

## Lecture XV.

First today let me compute the cohomology groups I have been talking about. Consider a general complex manifold

$$(*) \quad g = dx^2 + x^{2N} h.$$

If  $u = u_t + dx^1 u_h$  is a smooth form and  $n \in \mathbb{N}$  usual such,  $u \in C^\infty(X, \Lambda^k)$ , then

$$\underline{\underline{d=n-1}}$$

$$k < \frac{d+1}{2} \Rightarrow u \in L^2, \quad N=1$$

$$\text{or } h < \frac{d}{2} + \frac{1}{N} \Rightarrow u \in L^2, \quad \text{general } N.$$

To see this, check the norm and tangential part separately:  
 for  $N=1$ ,  $|x^k u_t|_{\Lambda^k} < \infty$  so  $u_t \in L^2$  if  $\int_0^1 x^{-2k+d+1} |du| < \infty$ .

for general  $N$ ,

$$|x^{kN} u_t|_{\Lambda^k} < \infty \Rightarrow \int_0^1 x^{-2kN+dN} |du| < \infty \Rightarrow u_t \in L^2.$$

Since  $dx$  has length around 1, the condition on  $u_h$  is  
 $k-1 < \frac{d}{2} + \frac{1}{N}$  which it was as weaker.

Thus, we have a map given by the

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Hodge decomposition, or factors into harmonic part

$$(E) \quad C^\infty(X; \Lambda^k) \rightarrow H_{\text{H}, A}^k(x), \quad k < \frac{d}{2} + \frac{1}{2}.$$

By definition the absolute de Rham cohomology is

$$H_{dR, A}^k(x) = \{ u \in C^\infty(X; \Lambda^k) ; du = 0 \} / \text{Im } d C^\infty(X; \Lambda^{k-1}).$$

The relative de Rham cohomology is

$$H_{dR, R}^k = \{ u \in \dot{C}^\infty(X; \Lambda^k) ; du = 0 \} / \text{Im } \dot{C}^\infty(X; \Lambda^{k-1}).$$

Prop For a conn.-comp metric ( $\omega$ ) ( $\text{so } N \geq 1$ ),

$$(E) \quad H_{\text{H}, A}^k \cong \begin{cases} H_{dR, A}^k(x) & k < \frac{n}{2} \\ \text{Im}(i : H_{dR, R}^k(x) \hookrightarrow H_{dR, A}^k(x)) & k = \frac{n}{2} \\ H_{dR, R}^k(x) & k \geq \frac{n}{2} + 1. \end{cases}$$

Here,  $i_R$  is the natural inclusion arising from

$$\dot{C}^\infty(X; \Lambda^k) \hookrightarrow C^\infty(X; \Lambda^k),$$

so  $[u] \in \text{Im}(i)$  represents (by  $u \in \dot{C}^\infty(X; \Lambda^k)$ ) also  $d\omega$ -nilpotent in  $\text{Im } d C^\infty(X; \Lambda^{k-1}) \cap \dot{C}^\infty(X; \Lambda^k)$ .

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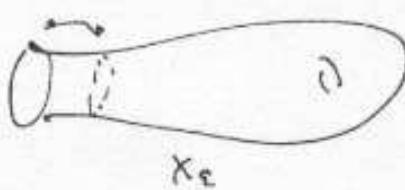
Proof from the inclusion  $C^\infty(X; \wedge^k) \hookrightarrow L_g^k(X; \wedge^k)$ ,  
 ~~$\forall k \leq \frac{n}{2}$~~  we not only have an inclusion (I) but

$$H_{\text{dR}, A}^k(X) \hookrightarrow H_{\text{dR}, A}^k(X) \quad k \leq \frac{n}{2}.$$

Indeed, if  $u$  is mapped to zero in  
 $(A) \quad u = d\varphi, \quad \varphi \in D_A^{k-1}.$

In fact we know considering more than this, since

$\varphi \in C^\infty(X^\circ; \wedge^k)$ , by elliptic regularity. This  
 however implies that  $[u] = 0 \in H_{\text{dR}, A}^k(X)$ ,  
 although not quite trivially. To do so we use a  
 retraction of  $X$  into its interior,  $X_\varepsilon = \{\beta \in X; x(\beta) \geq \varepsilon\}$ .



choose an inward-pointing vector field  $V$ ,  $V_x = 1$  near  $\partial X$   
 and consider the 1-parameter family  $F_t = \exp(tV)$  of flows.

$$\frac{d}{dt} F_t^* f = F_t^* V f, \quad \text{a force}$$

$$\frac{d}{dt} F_t^* u = F_t^* \mathcal{L}_V u = F_t^* (d\mathcal{L}_V + \mathcal{L}_V d) u$$

or forces.

