $\mathcal{C}^\infty(\mathbb{R})$ which preserve regularity of function on these sets. And what we do know is not encouraging!

5. Loop spaces

So, we need to be guided by some reasonably sensible examples. The simplest of these really are the loop manifolds. These are the manifolds I want to concentrate on in this course. So, just consider

$$\mathcal{L}M = \mathcal{C}^\infty(\mathrm{U}(1); M)$$

the space of all smooth maps from the circle into M, say a compact finite manifold, of finite dimension. I treat the circle here as the 1-dimensional Lie group for reasons that will become clear. Note that in the very imporant case, that appears often below, that $M = G$ is a Lie group, $\mathcal{L}G$ is also a group under 'pointwise product'.

The model is indeed essentially the smooth functions on a finite-dimensional manifold – in this case it is $\mathcal{C}^\infty(\mathrm{U}(1); \mathbb{R}^n)$ which is just the product of n copies of real-valued functions on U(1). So, what is the structure groupoid? We should perhaps ask what is the precise definition of smooth functions on open subsets of $\mathcal{C}^\infty(\mathrm{U}(1))$, but it is better to postpone that.

One of the first things I will go through carefully is the fact that the structure groupoid of the loop space 'is' meaning it is natural to take it to be, a groupoid which is very like the finite dimensional case. Not some huge thing at all but simply

$$\mathrm{Dff}(\mathcal{L}M) = \mathcal{C}^\infty(\mathrm{U}(1); \underline{\mathrm{Dff}}(\mathbb{R}^n))$$

just the loops in the groupoid for M. Basically this is becase $\mathcal{L}M$ is a space 'associated to M'.

6. Local coordinates

This needs to be properly justified, but let me give an outline of the 'proof' (I have not said exactly what the theorem is of course)! Take a Riemann metric on M and choose $\epsilon > 0$ smaller than the injectivity radius, so all the geodesic balls of this size are really balls. Now, for a loop $l : \mathrm{U}(1) \longrightarrow M$ look at the set of loops

$$N(l; \epsilon) = \{l' : \mathrm{U}(1) \longrightarrow M; d(l'(s), l(s)) < \epsilon \ \forall \ s \in \mathrm{U}(1)\}. \tag{1.6.1}$$

We can use the 'exponential' map of the metric on M to identify $B(l(s), \epsilon)$ with a ball around 0 in the tangent space $T_{l(s)}M$. This gives an identification of $N(l, \epsilon)$ with 'loops in the tangent bundle', meaning sections – with length less than ϵ everywhere – of l^*TM the pull-back to U(1) of the tangent bundle. Well, let's assume that M is orientable. Then this is actually a trivial bundle over the circle and we are really looking at $\mathcal{C}^\infty(\mathrm{U}(1); \mathbb{R}^n)$ as indicated above. What happens when we change base loop and look at the neighbourhood of another? Two $N(l_i, \epsilon)$ $i = 1, 2$ intersect only if there is a loop which is everywhere ϵ close to each – meaning that $d(l_1(s)l_2(s)) < 2\epsilon$ for all $s \in \mathrm{U}(1)$. When this is true the 'coordinate change' is a fibre-preserving map from TM pulled back to one to the pull-back to the other. So, you see it is a loop into the local diffeomorphisms of $\mathbb{R}^n$.

7. Whitehead tower

Now, for the rest of today and probably the next lecture, I want to discuss the question: Why? What makes loop manifolds interesting/important? For us the first reason – and something that I need to discuss in some detail, is because of the difficulties presented by the analytic properties (or lack thereof) of the Whitehead tower. So what is the Whitehead tower? For us it is about successive special properties of manifolds.

Let's go back and talk about what is really interesting, namely finite dimensional manifolds. Let me take the dimension n to be at least 5 so I don't have to keep making qualifying statements.

The tangent bundle of M can be given a Riemann metric, which reduces its 'structure group' to $\mathrm{O}(n)$, the space of orthogonal transformations. This just means that the bundle over M of orthogonal frames in the fibres of TM forms a principal $\mathrm{O}(n)$ bundle over M – the fibres are diffeomorphic to $\mathrm{O}(n)$ and $\mathrm{O}(n)$ acts freely and locally trivially.

