

last time I described, somewhat informally, the space of  $b$ -smoothing operators,  $\Psi_b^m(x)$ . I want to go on and discuss, in some detail the spaces  $\Psi_b^m(x)$  of finite order operators. I will do so, but first I will go ahead and examine some of the consequences of elliptic regularity in this sense.

Let me recall the setting. We work with a compact manifold with boundary,  $X$ , on which we consider the 'cone' structure epitomized by

$$(7.1) \quad \mathcal{Y}_C = \{u \in C^\infty(X); u|_{\partial X} = \text{const.}\}.$$

More precisely we consider a cone metric,  $g$ . This is a metric on the interior of  $X$  which near the boundary takes the form

$$(7.2) \quad g_C = dx^2 + t^2 h(x, y, dy, dt), \\ h_C = h(0, y, dy, 0) \gg 0.$$

This,  $h_C$  is a metric on  $\partial X$ . The question I want to make precise, and answer, is: -

1408 / VIII

What is the Hodge theory of  $g_c$ ?

Elliptic regularity is supposed to tell us the following things:

• If  $u \in L^2_{g_c}(X; \Lambda^p)$  and  $(d+\delta)u = 0$  then

$$(7.3) \quad u \in \mathcal{H}^{-\frac{1}{2}}_b(X; \Lambda^p).$$

• If  $u \in L^2_c(X; \Lambda^p) = \mathcal{L}^{-\frac{1}{2}}_b(X; \Lambda^p)$  and

$$(d+\delta)u \in L^2_c(X; \Lambda^p)$$

$$\text{then } u \in \mathcal{L}^{-\frac{1}{2}}_b(X; \Lambda^p).$$

We want to use this to get a Hodge decomposition

$$L^2_c = H^p_c \oplus \mathcal{L}^p_c \oplus \mathcal{L}^{p-1}_c$$

First I want to use the Yulea transform to see what we can say about  $u \in L^2_c(X; \Lambda^p)$

what solves  $u_t = u_x = 0$ , using (7.3)

149  
VIII/3

Let me remind you about the 1-dimensional  
Fourier transform, normalized by

$$\mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}} e^{-i\xi t} u(t) dt.$$

(7.4)  $\mathcal{F}: \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$  is an isomorphism

with inverse  $u(t) = (2\pi)^{-1} \int_{\mathbb{R}} e^{it\xi} \hat{u}(\xi) d\xi.$

(7.5)  $\mathcal{F}$  extends to an isomorphism  $L^2(\mathbb{R})$ .

Also recall the Paley-Wiener Theorem.

$$\mathcal{F} \cdot \{u \in L^2(\mathbb{R}); u \text{ compactly supported}\}$$

(7.6)  $\leftrightarrow \left\{ \hat{u} \in L^2(\mathbb{R}) \text{ s.t. } \hat{u} \text{ is holomorphic in } \right.$

$$\left. \text{Im } z < 0 \text{ \& } \sup_{\sigma \in \mathbb{R}} \int_{\mathbb{R}} |\hat{u}(z + i\sigma)|^2 dz < \infty \right\}.$$

100 VIII/4

By continuity, or directly,  $\mathcal{Y}: \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$  is an isomorphism as

$$\begin{cases} \mathcal{Y}(\mathcal{D}_t u) = \tau \mathcal{F}u, & \mathcal{Y} = \frac{1}{i} \frac{d}{dt} \\ \mathcal{Y}(tu) = -\mathcal{D}_\tau \mathcal{Y}u. \end{cases}$$

It is convenient to 'translate' these results to the Mellin transform, for functions on  $(0, \infty)$ .

$$v_H(\tau) = \int_0^\infty v(x) x^{i\tau} \frac{dx}{x}.$$

Since  $(0, \infty) \ni x \mapsto -\log x = t \in \mathbb{R}$

is a diffeomorphism, with  $\frac{dx}{x} = -dt$  the result is

for the Fourier transform:

$$v_H(\tau) = \int_{-\infty}^\infty v(e^{-t}) |e^{-t}| e^{-i\tau t} dt.$$

So  $v \mapsto v_H$  is an isomorphism of

$$L^2_b(0, \infty) = \left\{ u \text{ measurable on } (0, \infty), \int |u|^2 \frac{dx}{x} < \infty \right\} \rightarrow L^2(\mathbb{R})$$

Switch

$\{u \in L^2_{\downarrow}(0, \infty); u=0 \text{ in } x > 1\} \ni v \mapsto v_H$  has range as in (7.6).