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— Consider  $F_t^*: C^\infty(X; \Lambda^k) \rightarrow C^\infty(X; \Lambda^k)$  as  $dF_t = F_t^{-1}d$ ,

so  $F_t^*$  factors to a map on cohomology, in fact

$$F_t^*: H_{dR, X}^k(X) \rightarrow H_{dR, X}^k(X) \text{ is the identity.}$$

Indeed if  $u \in \text{clerk}$ ,  $\frac{d}{dt} F_t^* u = F_t^* d\sqrt{u} = dV_t$

so  $[u] = [u_t]$  in cohomology.

Now, from the  $t$  follows  $\text{ker } (\Delta)$  implies  $(u) = 0$ ,

since  $F_t^* u = dF_t^* v \in dC^\infty(X; \Lambda^{k-1})$  for  $t > 0$ .

Similarly, we can define

$$\Gamma: H_{H_0, X}^k(X) \rightarrow H_{dR, X}^k(X)$$

$$u \mapsto [F_t^* u], \quad t > 0 \text{ small.}$$

using the fact that  $H_{dR, X}^k(X) \subset C^\infty(X^0; \Lambda^k)$ . Since the composition  $\Gamma \circ i$  is the identity it suffices to see that  $\Gamma$  is injective, but this follows from the fact that  $\text{ker } i$  is close to the identity.

Next consider what happens for  $k \geq \frac{n}{2} + 1$ , we

observe here  $C^\infty(X; \Lambda^k) \hookrightarrow L_g$  as

$$H_{\delta R, \mathcal{R}}^h(x) \rightarrow H_{h, A}^L(x), \quad h \geq \frac{n}{2} + 1$$

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so again the first problem is effectively. The vanishing of  $[u]$  is the right way  $\|u\|_A \leq \|u\|_0 + D_A^{L-1}$ .  
as  $\|u\|_0 \rightarrow 0$ , this means  $(\text{for } N=1)$

$$u \in L^2, \quad d u_h \rightarrow u \in L^2$$

$\Rightarrow \langle u, u \rangle = \langle u, d u \rangle = \lim \langle u, d u_h \rangle = 0$ ,  
i.e.  $u$  is orthogonal to all  $d u_h$ 's. To get  $u$  as a solution, we

need to 'contract the other way'. We know (at least for  $N=1$ ) that  $u_t \in x^{-\frac{n}{2}+k+\delta} H_b^\infty(x; \wedge^k dx)$   
 $\Rightarrow u_t \in x^{-\frac{n}{2}+(k-1)+\delta} H_b^\infty(x; \wedge^{k-1} dx)$  are convergent,

$\delta > 0$  (in fact  $\delta < \frac{1}{2}$ ). Since  $k \geq \frac{n}{2} + 1$  this implies  
that a contraction of  $u_t$  is at least  $C$ . So if we extend  $u$  as  $\tilde{u} = \lim u_t$ ,



$$\tilde{u} = 0.$$

$x_\varepsilon$

This tells us that  $u \in H_{h, A}^L(x)$  does represent  
a relative dens. Thus passively look at

$\frac{\partial}{\partial t} u = 0$  near the boundary, hence

$$\frac{\partial^k u}{\partial t^k} = 0 \quad -\frac{\partial^k u}{\partial t^k} + x \partial_x^k u = 0$$

(where I have omitted the  $x^k$  boundary terms). The decay of the boundary is enough so that

$$w = \varphi \int_0^x u_n(x, \cdot) dx$$

makes sense, with  $\varphi$  a cut off at  $x=1$ . Then

$$\begin{aligned} dw &= \varphi' dx \wedge \int_0^x u_n dx \\ &\quad + \varphi \int_0^x dt^k u_n dx + \varphi dx \wedge u_n \\ &= \varphi' dx \wedge \int_0^x u_n dx + \varphi u. \end{aligned}$$

Therefore,  $u - dw \in C^\infty(X; \Lambda^k)$  is exact.

