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Last time I described, for a conic metric, the space

$$D = \{u \in L^2(X; \Lambda^*) ; du, d\bar{u} \in L^2_0(X; \Lambda^*)\}.$$

It consists of four pieces. The largest is simply the closure of $C^\infty(X; \Lambda^*)$ (or the Hilbert space $\alpha^{-\frac{n}{2}+1} H_{L^2}^1(X; c\Lambda^*)$) with respect to the norm

$$(ND) \quad \|u\|_D^2 = \|u^2\|_{L^2}^2 + \|du\|_{L^2}^2 + \|\bar{u}\|_{L^2}^2.$$

Apart from these there are three further dimensionless pieces, two associated to boundary cohomology

$$(*) \quad \begin{cases} E_A^{\frac{n}{2}-\frac{1}{2}} = \chi \cdot H_{H^1(\partial X)}^{\frac{n-1}{2}} (\partial X) \\ E_R^{\frac{n}{2}+\frac{1}{2}} = \chi \cdot d\chi \wedge H_{H^1(\partial X)}^{\frac{n+1}{2}} (\partial X) \end{cases}$$

which only exist for metrics on!

a 'non-coboundary' part G^* . I show, $\frac{S}{2}$
 rather briefly, that $G^* \subset D_A \cap D_R$, we can
 be approximately approximated.

We wish to show that $E_A^{\frac{n}{2}-\frac{1}{2}} \cap D_R = \{0\}$,
 which will complete the description of D_R and
 hence D_A . To do this we compute directly
 the quadratic form $\mathcal{Q}(u, v)$

$$\mathcal{Q}(u, v) = \int_{M \setminus D} (\langle \Delta u, v \rangle - \langle u, \Delta v \rangle) d\mu,$$

\times

$u, v \in D.$

Observe that this vanishes on D_R , since
 if $u_h, v_h \in C^\infty(X; \Lambda^k)$ then ~~$\mathcal{Q}(u_h, v_h) = 0$~~
 $\mathcal{Q}(u_h, v_h) = 0$.

Lemma. If $u = x \varphi \in E_A^{\frac{n}{2}-\frac{1}{2}}$ and
 $v = x \Delta \varphi \wedge \psi \in E_R^{\frac{n}{2}-\frac{1}{2}}$ then

$$\mathcal{Q}(u, v) = \int_X \langle \varphi, \psi \rangle dx.$$

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Proof for $v \in E_R^{\frac{n}{2} + \frac{1}{2}}$, $dv = 0$ at $u \in E_A^{\frac{n}{2} - \frac{1}{2}}$, $\delta v = 0$.

Thus

$$\begin{aligned} Q(u, v) &= \int_X (\langle du, v \rangle - \langle u, \delta v \rangle) dg \\ &= \int_0^\infty \int_{\mathbb{R}^n} 2xx' \langle \varphi_i \psi \rangle_{\mathbb{R}^n} dx dh \\ &= \int_{\mathbb{R}^n} \langle \varphi_i \psi \rangle_{\mathbb{R}^n} dh. \end{aligned}$$

From this it follows that $E_A^{\frac{n}{2} - \frac{1}{2}} \cap D = \{0\}$,

Since $E_R^{\frac{n}{2} + \frac{1}{2}} \subset D_R$.

Now we are in a position to prove the main proposition leading to the Hodge decomposition, namely that $d + \delta$ is self-adjoint and Fredholm on D_A and D_R .

For self-adjointness, recall that we define

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$$D_R^* = \{ u \in L_g^2; \exists \varphi \mapsto \langle (d+\delta)\varphi, u \rangle \text{ extends by continuity to } L_g^2 \}.$$

Recall that D_R is graded by degree so we see that $u \in D_R^*$ implies that each form component $u_{(h)} \in D_R^*$ and

$$D_R \ni \varphi \mapsto \langle d\varphi, u \rangle = \langle \varphi, v_d \rangle$$

$$D_R \ni \varphi \mapsto \langle \delta\varphi, u \rangle = \langle \varphi, v_\delta \rangle$$

but extends by continuity to L_g^2 — simply restrict to the appropriate form degree ± 1 . Thus, $u \in D$ since $d\varphi, \delta\varphi \in L_g^2$ so

$$D_R^* \subset D.$$

It is also clear that $D_R \subset D_R^*$ since if $u \in D_R$,

$$\langle (d+\delta)\varphi, u \rangle = \langle d\varphi, u \rangle + \langle \delta\varphi, u \rangle$$

$$= \lim_{n \rightarrow \infty} (\langle d\varphi_n, u \rangle + \langle \delta\varphi_n, u_n \rangle)$$

$$= \lim_{n \rightarrow \infty} \langle \varphi_n, du \rangle + \langle \varphi_n, \delta u_n \rangle = \langle \varphi, (d+\delta)u \rangle$$

$\frac{3}{5}$ using the approximations built to φ and u .
 Thus we only need show that $E_A \cap D_R^* = \{0\}$
 and this is the same argument as before.
 Namely, essentially by definition,

$$\begin{aligned} Q(\varphi, u) &= ((d+\delta)\varphi, u) - (\varphi, (d+\delta)u) \\ &= 0 \text{ on } D_R \times D_R^*. \end{aligned}$$

Taking $\varphi \in E_R^*$ it follows that $E_A \cap D_R^* = \{0\}$.