The orthogonal group $\mathrm{O}(n)$ has two components, the identity component being $\mathrm{SO}(n)$ for which the first few homotopy groups are

$$\pi_0(\mathrm{O}(n)) = \mathbb{Z}_2,\ \pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2,$$
$$\pi_2(\mathrm{SO}(n) = \{\mathrm{Id}\},\ \pi_3(\mathrm{SO}(n)) = \mathbb{Z},$$
$$\pi_i(\mathrm{SO}(n)) = \{\mathrm{Id}\},\ i = 4,5,\ pi_6(\mathrm{SO}(n)) = \mathbb{Z}.$$

The non-triviality of $\pi_1(\mathrm{SO}(n))$ corresponds to the existence of a non-trivial double cover – the spin group. In fact Whitehead's theorem says that there is a 'tower' of group homomorphisms – in general only topological groups

$$\mathrm{O}(n) \longleftarrow \mathrm{SO}(n) \longleftarrow \mathrm{Spin}(n) \longleftarrow \mathrm{String}(n) \dots \tag{1.7.1}$$

where for each successive group the bottom homotopy group is killed but the higher ones survive unscathed.

The first three groups here are (meaning have realizations as) Lie groups, the last cannot since $\pi_3(G) = \mathbb{Z}$ for any finite-dimensional connected Lie group.

8. Orientation

As I am confident is well-known to you, the first inclusion corresponds to the orienatility of M. Said formally:

When can the structure group of TM be reduced from $\mathrm{O}(n)$ to $\mathrm{SO}(n)$?

In this case we can associate with M another manifold $\mathrm{O}\,M$ which is just M with one of the two possible choices of orientation of T_pM at each point – and hence nearby. This double cover is trivial, has two components, if and only if M is orientable. As is very well known, this can be expressed by saying that M is orientable if and only if $w_1 \in H^1(M;\mathbb{Z}_2)$ the first Stiefel-Whitney class, vanishes.

9. Spin structure

As is also well-known, the second homomorphism in (1.7.1) corresponds to the question of the existence of a *spin structure* on M. Now we assume that M is oriented – connected, orientable and with an orientation chosen and now ask

Can the structure group $\mathrm{SO}(n)$ of TM be lifted to $\mathrm{Spin}(n)$?

This can be made precise by looking at $F_{\mathrm{SO}(n)}M$, the bundle of oriented orthonormal frames of M – by the choice of an orientation the structure group has been reduced from $\mathrm{O}(n)$ to $\mathrm{SO}(n)$ so this is a principal $\mathrm{SO}(n)$-bundle. The precise question then is – does there exists a principal $\mathrm{Spin}(n)$ bundle $F = F_{\mathrm{Spin}}$ (I denote the spin frame bundle just as F since it will appear often in the sequel) with fibre covering map

$F \longrightarrow F_{\mathrm{SO}(n)}$ giving a commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}(n) & \text{———} & F \\ \downarrow & & \downarrow \\ \mathrm{SO}(n) & \text{———} & F_{\mathrm{SO}(n)} \\ & & \downarrow \\ & & M. \end{array}$$

Again the answer is well-known, that this is possible – there exists a spin structure – if and only if $w_2 \in H^2(M;\mathbb{Z}_2)$, the second Stieffel-Whitney class, vanishes. There seems to be a pattern! There is, but it is not as simple as that.

10. Spin Dirac operator

Why should you be interested in spin structures? From the point of view of differential equations (remember this is, at least in principle, a course on differential equations) one important consequence of the existence of a spin structure is the existence of the spin Dirac operator. I will talk about this and the fact that it is elliptic and that its index is the $\hat{A}$-genus of the manifold. This is a number associated to an oriented compact manifold which is generally rational but in the case of a spin manifold is an integer – this was actually known before the index of the Dirac operator was understood by Atiyah and Singer and the existence of the Dirac operator serves as an explanation of the integrality of the genus

$$\hat{A} = \mathrm{ind}(\eth_{\mathrm{Spin}}). \tag{1.10.1}$$

Here the spin Dirac operator is defined by associating a vector bundle to F – using the spin representation of Spin – observing that it inherits the Levi-Civita connection and combining that with Clifford multiplication to get

$$\eth_{\mathrm{Spin}} = \mathrm{cl} \circ \nabla. \tag{1.10.2}$$

11. Spin and loops

What is the relationship between spin structures and $\mathcal{L}M$? The most obvious relationship with loops is that $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$ and this can be constructed as usual for the universal cover using loops. We do something similar for the spin structure.

