

## Lecture XVI

I had planned to go through the proof of the signature formula on an  $4k$ -dimensional manifold without boundary mentioned

$$(L) \quad \text{sgn}(x) = \int_x^L$$

and then discuss its generalization to the case  $\partial x \neq \emptyset$ :

$$(B) \quad \text{sgn}(x) = \int_x^L -\eta$$

due to Atiyah, Patodi & Singer. However, I am running short of time to go through the fairly extensive combinatorics behind (L). Instead I will concentrate mainly on the signature defect. I want to do this by going through Kirby!

Let me go back to the index in the general case of elliptic operators with local factors. Recall that  $A \in \Psi^{-m}(X; E, F)$  is elliptic if its principal symbol  $\sigma_m(A) \in C^\infty(SX; N_{\text{ell}}^{m+1})$  is elliptic if its principal symbol  $\sigma_m(A) \in C^\infty(SX; N_{\text{ell}}^{m+1})$  is elliptic. Then we showed that it was Fredholm with compact kernel. Then we showed that it was Fredholm with compact kernel. Let this  $AB - T_d$  and  $BA - T_d$  be smoothing operators. Let this  $AB - T_d$  and  $BA - T_d$  be smoothing operators. Let this  $I$  show that

$$\text{ind}(A) = \dim \ker(A) - \dim \ker(A^*) = \text{Tr}([B, A]).$$

We can deduce quite a few things from this formula.

Exer Show that if  $K: [0,1] \ni t \mapsto \Psi^t(X; E, F)$  is a continuous family of elliptic operators then it has constant index.

Suppose for the sake of definiteness that  $\text{ind}(A) \geq 0$ .

Since  $\text{dim null}(A) \geq \text{dim null}(A^*)$  as well as a finite dimensional complement to  $\text{Ran}(A)$ , we can find  $R \in \Psi^{-\infty}(X; E, F)$ , first order, such that  $A+R$  is surjective. In fact we can do a little more.

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Lemma Given a subspace  $\mathcal{S} \subset \Omega^k C^\infty(X; E)$  of dimension equal to the index of  $A \in \Psi^m(X; E, F)$  elliptic,  $\exists R \in \Psi^{-\infty}(X; E, F)$  such that  
 $\text{null}(A+R) = N$ .

Proof. For clarity, choose an hermitian fiber metric on  $E$  and a positive smooth density on  $X$  so that the  $L^2$  norm is defined by

$$\int_X |u|^2 v \quad u \in C^\infty(X; E).$$

Then choose orthonormal basis  $e_1, \dots, e_I$  of  $N$

and  $e_1, \dots, e_k, f_{k+1}, \dots, f_I$  of  $\text{null}(A)$  where

$k = \dim(N \cap \text{null}(A))$  (is most likely zero). Then define  $R$  to be the smoothing operator with kernel

$$R = \sum_{i>k} A e_i \cdot (f_i - e_i)^* \quad \in C^\infty(X; W_{\text{loc}}(E, F) \otimes \mathcal{D}_F^*)$$

where  $\cdot^*$  is the action of  $\sqrt{\Delta}$  through the inner prod.

then  $R$  is the action of  $\sqrt{\Delta}$  on the null space  $N$ .  $\square$

Now, with  $N$  fixed look at all such partition

$$\mathcal{P}_{A, \bar{N}} = \left\{ R \in \Psi^{-\infty}(X, F, E) ; A + R \text{ is surjective} \right. \\ \left. \text{on null space } N \right\}.$$

The lemma shows that  $\mathcal{P}_A \neq \emptyset$ . If  $A' \in \mathcal{P}_A$  then  $A'$  has a generalized inverse  $B$  which is injective & has range a complement to  $N$ ;  $B \in \Psi^{-\infty}(X, F, E)$ .

Lemma To fix  $N \subset C^\infty(X; E)$  with  $\text{ind}(N) = \text{ind}(A)$ ,  $A \in \Psi^{-\infty}(X; F, E)$  sufficient,

$$(P) \quad \mathcal{P}_{A, N} \cong \left\{ E \in \Psi^{-\infty}(X; F) ; (T_A + E)^{-1} \in \mathcal{I}_d + \mathcal{O}^{-\infty}(X; F) \right\}$$

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Proof Choose  $A' \in P_{A,N}$  with generalized inverse  $B$  as described above. Then for a general element  $\tilde{A} \in \tilde{P}_{A,N}$ ,

$$\tilde{A} \cdot B = \text{Id} + \tilde{E}, \quad \tilde{E} \in \tilde{\Psi}^{-\infty}(X; F)$$

in particular. Indeed,  $\tilde{A} = A' + R$  for some  $R \in \tilde{\Psi}^{-\infty}(X; F)$  so  $\tilde{A} \cdot B = A' \cdot B + R \cdot B = \text{Id} + \tilde{E}$ ,  $\tilde{E} \in \tilde{\Psi}^{-\infty}(X; F)$  as  $\tilde{A} \cdot B u = 0$  with  $Bu \in N$ , but  $N \cap \text{Ran}(B) = \{0\}$  so  $u = 0$  and  $\text{Id} + \tilde{E}$  must be injective. It follows that it is invertible with inverse  $\text{Id} + \tilde{E}'$ ,  $\tilde{E}' \in \tilde{\Psi}^{\infty}(X; F)$ .

