APPENDIX A

Finite-dimensional manifolds

First let me recall basic facts about compact manifolds, mainly to set up notation before specializing to the circle. Some of this I do not really need, but it seems a good idea to put things in context. In fact as you can see I got rather carried away and wrote down almost everything I could think of which might be relevant. Don't worry if you do not get ALL of this, since I generally do not need this much but if you want me to explain any of it a little more I am happy to do so. I do not plan to lecture on most of what is in this first section but am open to counter-proposals. In Lecture 3 I plan to start directly with §1.

Perhaps the most basic object associated with a compact $\mathcal{C}^\infty$ manifold, M, is the space $\mathcal{C}^\infty(M)$ of (real- or complex-valued, if necessary I can use the notation $\mathcal{C}^\infty(M;\mathbb{R})$, $\mathcal{C}^\infty(M;\mathbb{C})$) functions on M. From this one can construct the tangent and cotangent bundles, TM, T^*M – for instance the fibre of T^*M at $m \in M$ can be identified with $\mathcal{I}_m/\mathcal{I}_m^2$, $\mathcal{I}_m \subset \mathcal{C}^\infty(M)$ being the ideal of functions vanishing at m and $\mathcal{I}_M^2$ being the finite span of at two factors from $\mathcal{I}_m$.

I assume we are familiar with the notion of a vector bundle, V, real or complex, and of the operations on them giving the dual, V^*, the tensor product of two $V \otimes W$, the bundle of fibre homomorphisms from V to W, $\hom(V,W)$, that this is canonically $W \otimes V^*$, exterior products as antisymmetric parts of tensor products etc.

There is one construction which is maybe slightly less familiar than these standard ones, and which only works for rank one real bundles. Namely, if L is such a real line bundle then it can be identified (as in general) with $(L^*)^*$, so elements of L_m are linear maps

$$l : (L^*)_m \longrightarrow \mathbb{R}. \tag{A.0.1}$$

Instead one can consider maps which are absolutely homogeneous of any given degree $a \in \mathbb{R}$:

$$w : (L^*)_m \setminus \{0\} \longrightarrow \mathbb{R},\ w(s\mu) = |s|^a w(\mu),\ \forall\ s \in \mathbb{R} \setminus \{0\},\ \mu \in (L^*)_m \setminus \{0\}. \tag{A.0.2}$$

Clearly w is determined by its value at any one point and moreover the space of such w is actually linear and extends to give a smooth bundle. One could denote this bundle, somewhat confusingly, as $|L|^a$. It is *always trivial* so nothing much is going on with this construction, but note that $|L|^a \otimes |L|^b = |L|^{a+b}$ canonically and $|L|^0$ is canonically the trivial bundle. The notation comes from the fact that if $l \in \mathcal{C}^\infty(M;L)$ is a smooth section then $|l|^a \in \mathcal{C}^0(M;|L|^a)$ is a well-defined constinuous section. However the bundle $|L|^a$ always has a global smooth positive section.

The reason for interest is that we wish to set $\Omega^a(M) = |\Lambda^{\dim M} M|^a$ and call this the bundle of a-densities on M. Then the Riemann integral is well defined

$$\int : \mathcal{C}^0(M;\Omega) \longrightarrow \mathbb{R},\ \Omega M = \Omega^1 M. \tag{A.0.3}$$

These are the objects that can be integrated. One version of an orientation of M is that it is an isomorphisms $o : \Lambda^{\dim M} M \longrightarrow \Omega M$ such that any positive global section of the density bundle is the absolute value of its pull-back.

To be definite I will take $TM = (T^*M)^*$ to be the dual bundle of T^*M. The fibre $T_m M$ can also be identified with the space of derivations on $\mathcal{C}^\infty(M)$ at m. A Riemann metric on M is a smooth positive-definite quadratic form on the fibres of TM, so in particular a section of the symmetric part of $T^*M \otimes T^*M$. Any vector bundle (over a compact manifold) can be embedded in a trivial bundle.

The topology on $\mathcal{C}^\infty(M)$ is the Fréchet topology given by all the $\mathcal{C}^k$ semi-norms on compact subsets of coordinate patches, or equivalently the Sobolev norms on balls. It is countably normed and a Montel space – there is a sequence of norms $\|\cdot\|_k$, $k \in \mathbb{N}$, giving the topology such that a bounded set with respect to $\|\cdot\|_{k+1}$ is precompact with respect to $\|\cdot\|_k$. The space $\mathcal{C}^\infty(M;V)$ of sections of a vector bundle has a similar topology which is the same as the product topology in the trivial case and consistent with embedding.