Now, we are assuming that M is oriented, but *not* that it has a spin structure. The orientation implies that the orthonormal frame bundle is trivial over each loop in M, so the loops in $F_{\mathrm{SO}(n)}$ form a bundle, $\mathcal{L}F_{\mathrm{SO}(n)}$ over $\mathcal{L}M$ which is actually a principal bundle with structure group $\mathcal{L}\,\mathrm{SO}(n)$. This loop group has two components, so there is an orientation question – can the structure group be reduced to the connected component of $\mathcal{L}\,\mathrm{SO}(n)$? The universal cover construction shows that the connected component of $\mathcal{L}\,\mathrm{SO}(n)$ is canonically $\mathcal{L}\,\mathrm{Spin}(n)$ and so there is a natural choice of an 'orientation map' giving a short exact sequence

$$\mathcal{L}\,\mathrm{Spin} \longrightarrow \mathcal{L}\,\mathrm{SO} \longrightarrow \mathbb{Z}_2. \tag{1.11.1}$$

It was observed by Atiyah that if M is spin then $\mathcal{L}M$ is orientable in this sense of having a reduction of the structure group to $\mathcal{L}\,\mathrm{Spin}$. Conversely, it was proved by McLaughlin [**2**] that the orientability of $\mathcal{L}M$ implies that M is spin, provided M

is simply connected. However, in general it is not the case that orientability in the sense of the reduction of the structure group of $\mathcal{L}F_{\mathrm{SO}(n)}$ to the component of the identity is equivalent to the existence of a spin structure on M. One needs another condition on the orientation.

What one needs to add is the idea of *fusion*, introduced at least in this context by Stolz and Teichner; I will talk about this quite a bit. The combination, 'a fusion orientation' of $\mathcal{L}M$ ensures that it does correspond precisely to a spin structure on M. To see where the fusion condition comes from, suppose that M does have a spin structure; we proceed to construct an 'orientation' on $\mathcal{L}M$. Namely, a loop in $\mathcal{L}F_{\mathrm{SO}(n)}$ is 'positively oriented' if it can be covered by a section of $\mathcal{L}F$. This constructs a map [1]

$$o_F : \mathcal{L}F_{\mathrm{SO}(n)} \longrightarrow \mathbb{Z}_2 = \{1, -1\}. \tag{1.11.2}$$

It should be continuous, and that is what an orientation of $\mathcal{L}M$ is – a continuous map (1.11.2) which restricts to a fibre [2] of $\mathcal{L}F_{\mathrm{SO}(n)} \longrightarrow \mathcal{L}M$ to give an orientation of $\mathcal{L}\,\mathrm{SO}$ –meaning (1.11.1) or its opposite.

Instead of thinking about loops, consider paths in M; these are just smooth maps $[0, \pi] \longrightarrow M$. I take this interval because it is 'half a circle' but it really does not matter. In fact we will consider only flat-ended segments, meaning that all derivatives of the path vanish at the end-points, denote the collection of these as $\mathcal{I}M$. Clearly we have a map

$$\mathcal{I}M \longrightarrow M^2 \tag{1.11.3}$$

mapping to the two end-points of a segment. In fact this is a fibre bundle in an appropriate sense and we can consider the fibre-product (pairs of loops with the same ends). From this there is a 'join' map

$$J : \mathcal{I}^{[2]}M \longrightarrow \mathcal{L}M$$

obtained by following the first segment by the reverse of the second which maps into loops. Note that the flatness of the paths at the end-points means that they join up smoothly to a loop – not an arbitrary loop of course because it is 'flat' at the points 1 and -1 on the circle; other than that it is arbitrary.

From $\mathcal{I}M^{[3]}$ – triples of segments all three with the same ends – there are three maps into $\mathcal{L}M$, taking (f_1, f_2, f_3) to $J(f_1, f_2)$, $J(f_2, f_3)$ and $J(f_1, f_3)$ respectively.

