17. Lecture XVII, 4 November, 2003

Next lecture will be Thursday November 13 (and it will be a short one!)

Last time I constructed a principal bundle associated to any family $A \in \mathcal{C}^\infty(Y;\Psi^k(X;E,F))$ of elliptic pseudodifferential operators on a compact manifold without boundary, X, ofr a compact parameter space Y. The *structure group* of this bundle is $G^{-\infty}(X;E)$ or $G^{-\infty}(X;F)$ as the numerical index of the family if negative or positive. Assuming for the same of definiteness that $\# - \operatorname{ind}(A) \ge 0$, the bundle

$$\begin{tikzcd} G^{-\infty}(X;F) \arrow[r] & \mathcal{P}_{A,N} \arrow[d] \\ & Y \end{tikzcd} \tag{1}$$

has fibre at $y \in Y$

$$\mathcal{P}_{A,N,y} = \big\{B \in \Psi^{-\infty}(X;E,F); A_y + B \text{ has null space exactly } N\big\} \tag{2}$$

where $N \subset \mathcal{C}^\infty(X;E)$ is fixed but is chosen arbitrarily with dimension equal to $\# - \operatorname{ind}(A.)$.

Essentially by definition, this bundle is trivial if and only if there exists a smooth map $E \in \mathcal{C}^\infty(Y;\Psi^{-\infty}(X;E,F))$ such that $A_y + E_y$ is surjective for all $y \in Y$ and has null space N.

Exercise 29. Check this carefully, starting from the definition of triviality of a principal bundle.

Thus, the triviality of the principal bundle, together with the vanishing of the numerical index is *precisely* the obstruction to 'perturbative invertibility'.

Also recall that I *defined*

$$K^{-1}(Y) = [Y; G^{-\infty}] \tag{3}$$

$$K^{-2}(Y) = [Y \times \mathbb{S}, Y \times \{1\}; G^{-\infty}, \mathrm{Id}] \tag{4}$$

i.e. $K^{-1}(Y)$ is the set of homotopy classes of smooth maps into $G^{-\infty}$ (for any model) and $K^{-2}(Y)$ is similarly the set of homotopy classes of smooth maps from $Y \times \mathbb{S}$ into $G^{-\infty}$ taking $Y \times \{1\}$ to Id. We can also thing of (4) as

$$K^{-2}(Y) = [Y; \mathcal{L}G^{-\infty}, \mathrm{Id}] \tag{5}$$

where $\mathcal{L}G^{-\infty}$ is the loop group:

$$\mathcal{L}G^{-\infty} = \big\{F : \mathbb{S} \longrightarrow G^{-\infty}; F(1) = \mathrm{Id}\big\}\,. \tag{6}$$

The definitions (3) and (4) depend on the fact, which is the essential nature of *Bott Periodicity* that

$$\Pi_j(G^{-\infty}) = \begin{cases} \{0\} & j \text{ even} \\ \mathbb{Z} & j \text{ odd.} \end{cases} \tag{7}$$

Exercise 30. Assuming (7) show that

$$\Pi_j(\mathcal{L}G^{-\infty}) = \begin{cases} \mathbb{Z} & j \text{ even} \\ \{0\} & j \text{ odd} \end{cases} \tag{8}$$

where the higher homotopy groups can be considered as maps into the component of the identity.

Exercise 31. What are $K^{-1}(\mathbb{S}^n)$ and $K^{-2}(\mathbb{S}^n)$?

I will prove the first part of (7). The first claim is

Lemma 7. *$G^{-\infty}$ is connected.*

Proof. This is a direct consequence of the fact that for any smoothing operators $A \in \Psi^{-\infty}(X;E)$ the family $\operatorname{Id} + zA$ is invertible for $A \in \mathbb{C} \setminus D$ where $D \subset \mathbb{C}$ is discrete (i.e. countable and without points of accumulation). We can either use the Fredholm determinant to prove this or proceed directly. Fix a value $\bar{z}$ of z. If $\operatorname{Id} + zA$ is invertible then we know it has a bounded inverse as an operator on $L^2(X;E)$ and by the openness of the set of invertible operators (i.e. convergence of the Neumann series) it remains invertible for $|z-\bar{z}|\|A\|\|(\operatorname{Id} - \bar{z}A)^{-1}\| < 1$. Thus D is closed. If $\operatorname{Id} + \bar{z}A$ is not invertible then we use finite rank approximation to write $A = A_1 + A_2$ where A_1 is a finite rank smoothing operator and A_2 has small norm, for instance $|\bar{z}|\|A_2\|_{L^2} < \frac{1}{2}$. Then $\operatorname{Id} + zA_2$ has inverse $\operatorname{Id} + B(z)$ for $|z - \bar{z}| < \epsilon$ with $B(z)$ holomorphic with values in the smoothing operators and we are reduced to considering

$$(\operatorname{Id} + B(z))(\operatorname{Id} + zA) = \operatorname{Id} + A'(z),\ A'(z) = (\operatorname{Id} + B(z))zA_1$$

Thus $A'(z)$ has finite rank, at most the rank of A_1, and is holomorphic near z so $\operatorname{Id}_N + A'(z)$ is invertible for $0 < |z - \bar{z}| < \epsilon'$ for some $\epsilon > 0$ and D is discrete.