The space on the right in (P) is a group

$$G^{-\infty}(X; E) = \left\{ \text{Id} + E; E \in \tilde{\Psi}^{-\infty}(X; E) \text{ invertible}, \right.$$

$$\left. (\text{Id} + E)^{-1} = \text{Id} + E', E' \in \tilde{\Psi}^{\infty}(X; F) \right\}$$

Proposition If  $X$  is a compact (connected) manifold without boundary (actually with boundary or empty too) and  $E$  is any complex vector bundle over  $X$  then

$$G^{-\infty}(X; E) \hookrightarrow \tilde{\Psi}^{-\infty}(X; E)$$

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is open and dense in the  $C^\infty$  topology and is  
isomorphic to a fixed group, and so

$$G^{-\omega} = \left\{ a_{ij} : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{C}, \sum_{i,j} |a_{ij}|^2 < \infty \right. \\ \left. \text{and } \forall i, j, (Id + a)^{-1} = Id + b \text{ exists} \right\}.$$

Proof. It is convenient to use the eigenbases for a  
self-adjoint elliptic (pseudo) differential operator of  
positive order. Say an unit system  $\{\varphi_i\}_1^n$  for some  
positive order. The eigenvectors form a complete  
orthonormal basis

$$e_i \in C^\infty(X, E), \quad a$$

$$n \in C^\infty(X, E) \iff n = \sum_{i=1}^n c_i e_i,$$

$$\sum_i |c_i| i^\beta < \infty \quad \forall \beta.$$

Then  $A \in C^\infty(X, E)$  may be defined with an  
input matrix

$$a_{ij} = \overline{\langle e_i, A e_j \rangle}$$

The  $e_i \otimes e_j^*$  form a basis for  $C^*(X; \text{hol}(E))$   
 and this is equivalent to the 'Fourier' expansion of the  
 kernel, i.e.

$$x \in \tilde{C}^*(X, E) \iff \sum_{i,j} |i|^{\frac{1}{2}} |j|^{-\frac{1}{2}} |c_{ij}|^2 < \infty$$

by 2.

More generally in complex algebras, so the  
 result follows.

Thus these groups are all 'the same'.

Definition For any compact manifold (with corner)  
 its K-space is defined as

$$K^*(Y) = [Y \times S^1; G^*(X, E)]$$

the homotopy classes of smooth maps into  $G^*(X, E)$ .

Note that even though the group has no clearly  
 infinite dimensional, there is no problem understanding

$$\therefore T^* G^*(X, E) \subset C^*(X, \text{hol}(E))$$

$\Rightarrow$  a  $C^\infty$  up to  $G^{-\infty}(X; \mathbb{F})$  is just a  $C^\infty$  XVI/2  
series &

$$F \in C^\infty(Y \times \mathbb{S}^1 \times X^2; \text{ker}(E|_{\mathcal{Q}}))$$

that happens always to have initial value (when  
we add the identity). So we can take homotopy  
to mean smooth homotopy :-

$$F_0 \sim F_1 \Leftrightarrow \exists F \in C^\infty([0,1] \times Y \times \mathbb{S}^1 \times X^2; \\ \text{ker}(E|_{\mathcal{Q}}))$$

$$\text{s.t. } F(0, \cdot) = F_0, \quad F(1, \cdot) = F_1 \text{ or}$$

$$(D + F(t, y, 0))^{-1} \text{ exists } \forall t \neq 0.$$

Going back to  $A \in \Psi^\infty(X; E, \mathbb{F})$  what  
is left, we see that

$P_{A,N}$  is a principal  $G^{-\infty}(X; \mathbb{F})$ -space  
(if  $\text{ind}(A) = \dim N \geq 0$ )

Exn Go through it you can find  $\text{ind}(A) \leq 0$  (a)  
show a smaller  $G^{-\infty}(X; \mathbb{F})$ -space can be  
constructed

XVI/<sup>a</sup> If we fix a point,  $1 \in S'$  we can consider two subgroups of  $K^*(Y)$ :

$$K'^*(Y) = \{[F]; F: Y \rightarrow G^{-\infty}(X; E) \text{ is }\underline{\text{continuous}} \text{ on } S'\}$$

$$K^{-2}(Y) = \{[F]; \overset{(\exists)}{F: Y \times S' \rightarrow G^{-\infty}(X; E)}$$

$$\text{ s.t. } F(y, 1) = \text{id} \quad \forall y \in Y\}.$$

Lemma  $K^*(Y) = K'^*(Y) + K^{-2}(Y)$  is an abelian group.