This discussion applies to any manifold, so we can consider $\mathcal{I}F_{\mathrm{SO}(n)}$, the flat-ended loops into $F_{\mathrm{SO}(n)}$ with its map to $F^2_{\mathrm{SO}(n)}$. So 'join' becomes a map

$$J : \mathcal{I}^{[2]}F_{\mathrm{SO}(n)} \longrightarrow \mathcal{L}F_{\mathrm{SO}(n)}. \tag{1.11.4}$$

As noted above, a spin structure on M allows us to assign an orientation (1.11.2) to each element of $\mathcal{L}F_{\mathrm{SO}(n)}$. This assignment is by 'holonomy' – lift the intial point into the spin frame bundle and then travel around the curve (there is a unique local lift to F because it is just a double cover) and ask whether you have come back to the same point or to the other lift. The three paths determined by a triple, an element of $\mathcal{I}^{[3]}F_{\mathrm{SO}(n)}$, each have an orientation and the construction by holonomy means that the product of the orientation of two of them is the orientation of the

[1] This is where $F_{\mathrm{SO}(n)}$ became F in v3

[2] If this happens on one fibre it happens on all

third. We can think of this in a fancier way, useful for later generalization, that there are simplicial projection maps

$$\pi_{12},\ \pi_{23},\ \pi_{13} : \mathcal{I}^{[3]}F_{\mathrm{SO}(n)} \longrightarrow \mathcal{I}^{[2]}F_{\mathrm{SO}(n)}, \tag{1.11.5}$$

where π_{ij} drops the missing index. Now the compatibility condition on the orientation is that the product of the three maps

$$(\pi_{12} \circ J)^* o_F \cdot (\pi_{23} \circ J)^* o_F \cdot (\pi_{13} \circ J)^* o_F \equiv 1. \tag{1.11.6}$$

[There should really be an inverse on the third factor but of course here it makes no difference.]

Now, we can say a loop-orientation of $\mathcal{L}M$ – meaning a continuous map (1.11.2) with the right behaviour on each fibre, is 'fusion' if it satisfies (1.11.6).

THEOREM 1 (Stolz and Teichner [**8**], 2005). *There is a 1-1 correspondence between fusion orientations and equivalence classes (up to smooth principal-bundle isomorphism) of spin structures on M and each is classified by $H^1(M;\mathbb{Z})$.*[3]

PROOF. [4] You can of course consult [**8**]. In fact this proof is quite illustrative.

The passage from spin structures to fusion orientations is discussed above, but let me repeat it briefly. The principal Spin(n) bundle, F, associated to a spin structure is a double cover of $F_{\mathrm{SO}(n)}$, the oriented orthonormal frame bundle. So, given a loop $l \in \mathcal{L}F_{\mathrm{SO}(n)}$ we may lift the initial point $l(1)$ to a point $l'(1) \in F$ above it; there are two choices. Once the initial point is chosen there is a unique path in F covering the loop in $F_{\mathrm{SO}(n)}$ and we assign $o(l) = \pm 1$ corresponding to whether the lifted path does, or does not, return to its starting point. It follows that this assignment is independent of the choice of initial point, it is the *holonomy* of the curve corresponding to the $\mathbb{Z}_2$ bundle which is $F \longrightarrow F_{\mathrm{SO}(n)}$. The fact that the fibre F_m above $m \in M$ has a spin action covering the SO(n) action on $(F_{\mathrm{SO}(n)})_m$ means that the map o on loops in a given fibre takes both signs, corresponding to the fact that $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ but that Spin(n) is simply connected. Thus o is an orientation of $\mathcal{L}M$ in the sense of (1.11.2). That the spin structure leads to a *fusion* loop-orientation follows from the definition of fusion. Namely at any point in $\mathcal{I}^{[3]}F_{\mathrm{SO}(n)}$ the three loops in (1.11.6) come from three paths i_1 with the same endpoints. The orientation of the first loop is the holonomy along i_1 followed by the reverse of i_2, the orientation of the second is obtained by going along i_2 and then i_3 reversed. We can certainly use as initial point in the second case the end-point of the lift used in the first case. Then the holonomy along the 'fused' loop is obtained by lifting above i_1 and then i_3 in reverse. However, adding the detour along i_2 and then reversed along i_2 does not do anything so the holonomy along the fusion is indeed the product which gives (1.11.6).