Now, once we have introduced the density bundle we can recall the standard notation for distributions. Namely 'distributional functions' form the dual space

$$\begin{gathered} \mathcal{C}^{-\infty}(M) = (\mathcal{C}^\infty(M;\Omega))' \text{ or more generally} \\ \mathcal{C}^{-\infty}(M;V) = (\mathcal{C}^\infty(M;\Omega \otimes V^*))'. \end{gathered} \tag{A.0.4}$$

This is done so that the smooth sections map naturally into the 'distributional sections' (not quite sections of course)

$$\mathcal{C}^\infty(M;V) \longrightarrow \mathcal{C}^{-\infty}(M;V),\ \phi \longmapsto i(\phi) \in (\mathcal{C}^\infty(M;\Omega \otimes V^*))',\ i(\phi)(\psi) = \int \langle \phi, \psi \langle \tag{A.0.5}$$

since the pointwise pairing of $\phi \in \mathcal{C}^\infty(M;V)$ and $\psi \in \mathcal{C}^\infty(M;\Omega \otimes V^*)$ gives a section of the density bundle.

The space of smooth vector fields

$$\mathcal{V}(M) = \{V : M \longrightarrow TM; V(m) \in T_m M\} \tag{A.0.6}$$

is a Lie algebra and its universal enveloping algebra consists of the (smooth, linear) differential operators on $\mathcal{C}^\infty(M)$. There is a corresponding space of differential operators, $\mathrm{Diff}^k(M;V,W)$, between the sections of any two smooth (finite-dimensional) vector bundles V and W over M with $\mathrm{Diff}^0(M;V,W) = \hom(V,W)$. Note that this is not a vector bundle over M (if $k > 0$) because the transition maps are not bundle maps.

Although seldom really used it is good to know the basic theorems about differential operators and distributions. Namely the embedding (A.0.5) of smooth functions into distributins extends to map $L^1(M;V)$ injectively into $\mathcal{C}^{-\infty}(M;V)$ (and we regard this injection as an identification). In particular $\mathcal{C}^0(M;V) \longrightarrow L^2(M;V) \longrightarrow L^1(M;V)$ are all identified with subspaces of distributions. Schwartz' representation theorem gives a partial inverse of this. Namely the action of smooth differential operators extends uniquely

$$\begin{array}{ccc} \mathcal{C}^\infty(M;V) & \xrightarrow{P} & \mathcal{C}^\infty(M;W) \\ \big\downarrow & & \big\downarrow \\ \mathcal{C}^{-\infty}(M;V) & \xrightarrow[P]{} & \mathcal{C}^{-\infty}(M;W) \end{array} \quad \forall\ P \in \operatorname{Diff}^k(M;V,W). \tag{A.0.7}$$

Then for any $u \in \mathcal{C}^{-\infty}(M;V)$ there exists $P \in \operatorname{Diff}^k(M;V) = \operatorname{Diff}^k(M;V,V)$ (where k depends on u) such that $u = Pv$, $v \in L^2(M;V)$. Pushing this a bit further one can define global Sobolev spaces so that

$$\begin{gathered} \mathcal{C}^{-\infty}(M;V) = \bigcup_{m\in\mathbb{R}} H^m(M;V), \ \bigcap_{m\in\mathbb{R}} H^m(M;V) = \mathcal{C}^\infty(M;V), \\ P \in \operatorname{Diff}^k(M;V,W) \Longrightarrow P : H^m(M;V) \longrightarrow H^{m-k}(M;W). \end{gathered} \tag{A.0.8}$$

The other big, related, theorem is the Schwartz kernel theorem. It can be interpreted in terms of completion of tensor products (and spawned a big industry in the 1960s along these lines). To state it we need to give $\mathcal{C}^{-\infty}(M;V)$ a topology – the weak (or is it weak*) topology is given by the seminorms $|u(\cdot)|$ for $u \in \mathcal{C}^\infty(Ml\Omega \otimes V^*)$ acting through the duality pairing. There are other topologies but this is okay for this purpose. Then we know what a continuous linear map

$$Q : \mathcal{C}^\infty(M;V) \longrightarrow \mathcal{C}^{-\infty}(M;W) \tag{A.0.9}$$

is. Schwartz' kernel theorem says there is a bijection (tological too) between such continuous linear 'operators' and $\mathcal{C}^{-\infty}(M^2; \operatorname{Hom}(W,V) \otimes \pi_R^*\Omega)$. Here, π_R is the projection onto the right factor of M and $\operatorname{Hom}(V,W)$ is the two-point homorphism bundle, with fibre over $(m,m') \in M^2$ the space $\hom(V_{m'}, W_m)$. The map Q assocatiated to $\tilde{Q} \in \mathcal{C}^{-\infty}(M^2; \operatorname{Hom}(W,V) \otimes \pi_R^*\Omega)$ is determined by the condition

$$Q(v)(w) = \tilde{Q}(w \boxtimes v) \tag{A.0.10}$$

where you need to sort out the pairing over M^2 on the right to get the pairing over M on the left.