Proof For a given  $[F] \in K^*(Y)$  define

$$F_{-2}(y, 0) = F(y, 1)$$

$\Rightarrow [F_{-2}] \in K^{-2}(Y)$ , being constant on the curve,

as  $F_{-1} = F F_{-2}^{-1}$  is the identity at  $(y, 1)$ ,

$\Rightarrow [F_{-1}] \in K'^*(Y)$ . The group law should clearly be

$$[F] + [G] = [FG]$$

using the group composition in  $G^{-\infty}$ .

Exa Show that  $K^{-1}(Y) \cap K^{-2}(Y) = \{0\} = [\text{Id}]$ .

Why is this 'natural' observation? This comes from our ability to approximate by finite rank operators. The easiest to see is the  $G^{-\infty}$  model.

Let  $\pi_N$  be the projection onto say the span of the first  $N$  eigensections for our first elliptic self-adjoint operator,  $\pi_N: C^\infty(X, E) \rightarrow C^\infty(X, E)$ . Then for any compact subset  $K \subset \Psi^\infty(X, E)$   $\exists N$ , sufficiently large, such that

$$\forall A \in K, \|A - \pi_N A \pi_N\|_{L^2} < \frac{1}{2}.$$

So given  $F: Y \times S^1 \rightarrow G^{-\infty}(X, E)$  we can choose  $N$  so large that

$$F_t = (1-t)F + t \pi_N F \pi_N$$

is a smooth map  $F: [0, 1] \times Y \times S^1 \rightarrow G^{-\infty}(X, E)$ .

This  $[F] = [F_1]$  where  $\pi_N F_1 = F_1 \pi_N = F_1$ .

$\sum_{i=1}^N$  Given two elements,  $[F], [G] \in K^*(Y)$  we can  
 choose the same  $N$  for both, w  $[F] = [F_1], [G] = [G_1]$ .  
 Now consider  $\pi_{2N}(I) - \pi_N$ . Then as the projection onto  
 the eigenvalues number  $N+1, \dots, 2N$ . This  
 gives us an isomorphism:  $S^{\frac{1}{2}} e_i = S e_{N+i}$ ,  $i=1, \dots, N$   
 $S^{\frac{1}{2}} e_i = 0$  also, as  $S \in \Psi^{\infty}(X, E)$ . Consider the  
 "rotation"

$$S_z = \begin{bmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{bmatrix}$$

like the "blobs" a square  $\Delta f_1, \dots, \Delta f_N \{e_{N+1}, \dots, e_{2N}\}$   
 of  $S^{\frac{1}{2}} e_{N+i} = e_i$ ,  $i=1, \dots, N$  at zero there.

$$\tilde{G}_z = S_z G_1 S_z$$

define a homotopy from  $G_1$  (at  $z=0$ ) to  $\tilde{G}_z$  at  
 $z=\pi/2$  in  $\tilde{G}_z$  get  $G$  and  $e_{N+1} \dots e_{2N}$ .

Then  $[G_1] = [\tilde{G}]$  but  $F_1 \tilde{G} = \tilde{G} F_1$ .

Now suppose  $Ay \in \Psi^m(X; E, F)$  is a smooth family of elliptic operators, i.e. a map  $Y \rightarrow \Psi^m(X; E, F)$  for some compact manifold  $Y$ . The families under further analysis: Is it possible to find a smooth map

$$(FJ) \quad \left\{ \begin{array}{l} R: Y \rightarrow \Psi^{-m}(X; E, F) \text{ s.t.} \\ A_y + R_y \text{ is invertible } \forall y? \end{array} \right.$$

We shall see, I hope, that there is a well-defined answer.

$$\text{Ind}(A.) \in K^2(Y) \text{ s.t.}$$

$$\text{Ind}(A.) = 0 \iff R \text{ is (FJ) exists.}$$

How to construct  $R$ ? How compute  $\text{Ind}(A.)$ ?

We can ask for a slightly weaker condition (FT).

Now, if  $N \subset C^\infty(X; E)$  is fixed we can also find

$$(FT) \quad \left\{ \begin{array}{l} R: Y \rightarrow \Psi^{-m}(X; E, F) \text{ s.t.} \\ A_y + R_y \text{ is surjective onto metospace } N. \end{array} \right.$$