Now, to go in the opposite direction, i.e. to construct a spin structure from a fusion loop-orientation, choose a base point in $F_{\mathrm{SO}(n)}$ and consider the 'pointed paths' $\dot{\mathcal{I}}F_{\mathrm{SO}(n)} \subset \mathcal{I}F_{\mathrm{SO}(n)}$, just those flat-ended paths which start at the base point. Evaluating at π gives a map $\dot{\mathcal{I}}F_{\mathrm{SO}(n)} \longrightarrow F_{\mathrm{SO}(n)}$ which is surjective by the connectedness of M and SO(n).

Now, look at the product $\dot{\mathcal{I}}F_{\mathrm{SO}(n)} \times \mathbb{Z}_2$ and define a relation on it:

$$(i,s) \sim (i',s') \Longleftrightarrow i(\pi) = i'(\pi),\ o(J(i,i'))s' = s. \tag{1.11.7}$$

[3] Also a typo here in v3

[4] There were missing SO(n) subscripts in v3

The fusion condition is exactly what is needed to see that this is an equivalence relation. First, we need to know that $(i,s) \sim (i,s)$ which reduces to $o(J(i,i)) = 1$. That is, the orientation of a 'there-and-back' loop is $+1$. This follows from (1.11.6) since $(o(J(i,i)))^3 = 1$. Similarly symmetry reduces to $o(J(i,i')) = o(J(i',i))$. Again this follows from (1.11.6) applied to the three paths i, i, i' since

$$o(J(i,i))o(J(i,i')) = o(J(i',i)). \tag{1.11.8}$$

The transitivity of $\sim$ is the full fusion condition.

Now, the quotient $F = \dot{\mathcal{I}}F_{\mathrm{SO}(n)} \times \mathbb{Z}_2/\sim$ is a double cover $F \longrightarrow F_{\mathrm{SO}(n)}$ since locally near a point of $F_{\mathrm{SO}(n)}$ it is just the product of a neighbourhood with $\mathbb{Z}_2$. In fact this construction applied to $\mathrm{SO}(n)$ is one of the standard constructions of $\mathrm{Spin}(n)$.

Note that there is a slightly sticky issue here, that the loop-orientation construction applied to the spin structure just associated to a fusion loop-orientation should reconstruct the original loop-orientation. This is certainly true on loops obtained by joining two flat-ended paths since a path in the new F is really just a path in $F_{\mathrm{SO}(n)}$ with some choice of $\pm \in \mathbb{Z}_2$. However, an orientation is determined by its restriction to the image of the joint map. This is a strengthened form of the independence of the parameterization and we need to deal with it seriously as 'flattening' below.

Note that we are also using the fact that the holonomy is unchanged under a principal bundle isomorphism but this is clear from the definition. The last statement in the theorem, follows from the fact that spin structures are so classified. Namely a spin structure is a $\mathbb{Z}_2$ bundle over $F_{\mathrm{SO}(n)}$ which encodes the $\mathrm{Spin}(n)$ action. It follows that the tensor product of these two $\mathbb{Z}_2$ bundles is the pull-back of a $\mathbb{Z}_2$ bundle from M under the projection and conversely. That $H^1(M;\mathbb{Z}_2)$ may be identified with the equivalence classes of $\mathbb{Z}_2$ bundles over M is standard. $\square$

It is worth thinking a little about the last part of this proof to see how the classification of loop-orientation structures by $H^1(M;\mathbb{Z})$ arises directly. Namely for two loop-orientations the product is the pull-back of a continuous map

$$\mathcal{L}M \longrightarrow \mathbb{Z}_2 \tag{1.11.9}$$

under the projection map $\mathcal{L}F_{\mathrm{SO}(n)} \longmapsto \mathcal{L}M$ since it is constant on the fibres. This map also has the fusion property for loops on M since this follows by looking at lifts to paths and loops in $F_{\mathrm{SO}(n)}$ for loop-orientations – i.e. follows by taking the product of the identity (1.11.6) for the two loop-orientations (so you need to check that each element of $\mathcal{L}^{[3]}M$ does have a lift to $\mathcal{L}^{[3]}F_{\mathrm{SO}(n)}$. So what this comes down to is the identification

$$H^1(M;\mathbb{Z}_2) = \{\text{Fusion maps (1.11.9)}\}. \tag{1.11.10}$$

Exercise 1. Check (1.11.10) – it is a simpler version of the proof above.