Then we can define

$$r: H_{H_0, X}^k(X) \rightarrow H_{dR, R}^k(\omega) \quad k \geq \frac{n+1}{2}$$

such that  $u \mapsto [u - dw]$ . Again it is a multiple of  $d\omega$  projects into  $d\omega_A$ .

Finally for the "middle" one. Here, what as  
 to say  $k = \frac{n}{2}$  or  $k = \frac{n+1}{2}$  we cannot forget smooth  
 forms also  $H_{H_0, A}^k(x)$ , since they are not  $L^2$ . Then  
 we do have

$$H_{dR, R}^k(x) \rightarrow H_{H_0, A}^k(x) \quad k = \frac{n}{2}, \frac{n+1}{2}.$$

going the other way, we can still find one for  $d\bar{x}$   
 so we do get

$$H_{H_0, A}^k(x) \rightarrow H_{dR, A}^k(x).$$

Now this does not fit its counterpart as the map  
where  $\delta$  written in  $dR_A$  is actually  
 the same as  $\delta$  written in  $H_0 A$ . If we  $H_{dR, R}^k$  was to give  
 $\delta$  in  $H_{dR, A}^k$ ,  $n = d\varphi$ ,  $\varphi \in C^\infty(X, \Lambda^{k-1})$ . Now  
 $k = \frac{n}{2} - 1 = \frac{k}{2} - \frac{1}{2}$ , so then does forget about  $\delta$ ?  
 so  $\delta$   $\varphi \mapsto 0$  in  $H_{H_0, A}^k$ . Then we do get

$$\text{Two maps: } H_{dR, R}^k(x) \rightarrow H_{dR, A}^k(x) \leftarrow H_{H_0, A}^k.$$

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Injutor's file for the fact that we can arrange (using our retraction) that  $u \in C_c^\infty(X^0; \Lambda^k)$  as the  $[u] = [F_t^* u]$  as  $F_t^* \tilde{u} = d\varphi$  if  $u \mapsto 0$ .

Exercice: Complete the last step of the proof, to show that the map  $u \mapsto u - du$  is injective.

I have made rather heavy work of this part, it would be better to define some multidimensional objects, in particular compute the coboundary groups for forms with constant coefficients at various weights.

and

Definition: The signature of a compact manifold with boundary of dimension  $4k$  is the signature of the quadratic form

$$(89) \quad H_{H_1, g}^{2k}(X) \times H_{H_1, g}^{2k}(X) \ni (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$$

for any of these metrics. [This is really a /brane!]

But what, you say a definition should not require a proof? This is a sign of good how disorganized the notes are! Of course I do have to show that the signature (sing) is independent of the metric ad, more precisely, that it is well-defined - will then be clear since it follows from  $L^2$ . In fact the same definition works for  $H^{2k} = i(H_{\text{SR},R}^{2k}(x) \rightarrow H_{\text{SR},A}^{2k}(x))$  since it is symmetric ( $\gamma h$  is even) and if  $\alpha \in C^\infty(X; \Lambda^{2k})$  and  $\beta = d\varphi$ ,  $\varphi \in C^\infty(X; \Lambda^{2k})$  we see that

$$\alpha \wedge d\varphi = d(\alpha \wedge \varphi) - d\alpha \wedge \varphi$$

so the integral vanishes if  $d\alpha = 0$ .

In fact we can restate (sing) in terms of \*

as

$$H_{H,S}^{2k}(x) \times H_{H,S}^{2k}(x) \ni (\alpha, \beta) \mapsto \int_X \langle \alpha, \beta \rangle \, d\gamma.$$

Since  $n=4k$ ,  $*^2 = * (-1)^{\frac{n(n-2k)}{2}} = \text{Id}$  we can identify a  $H_{H,g}^{2k}$  as the signature is  $1k$ .

$H$  has ev. —  $H$  has od.