One class of operators we will consider (mostly over the circle as with everything else) are the smoothing operators. These correspond to the subspace

$$\mathcal{C}^\infty(M^2; \operatorname{Hom}(W,V) \otimes \pi_R^*\Omega) \subset \mathcal{C}^{-\infty}(M^2; \operatorname{Hom}(W,V) \otimes \pi_R^*\Omega) \tag{A.0.11}$$

in Schwartz' theorem and form an algebra. The action then is really give by an integral so that $\tilde{Q} \in \mathcal{C}^\infty(M^2; \operatorname{Hom}(W,V) \otimes \pi_R^*\Omega)$ defines

$$Q : \mathcal{C}^\infty(M;V) \longrightarrow \mathcal{C}^\infty(M;W),\ (Qv)(m) = \int_M \langle \tilde{Q}(m,\cdot), v(\cdot)\rangle \tag{A.0.12}$$

where again the pairing leading to a density needs to be sorted out. I will denote the algebra of smoothing operators on V by $\Psi^{-\infty}(M;V)$ for reasons that will become clear below – the module of smoothing operators between sections of different bundles will be denoted $\Psi^{-\infty}(M;V,W)$.

One important (fairly elementary) result is that the group of operators

$$\begin{aligned} G^{-\infty}(M;V) = \big\{ \operatorname{Id} + A,\ A \in \Psi^{-\infty}(M;V); \\ \exists\ B \in \Psi^{-\infty}(M;V),\ (\operatorname{Id} + A)(\operatorname{Id} + B) = \operatorname{Id} \big\} \end{aligned} \tag{A.0.13}$$

is a classifying space for odd K-theory. For our purposes it is more important for the moment that it carries the Fredholm determinant. In fact this is a well-defined (entire analytic) function on $\Psi^{-\infty}(M;V)$, which we write as $\det_{\mathrm{Fr}}(\mathrm{Id}+A)$, $A\in\Psi^{-\infty}(M;V)$, and which has many of the properties of the finite-dimensional determinant so

$$\begin{gathered}\mathrm{Id}+A\in G^{-\infty}(M;V)\Longleftrightarrow \det_{\mathrm{Fr}}(\mathrm{Id}+A)\neq 0,\\ \det_{\mathrm{Fr}}\left((\mathrm{Id}+A)(\mathrm{Id}+B)\right)=\det_{\mathrm{Fr}}(\mathrm{Id}+A)\det_{\mathrm{Fr}}(\mathrm{Id}+B),\\ \frac{d}{ds}\det_{\mathrm{Fr}}(\mathrm{Id}+sA)=\det_{\mathrm{Fr}}(\mathrm{Id}+sA)\,\mathrm{Tr}(A)\end{gathered}\tag{A.0.14}$$

where the trace functional $\mathrm{Tr}:\Psi^{-\infty}(M;V)\longrightarrow\mathbb{C}$ is the unique continuous linear map vanishing on commutators and such that on finite rank projections (or idempotents) in $\Psi^{-\infty}(M;V)$ it reduces to the rank.

The relationship between $\det_{\mathrm{Fr}}$ and the K-theoretic statement above is that $\frac{1}{2\pi i}\det_{\mathrm{Fr}}:G^{-\infty}(M;V)\longrightarrow\mathbb{C}^*$ generates the 1-dimensional integral cohomology of $G^{-\infty}(M;V)$ and this is the bottom part of the odd Chern character – the rest can be written down similarly. Remember that one of the things that this course is at least related to is 'smooth cohomology' and this is somewhat epitomized by $G^{-\infty}(M;V)$ which carries smooth universal odd Chern classes. The even version is not much more complicated.

Although we will not need (at least I don't think it will come up) the general case, it seems appropriate to understand a little about the space of pseudodifferential operators over M. These are operators like the smoothing operators in that they map $\mathcal{C}^\infty(M;V)$ to $\mathcal{C}^\infty(M;W)$ for any two vector bundles. In fact it is useful to consider (classical) pseudodifferential operators of complex order $\Psi^z(M;V,W)$. These compose sensibly

$$\Psi^z(M;V_2,V_3)\circ\Psi^{z'}(M;V_1,V_2)=\Psi^{z+z'}(M;V_1,V_3)\tag{A.0.15}$$

and have lots of other properties too. The main point though is that they, as a space of operators, are much more like $\mathcal{C}^\infty$ rather than $\mathcal{C}^{-\infty}$.

More precisely, there is a symbol map giving a short exact sequence for any z :

$$\Psi^{z-1}(M;V,W)\longrightarrow\Psi^z(M;V,W)\longrightarrow\mathcal{C}^\infty(S^*M;\pi^*\hom(V,W)\otimes N^z).\tag{A.0.16}$$

Here $S^*M=(T^*M\setminus O_M)/\mathbb{R}^+$ is the cosphere bundle of M and N^z is the bundle with sections over S^*M which are functions on $T^*M\setminus O_M$ which are positively (not absolutely) homogeneous of degree z. So this is a trivial bundle with section given by a metric for instance, $|\xi|^z$.