Thus if $\operatorname{Id} + A$ is invertible then it can be connected to the identity by an inverible family $\operatorname{Id} + z(s)A$. $\square$

Exercise 32. Use such a finite rank approximation to define the Fredholm determinant $\det(\operatorname{Id} + A)$ as an entire function of A, extending the usual definition for finite rank operators, such that $(\operatorname{Id} + A)^{-1}$ exists if and only if $\det(\operatorname{Id} + A) \neq 0$ with the usual multiplicative and differential properties

$$\begin{gathered}\det((\operatorname{Id} + A)(\operatorname{Id} + B)) = \det(\operatorname{Id} + A)\det(\operatorname{Id} + B),\\ \frac{d}{dz}\det(\operatorname{Id} + zA) = \det(\operatorname{Id} + zA)\operatorname{Tr}(\operatorname{Id} + zA)^{-1}A) \text{ where } \det(\operatorname{Id} + zA) \neq 0.\end{gathered} \tag{9}$$

The second, and more substantial, part of (7) is

Proposition 11. *If $F : \mathbb{S} \longrightarrow G^{-\infty}(X;E)$ is a smooth loop then*

$$w(F) = \frac{1}{2\pi i}\int_{\mathbb{S}} \operatorname{Tr}\left(F(\theta)^{-1}\frac{dF(\theta)}{d\theta}\right) d\theta \in \mathbb{Z} \tag{10}$$

there exists a smooth map

$$\tilde{F} : [0,1] \times \mathbb{S} \longrightarrow G^{-\infty}(X;E) \text{ with } \tilde{F}(0) = F \text{ and } \tilde{F}(1) = (\operatorname{Id} - \pi) + z^{w(F)}\pi \tag{11}$$

where π is a projection of rank one.

We start off with a simple case, where the family is actually affine.

Lemma 8. *If A, B are $N \times N$ complex matrices and $A + zB$ is invertible on $|z| = 1$ then, for $|z| = 1$, it is homotopic to $(\operatorname{Id} - \pi) + z\pi$ where π is a projection of rank $w(A + zB)$.*

Proof. If A is not invertible, we may perturb the family slightly and so deform it to $(A + t\operatorname{Id}) + zB$ where the constant term is invertible. Then, using the connectedness of $\mathrm{GL}(N)$, which follows from the proof above, we may deform away the constant term and replace the family by $\operatorname{Id} + zB'$, $B' = (A + t\operatorname{Id})^{-1}B$. On the circle $z = e^{i\theta}$,

$dz = ie^{i\theta}d\theta$ so for a family which is holomorphic near the circle the integral in (10) can be written as a contour integral

$$w(F) = \frac{1}{2\pi i}\int_{|z|=1} \operatorname{Tr}\left(F(\theta)^{-1}\frac{dF(z)}{dz}\right)dz \tag{12}$$

In this case $dF/dz = B$ and the integral, without the trace, becomes

$$M = \frac{1}{2\pi i}\int_{|z|=1}(\operatorname{Id}+zB')^{-1}B'dz \tag{13}$$

which is in fact the projection onto the span of the generalized eigenspaces of B for outside the unit circle (and with null space the span of those inside). We don't need all of this information but we do need to see that M is a projection (or perhaps better to say an idempotent, $M^2 = M$). Indeed the square can be written as the double integral

$$M^2 = \frac{1}{(2\pi i)^2}\int_{|z|=1}\int_{|z'|=1+\epsilon}(\operatorname{Id}+zB')^{-1}(\operatorname{Id}+z'B')^{-1}(B')^2dzdz' \tag{14}$$

for $\epsilon > 0$ small (using Cauchy's theorem). Now the resolvent identity can be written

$$(\operatorname{Id}+zB')^{-1}(\operatorname{Id}+z'B')^{-1}B' = (z'-z)^{-1}\left((\operatorname{Id}+zB)^{-1} - (\operatorname{Id}+z'B')\right),\ z \neq z'. \tag{15}$$

Inserting this into (14), one of the integrals can be carried out for each term. Indeed the second is holomorphic in $|z| \le 1$ so integrates to zero, while for the first has a simple pole in z' at $z' = z$ and so the z' integral may be replaced by the residue which is just M.

Furthermore M and B' commute, since B' commutes with $(\operatorname{Id}+zB')^{-1}$ and

$$(\operatorname{Id}+zB')^{-1}M \text{ is holomorphic in } |z| \ge 1,\ (\operatorname{Id}+zB')^{-1}(\operatorname{Id}-M) \text{ is holomorphic in } |z| \le 1. \tag{16}$$

This involves an argument similar to that above, to prove the first write

$$\begin{aligned}(\operatorname{Id}+zB')^{-1}M &= \frac{1}{2\pi i}\int_{|s|=1}(\operatorname{Id}+zB')^{-1}(\operatorname{Id}+sB')^{-1}B'ds \\ &= \frac{1}{2\pi i}\int_{|s|=1}(s-z)^{-1}\left((\operatorname{Id}+zB')^{-1} - (\operatorname{Id}+sB')^{-1}\right)ds.\end{aligned}$$

Here the first term vanishes (for $|z| > 1$), by Cauchy's theorem, and the second is holomorphic. The other case is similar.

Finally, we conclude that under the deformation $B'_t = t(\operatorname{Id}-M)B' + M(tB' + 2(1-t))$ $(\operatorname{Id}+zB'_t)^{-1}$ remains holomorphic near $|z| = 1$ and results in a family as desired. □

18. Lecture XVIII, 13 November, 2003

Handwritten notes: Pages 1-11

19. Lecture XIX, 18 November, 2003

Handwritten notes: Pages 1-10

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