Finally then, we need to check the
 Fredholm property for $d+\delta$ on D_R , say.
 The main point here is that D (or D_R)
 will be norm (NS) inject compactly into
 L^2_D :

(T) $I: D \hookrightarrow L_g^2, \overline{I(B)} \subset L_g^2$ is
 compact if $B \subset D$ is bdd.

This is the L^2 Ascoli-Arzelà theorem,
 namely

$$D \subset x^{-\frac{n}{2}} H_b^4(X, X^*) \cap x^{-\frac{n}{2} + \epsilon} L_b^2(X, X^*)$$

and the latter already rejects compactly in L^2_g .

Exercise Check this!

From this compactness we deduce immediately that

$$H_{H(R,g)}^*(X) = \{u \in D_R; (d+\delta)u=0\}$$

is finite dimensional, since it is closed in L^2_g (by the comp content of $d+\delta$ into deobtions) and has compact unit ball.

Similarly, the range

$$(d+\delta)D_R \subset L^2_g \text{ is closed.}$$

Indeed, if $(d+\delta)u_n \rightarrow v$ in L^2_g then we can assume $u_n \perp H_{H(R,g)}^*(X)$. The orthogonal due to $u \in D_R$, shows $\|d u_n\| \rightarrow \|v\|$, $\|u_n\| \rightarrow \|v\|$, $v = v_d + v_g$.

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If we go back to the derivation of the structure of D we can apply the same argument, first concluding that $u_n \rightarrow u$ in $\alpha^{-\frac{n}{2}} H_b^1(X; \mathbb{A})$, using elliptic regularity. From this we deduce that $(d + \delta_0) u_n \rightarrow (d + \delta_0) u \in L^2$ and hence that $u_n \rightarrow u$ in D (as have D_Γ), using the Mellin transform.

Finally then we have most of what we set out to get for the cones — Hodge decomposition as identification of Hodge as L^2 -cohomology, subject however to elliptic regularity (which despite my delaying the part, is not supposed to be hard!). We still need to check the identification with intersection cohomology, but I will get back to that.

So, back to more geometric analysis.

If you recall I had introduced a space X_b^2 by blowing up the corner in X^2 , where X is a (compact) manifold with boundary. If we go back to the beginning when we thought a little about identifying X with, or from, $C^\infty(X)$. From this point of view we can define

$$C^\infty(X_b^2) = \left\{ u \in C^\infty(X^2 \setminus (\partial X)^2); u \text{ is } C^\infty \text{ in polar coords locally around } \partial X^2 = \{x=s=0\} \right\}.$$

Of course we either have to say all polar coords coincide or show that this condition is independent of what polar coords we take (that is, what coords we take before we introduce polar coords). I already did this.

Naively, if we just take a compact

8/9 manifold with corners at-faces on a particular border face F_i of codimension k , we are reduced to the following

Lemma If $F: U, 0 \rightarrow \mathbb{R}^{n,k}, 0$ is a diffeomorphism from a neighbourhood U of $0 \in \mathbb{R}^{n,k}$ onto a neighbourhood of 0 then

$$\text{if } C^\infty \text{ is } x_1 + \dots + x_k, \frac{x_i - x_{i+1}}{x_1 + \dots + x_k}, i=1, \dots, k-1$$

as y_1, \dots, y_{n-k} implies the same restriction of $F|_M$.

Proof It suffices to show that the 'pole' coordinate function $x_1 + \dots + x_k = r$ and $t_i = (x_i - x_{i+1})/r$ pull back to C^∞ functions (of the new coordinate). For the y_i 's the original. Reversing the coordinate change we see that

$$x_i - x_{i+1} = t_i r \quad i=1, \dots, k-1$$

$$\Rightarrow \quad x_i = (L_i t) r, \quad L_i t = l_i t + l_i'$$

For vectors b_i, b_i^t which I will leave you to evaluate, however they do form a basis of \mathbb{R}^k , with $L_i t \geq 0$ of all types ~~except~~.

By assumption, F preserves $\mathbb{R}^{n,k}$ really; let's assume for simplicity that

$$F^* x_j = a_j x_j, \quad \alpha a_j \in C^\infty$$

Since this must be true w/o be rearrangement.

Thus

$$F^* r = \sum_{\alpha \in C^\infty} (\alpha_j L_i t) \cdot r = \alpha r,$$

where I leave you to check the start position of α . Then this at follows (or

$$F^* t_i = \frac{F^* x_i - F^* x_{i+1}}{\alpha r}$$

$$= \frac{1}{\alpha} (a_i L_i t - a_{i+1} L_{i+1} t_i)$$

$\in C^\infty$ as claimed.

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Then we know that $[X; F]$ is a well-defined compact manifold with corners when

$$C^\infty([X; F]) = \{u \in C^\infty(X \setminus F) ;$$

$u \in C^\infty$ w.r.t. polar coords near each pt of $F\}$

(normal)

It is important to note that $[X; F]$ comes with a smooth boundary w.r.t.

$$\beta: [X; F] \rightarrow X$$

as that it is, as a set,

$$X \setminus F \cup F \times \bigcup_{k=1}^{k-1} \bigcup_{j=1}^{k-1} S^{k-1, k-1}$$