The main point about this sequence, which will show up mostly when $z\in\mathbb{Z}$, is that we can 'iterate' it and the notation is consistent

$$\Psi^{k-1}(M;V,W)\subset\Psi^k(M;V,W)\tag{A.0.17}$$

is the subspace of operators with vanishing symbol of order k and

$$\Psi^{-\infty}(M;V,W)=\bigcap_{k\in\mathbb{Z}}\Psi^k(M;V,W),\tag{A.0.18}$$

the 'residual' space is indeed the space of smoothing operators.

The pseudodifferential operators can be characterized quite explicitly in terms of their Schwartz kernels, and I will talk more about this in the case of the circle. However, in brief, the Schwartz kernels of the elements of say $\Psi^k(M;V,W)$ are sections over M^2 of the appropriate bundle $\operatorname{Hom}(V,W)\otimes\pi_R^*\Omega$ with the following special properties

(1) The kernels are smooth away from the diagonal
(2) A neighbourhood of the diagonal is diffeomorphic to a neighbourhood of the zero section of TM and the bundles to the pull-backs of their restrictions to the diagonal, which is identified with the zero section so trivial on the fibres. Cutting off the kernels by a smooth function of compact support in the neighourhood the Fourier t.ransform in the fibre directions reduces the kernels to 'Laurent' sections of the bundles over $\overline{T^*M}$ – the radial compactification of the cotangent bundle to a (closed) ball bundle. This means that these Fourier transforms are precisely of the form $|\xi|^{-z}a$ where a is a smooth section of the bundle including up to the boundary of the ball. The symbol is (taking care of densities correctly) the restriction of a to 'infinity'.

The differential operators are the subspace of the pseudodifferential operators with Schwartz kernels supported in the diagonal (this is 'locality' of differential operators). Their symbols are homogeneous polynomial sections of the appropriate bundle on T^*M.

Now, I went as far as including classical pseudodifferential operators of complex order so as to be able to describe the residue trace introduced by Wodzicki and later by Guillemin. One can actually find an entire family

$$R(z)\in\Psi^z(M;V) \tag{A.0.19}$$

which are invertible, with $R(-z)$ the inverse of $R(z)$. In fact Seeley did this, just find a positive-definite self-adjoint differential operator, P, of order 2 on sections of V with respect to some Hermitian inner product – such a thing is like a quantize metric and always exists (nothing very natural about the choice). Then the complex powers exists and $R(z)=P^{z/2}$ is of the form (A.0.19).

Now, if we take an element $A\in\Psi^k(M;V)$ of *integral* order we can form

$$R(z)A\in\Psi^{z+k}(M;V). \tag{A.0.20}$$

The trace functional $\operatorname{Tr}(A)$ discussed above on smoothing operators actually extends by continuity to $\Psi^z(M;V)$ provided $z<-\dim M$ – the elements of this space are trace class operators on $L^2(M;V)$. Seeley already observed that the trace has an analytic extension so one can say

(A.0.21)

$\operatorname{Tr}(R(z)A)$ is meromorphic with poles only at $z=-\dim M-k-j,\ j\in\mathbb{Z},\ j\geq 0.$

The first poles just corresponds to the point where the operator stops being trace class (assuming its symbol doesn't vanish). What we are particularly interested in is the pole at $z=0$ which might occur by (A.0.21) if $k\geq-\dim M$.

The pole at $z=0$ is always simple and the *residue*

$$\operatorname{Tr}_{\mathrm{R}}(A)=\lim_{z\to 0}z\operatorname{Tr}(R(z)A) \tag{A.0.22}$$

is called the residue trace. It does not depend on the choice of R with the properties above – not so surprising since $R(0)=\operatorname{Id}$. By construction it vanishes on $\Psi^{-\dim M-1}(M;V)$. It can be explicitly computed as

$$\operatorname{Tr}_{\mathrm{R}}(A)=c\int_{S^*M}\sigma_{-\dim M}(A),\ A\in\Psi^{-\dim M}(M;V) \tag{A.0.23}$$

and really for all the integral-order pseudodifferential operators.

The residue trace *is* a trace. We will also be interested in the functional, the 'regularized trace'

$$\mathrm{Tr}_R(A) = \lim_{z\to 0}(\mathrm{Tr}(R(z)A) - \mathrm{Tr}_{\mathrm{R}}(A)/z) \tag{A.0.24}$$

which does depend on R and which is not a trace, but does restrict to Tr on operators of order $< -\dim M$.

We will use the regularized trace later (for the circle) to define connections on bundles.