Now,  $u \in L^2_c(X, \chi^{\sigma})$  means that  $u$  has (locally) square-integrable coefficients w.r.t. an orthonormal basis, for  $\mathcal{G}_c$ , of  $\chi^{\sigma}$ . The  $v, dx, x dy$ , as above enough to orthonormal. However the Riemannian volume form is  $dy \simeq x^{n-1} dx dy \simeq x^n \frac{dx}{x} dy$ , so

$$u \in L^2_c(X, \chi^k) \Leftrightarrow u = \sum_k x^k u_k + \sum_{n \neq k} x^{k-1} dx u_n, \quad u_k, u_n \in x^{-\frac{n}{2}} L^2_{\downarrow}$$

+ integrability in the volume, show

$$\iint_{\partial X} |x^k u_k|^2 x^n \frac{dx}{x} dy < \infty$$

$$\Rightarrow \int_0^1 \int_{\mathbb{R}^n} |x^{\frac{n}{2}} u_k|^2 \frac{dx}{x} dy < \infty.$$

So this is how we will write forms near the boundary:

$$u = x^L u_t + x^{k-1} dx \wedge u_n$$

$u_t, u_n$  tangent  $k, k-1$  forms (depending on  $x$ ).

I computed  $d$  in terms of this decomposition before:

$$du = x^k \frac{1}{x} du_t + x^{k-1} dx \wedge (-d_t u_n +$$

$$= x^{k+1} \left( \frac{1}{x} dx u_t \right) + x^k \left( -\frac{1}{x} d_t u_t + \left( \frac{x \partial_x^2 u_t + k u_t}{x} + \frac{L}{x} \right) u_t \right)$$

as a matrix:

$$d \begin{pmatrix} u_n \\ u_t \end{pmatrix} = \frac{1}{x} \begin{pmatrix} -d_t & x \partial_x^2 + k \\ 0 & d_t \end{pmatrix} \begin{pmatrix} u_n \\ u_t \end{pmatrix}.$$

So,  $du = 0$  near  $\partial X$  becomes

$$\begin{cases} d_t u_t = 0 \\ -d_t u_n + (x \partial_x^2 + k) u_t = 0 \end{cases}$$

Let us compute the form of  $\int$  following the idea of Hodge. It suffices to work locally at each the defining ideal

$$(IBP) \int_U \langle d\varphi, u \rangle dg = \int_U \langle \varphi, du \rangle dg$$

then we can assume  $U$  is open and oriented.

If  $e_1, \dots, e_n$  is an orthonormal basis of forms,

Hodge define

$$* e_{i_1} \wedge \dots \wedge e_{i_k} = \pm e_{j_1} \wedge \dots \wedge e_{j_{n-k}}$$

where  $\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$

and the sign is given by the associated permutation.

Thus,

$$* (e_{i_1} \wedge \dots \wedge e_{i_k}) \wedge (e_{j_1} \wedge \dots \wedge e_{j_{n-k}}) = e_{i_1} \wedge \dots \wedge e_{j_n} = dg$$



This just means that

$$* u \wedge v = \langle u, v \rangle dg \quad \forall k \text{ forms } u, v \\ = \langle v, u \rangle dg \equiv * v \wedge u.$$

15/8 The identity (IBP) can be written. 11/8

$$\int_U *d\varphi \wedge u = \int * \varphi \wedge du = \int * du \wedge \varphi$$

$\forall k-1$  form  $\varphi \in C_c^\infty$ ,  $u$   $k$ -form.

$$= \int_U *u \wedge d\varphi$$

$$= \int_U (-1)^{n-k} d(*u \wedge \varphi) - (-1)^{n-k} (d*u) \wedge \varphi$$

$$= \int_U \varphi \wedge (d*u)$$

$$\Leftrightarrow *du = (-1)^{n-k+1} d*u.$$

since  $*^2 = \cancel{k(k-1)} (-1)^{k(n-k)}$  on  $k$ -form.

$$du = (-1)^{n-k+1 + (k-1)(n-k+1)} *d*u.$$

$$du = (-1)^{k(n-k+1)} *d*u.$$

For an conic metric, or the under conic  
metric

155  
11/9

$$dx^2 + h_0(x, dx)$$

$dx$ ,  $\alpha e_i$  is a kernel of  $e_i$  is o.n. for  $h_0$ .

$$\begin{aligned} \Rightarrow *u &= * (x^k u_t + x^{k-1} dx \wedge u_n) \\ &= x^{n-k-1} (*_t(dx \wedge x^k u_t)) + x^{n-k} (-1)^{n-k} *_t u_n. \end{aligned}$$

$$\begin{aligned} \text{so } d*u &= x^{n-k} (-1)^{n-k} d*_t u_n + x^{n-k-1} dx \wedge \\ &\quad \left( -d*_t u_t + (-1)^{n-k} \left( \partial_x *_t u_n + \frac{n-k}{x} x^k u_t \right) \right). \end{aligned}$$

$$\begin{aligned} * * d*u &= (-1)^{k-1} x^{k-2} \left( -d*_t *_t u_t + (-1)^{k-1} \left( \partial_x x^2 u_n + \frac{n-k}{x} x^2 *_t u_t \right) \right) \\ &\quad + x^{k-1} (-1)^{n-k} dx \wedge *_t d*_t u_n. \end{aligned}$$