This is all part of the general principal that 'transgression' and 'regression' which are maps from objects on M to objects on $\mathcal{L}M$ and conversely become isomorphisms (or functors) provided the correct 'fusion' condition is added on the loop side. You might, by the way, complain that the notation is messed up and that (1.11.10) should really be the space of 'loop-orientations' on $\mathcal{L}M$ (or M) but it is too late to try to reverse history.

12. Regularity

Let me just note some things about Theorem 1. The fact that we can away without having to worry about regularity is due to the discreteness of $\mathbb{Z}_2$. A map into $\mathbb{Z}_2$ is as smooth as it can be as soon as it is continuous – which is to say it is locally constant. The same applies to the spin structure, the bundle F when it exists has the same regularity as $F_{\mathrm{SO}(n)}$ since it is locally the same. When we go to more serious questions we will have to tackle regularity head on.

Just think for a moment what a continuously differentiable function on some Fréchet manifold modelled on say $\mathcal{C}^\infty(M)$ for a smooth manifold M should be. There are different notions of derivative but in any case such a function should have a derivative at each point. What should that be? The minimal condition would that it be a linear function on the tangent space at each point. However, the tangent space – given the patching definition outlined briefly above – would usually be interpreted as essentially the model space, $\mathcal{C}^\infty(M)$, although not canonically. Still, this means the derivative should be a functional on $\mathcal{C}^\infty(M)$ which is to say a distribution – given that it is continous which we would surely want. So, hidden in a continously differentiable function is a distribution at each point. This is a bit of a problem as soon as we try to do something as is implicit in (1.10.2) which requires some sort of multiplication operation.

Of course there is an implicit bias at work here, 'preferring' the tangent to the cotangent space. It would be reasonable to demand (and we shall) much more regularity than this and claim that the derivative should itself be (or if you want to think in terms of distributions, be given by) an element of $\mathcal{C}^\infty(M)$. Such a function would be 'very smooth'. Still, in order for this to make sense we have to make sure that such notions, and corresponding issues for higher derivatives, transform correctly under the transformation groupoid.

13. Reparameterization

Another property of the holonomy definition of the orientation map on $\mathcal{L}F_{\mathrm{SO}(n)}$ defined from a spin structure is that it is independent of the parameterization of a loop. This is easy to see from the definition and suggests what is natural anyway, that the 'best' objects on a loop space will be independent (in some sense) of the parameterization of the curves. In fact the group $\mathrm{Dff}^+(\mathrm{U}(1))$ of oriented diffeomorphisms of the circle (or the unoriented ones for that matter) act on say $\mathcal{L}M$ by reparameterizing loops and we can therefore think about the quotient

$$\mathcal{L}M/\,\mathrm{Dff}^+(\mathrm{U}(1)). \tag{1.13.1}$$

The problem is that this is quite singular, since for instance the constant loops are fixed points for the action which is very much non-free. Still, it is natural to look for invariance, or equivariance, under this action as we certainly have in the case of the orientation.

There is however a tension between reparameterization and fusion, both of which are clearly important. Namely, the fusion operation from paths to loops only really makes sense if we have flat-ended paths so that the resulting loops are smooth. One can go along way with piecewise-smooth loops, which is what you get by joining smooth but not flat-ended paths with the same ends, but then the topology is going to get out of hand since one needs to allow the breaks to occur anywhere.

These are the sort of issues that I hope to nail down properly. They do not arise above, but they immediately come up below and that is one reason these problems have remained open for quite a long time.

14. String structures

Now, on to the main topic of the first part of these lectures. Namely the next structure in the Whitehead tower, the notion of a *string structure* on M. That is, when does a manifold with a spin structure have a $\mathrm{String}(n)$ principal bundle covering the $\mathrm{Spin}(n)$ bundle:-

$$\begin{array}{ccc} \mathrm{String}(n) & \text{——} & F_{\mathrm{String}} \\ \downarrow & & \downarrow \\ \mathrm{Spin}(n) & \text{——} & F \\ & & \downarrow \\ & & M. \end{array} \tag{1.14.1}$$

The existence of a decent model for $\mathrm{String}(n)$ is not so trivial.