Answer,  $\delta u = 0 \Leftrightarrow d*u = 0 \Leftrightarrow$

$$\delta_t u_n = 0 \quad \& \quad \delta_t u_t + (x^2 + n-k) u_n = 0.$$

To understand the behavior of  $u \in L^2_c$  satisfying  $du = d\bar{u} = 0$ , we cut it off near  $x=1$  at  $\frac{1}{2}$  by the Kähler metric:

$$u_H(s, y) = \int_0^{\infty} \varphi u(x, y) x^{\epsilon s} \frac{dx}{x} \quad \begin{array}{c} \uparrow \\ \frac{1}{2} \\ \downarrow \end{array}$$

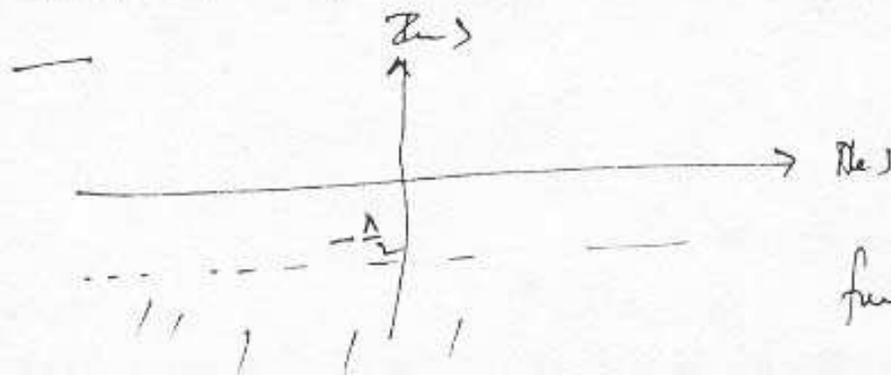
$$(x \partial_x u)_H = -\epsilon s u_H + \varphi_H$$

Here we know that  $v \in C^\infty$  on support in  $\frac{1}{2} < x < 1$ .

$$\underline{\text{Prop}}^h: u \in L^2_c(X; \mathcal{L}^{\epsilon}) \xrightarrow{\Delta} du = d\bar{u} = 0 \Rightarrow$$

The Kähler metric  $(u_E)_H, (u_u)_H$  are meromorphic in the entire complex plane, holomorphic in

Im  $s < -\frac{n}{2}$  with values in  $C^\infty$  and rapidly decreasing as  $|\text{Re } s| \rightarrow \infty$  with fixed Im  $s$ .



for Poly-Weyl

$\mathcal{U}_n$  take as entire.

$$d(\mathcal{U}_t)_n = 0$$

$$(-is+k)(\mathcal{U}_t)_n - d_t(\mathcal{U}_n)_n \text{ entire.}$$

$$d(\mathcal{U}_n)_n = 0$$

$$(-is+(n-k))(\mathcal{U}_n)_n \pm d_t(\mathcal{U}_t)_n \text{ entire.}$$

$$(-is+k)(-is+(n-k))(\mathcal{U}_t)_n \pm d_t d_t(\mathcal{U}_t)_n \\ \text{is entire.}$$

$$d_t(\mathcal{U}_t)_n = 0.$$

$$\Rightarrow (\mathcal{U}_t)_n = (\pm \Delta_t + (-is+k)(-is+(n-k))^{-1} \text{ent.}$$

is meromorphic.

When are the poles exactly? Next interesting poles  
 are in the strip  $-\frac{n}{2} < \text{Re } s \leq -\frac{n}{2} + 1$ . They cannot  
 be any as  $\text{Re } s = -\frac{n}{2}$  by  $L^2$  condition.

1st  
vii/12

The domain of  $d+d^*$

We consider two distinct domains for  $d+d^*$ , the absolute & relative domains:-

$$\text{Dom}_A (d+d^*) = \left\{ u \in L_c^2(X; \Lambda^k); du \in L_c^2 \text{ and } \exists \varphi_j \in C^\infty(X; \Lambda^{k-1}), \varphi_j \rightarrow u \text{ in } L_c^2(X; \Lambda^k), d\varphi_j \rightarrow du \text{ in } L_c^2(X; \Lambda^k) \right\}$$

$$\text{Dom}_R (d+d^*) = \left\{ u \in L_c^2(X; \Lambda^k); du \in L_c^2 \text{ and } \exists \varphi_j \in C^\infty(X; \Lambda^{k-1}), \varphi_j \rightarrow u \text{ in } L_c^2, d\varphi_j \rightarrow du \text{ in } L_c^2 \right\}$$

Theorem (Cheeger-Croke-MacPherson) The null space of  $d+d^*$ :  $\text{Dom}_R \rightarrow L_c^2(X; \Lambda^k)$  is isomorphic to the  $L^2$  cohomology and dual to the lower-middle middle cohomology of  $X/\partial X$ ;

$$H_{L^2}^k = \left\{ u \in L_c^2(X; \Lambda^k); du=0 \right\} / \left\{ u \in L_c^2(X; \Lambda^{k-1}), du \in L_c^2(X; \Lambda^k) \right\}$$