In fact the answer is also well-established. The spin structure defines a characteristic class in $H^4(M;\mathbb{Z})$ twice which is the usual p_1, the first Pontryagin class. So it is generally denoted $\frac{1}{2}p_1$ but it is integral. The statement here depends a bit on finding a decent model for String so I will not go into it at precisely at this stage but a lift (1.14.1) exists if and only if $\frac{1}{2}p_1 = 0$. Note that the pattern

Orientation — Spin — String — (Fivebrane)

does continue in the sense that the extension to successive groups in the Whitehead tower is obstructed by a cohomology class, not the class involved in the third step (and in the next too) is an integral cohomology class, not a $\mathbb{Z}_2$ class.

At this point you might well have a couple of big questions. Why would we care about string structures? And in any case, what have they to do with loops? What is this thing that comes next? I will not get into the next step!

From looping behaviour,

$$\pi_2(\mathcal{L}\,\mathrm{Spin}(n)) = \pi_3(\mathrm{Spin}(n)) = \mathbb{Z}.$$

In the absence of other groups nearby this implies that $H^2(\mathcal{L}\,\mathrm{Spin}(n)) = \mathbb{Z}$, so there is a line bundle over $\mathcal{L}\,\mathrm{Spin}(n)$. This line bundle is in fact 'primitive', i.e. corresponds to a central extension by a circle

$$\mathrm{U}(1) \longrightarrow E\mathcal{L}\,\mathrm{Spin}(n) \longrightarrow \mathcal{L}\,\mathrm{Spin}(n).$$

At the level of Lie algebras this is the Kac-Moody extension.

15. Loop-spin structures

The existence of a string structure on M is then related to the existence of a 'loop-spin' structure on $\mathcal{L}M$ in the sense of a covering of the $\mathcal{L}\,\mathrm{Spin}(n)$ principal bundle by an $E\mathcal{L}\operatorname{Spin}(n)$-principal bundle

(1.15.1)
$$\begin{array}{ccc} E\mathcal{L}\operatorname{Spin}(n) & \text{———} & D\mathcal{L}F \\ \downarrow & & \downarrow \\ \mathcal{L}\operatorname{Spin}(n) & \text{———} & \mathcal{L}F \\ & & \downarrow \\ & & M, \end{array}$$

where D is an 'appropriate' circle bundle over $\mathcal{L}F$.

The situation is similar to spin and loop-orientation above. There is an obstruction (Dixmier-Douady) class in $H^3(\mathcal{L}M;\mathbb{Z})$ to the existence of an extension principal bundle. Recently Waldorf has shown that there is a notion of 'fusion' structure for a loop-spin structure and the existence of such a structure is equivalent to the existence of a string structure. With any luck (i.e. it does not blow up in the writing) Chris Kottke and I have shown that there is a 1-1 correspondence between string and fusive (this is a strengthening of the fusion condition that we will get to) loop-spin structures up to natural equivalence; these are both classified by $H^3(M;\mathbb{Z})$. Moreover there are analogues in this case of the direct constructions outlined above.

16. Bundle gerbes

What more do we need to carry through this construction? One thing we will use is a geometci realization of 3-dimensional integral cohomology, in the form of bundle gerbes [**3**]. I will develop the theory of these as needed. In particular they apply directly to analyse extensions of principal bundles corresponding to central extensions of their structure groups as in (1.15.1). In fact we need some sort of geometric realization of 4-dimensional integral cohomology, to capture the obstructure class $\frac{1}{2}p_1$. These are '2-gerbes' in this case the Brylinski-McLaughlin bundle 2-gerbe.

17. Witten genus

So, why is this interesting? The basic 'claim' is that analysis is much easier on the loop-spin side of this correspondence than on the string side. In particular, Witten has given a Physical discussion of the index of a differential operator, the Dirac-Raymond operator, on $\mathcal{L}M$, which is associated to the loop-spin bundle. This index is not a number, but is rather a formal power series with integer coefficients – the Witten genus. Again this is analogous to the $\hat{A}$ genus, with integrality of the coefficients a consequence of the existence of a spin structure. To discuss all this properly we would need an analytic-geometric theory of elliptic cohomology, in which the Witten genus resides. I can hope to do this during the semester, but I do not know how to do it now!

18. In the sky

What else would I *like* to (be able to) do? Full analysis of the Dirac-Ramond operator to derive the Witten genus as the equivariant index. Discuss the relationship to quantum field theory and topological quantum field theory. Give a geometric realization of elliptic cohomology and topological modular forms. Don't hold your breath on this.