14. Lecture XIV, 23 October, 2003

What have I not done to complete the treatment of the 'conic case' at least as far as the identification of the L^2 cohomology, the relative Hodge cohomology and the appropriate intersection cohomology is concerned? I have not

(1) Treated the composition of finite-order b-pseudodifferential operators.
(2) This in turn is only really needed (for the moment) for the proof of the Sobolev continuity of such operators, that $A \in \Psi_{\mathrm{b}}^k(X)$ always defines a bounded linear operator from $x^s H_{\mathrm{b}}^m(X)$ to $x^s H_{\mathrm{b}}^{m-k}(X)$ for any m, s. This is what gives us elliptic regularity.
(3) I have not yet discussed intersection cohomology at all.

I will add to the notes, but probably not devote a lecture to, the first two of these. My reasoning here is that these reduce, given what we already know, to the same issues in the boundaryless case, so I do not feel the need to go through the discussion fully here.

Rather than go through the conic case again, I will now quickly describe the same sort of approach to another class of degenerate metrics which I will call 'cusps' but are often called 'horns'. I do not want to take the time to go through all the details, but I will attempt to write down everything to the point where it is 'straightforward' to check the claims that I make.

The metrics we consider again exist on any compact manifold with boundary, but with a somewhat different degeneration than for conic metrics. Thus, we suppose that in the interior, g is a metric and near the boundary there is a boundary defining function ρ such that

$$g = d\rho^2 + \rho^{2N} h \tag{1}$$

where h is as before, a smooth symmetric 2-cotensor which restricts to the boundary to a metric h_0 and $N \geq 2$; the conic case corresponds to $N = 1$. The extreme case, $N = 0$, is that of a regular boundary problem, which can also be handled the same way but leads to somewhat different analytic issues (and a different L^2 cohomology of course, namely the absolute cohomology).

Exercise 21. Let $Y^n \subset X^{n+1}$ be a singular submanifold of a compact manifold without boundary where Y has just an isolated singular point (or perhaps several) near which there are local coordinates z_j, in which it takes the form

$$z_0^{2N} = \sum_{j=1}^{n} z_j^2 + f(z_1, \dots, z_n),\ z_0 \geq 0, \tag{2}$$

where f (real-valued) vanishes to order 3 at least at 0. Show that the introduction of the singular coordinates z_0, z_j/z_0^N resolves Y to a manifold with boundary to which a metric on X restricts to a 'horn' metric (note that z_0 might not quite be x.)

For extra credit (!) show that the same thing can be accomplished by repeatedly blowing up the singular point, namely it needs to be blown up N times.

Problem 1. Describe the L^2 and Hodge cohomology for a metric of this 'cusp' type. In fact we want to do 'everything' in a sense that should be getting clearer by now.

The approach I will use is, and of course this is one of the main points, essentially the same as in the conic case although some of the 'details' are necessarily different.

Namely, first look at the structure of $d+\delta$ in terms of a Lie algebra of vector fields such that we can give an elliptic regularity result for the associated enveloping algebra (and develop a full calculus of pseudodifferential operators to go along with this). This is used to analyse the relative and absolute domains, which have the same definitions as before, deduce self-adjointness and the Fredholm property and hence get the Hodge decomposition and identity of L^2 cohomology and relative Hodge cohomology. It turns out that in this case the Hodge cohomologies can be identified in terms of the usual relative/absolute cohomology and subsequently in terms of appropriate intersection cohomology. Rather surprisingly perhaps the spaces (say L^2 cohomology) on a fixed manifold for different metrics and different values of $N \ge 1$ turn out to be canonically isomorphic.

So, first we look for a Lie algebra of smooth vector fields with which to describe the Laplacian and $d+\delta$. If we look at vector fields of finite length they will generally be singular at the boundary, with the worst singularity being $O(\rho^{-N})$ (take $N=2$ if you want). So we can look at the vector fields V which are smooth and satisfy

$$|V|_g = O(\rho^N). \tag{3}$$

Since the part $\rho^{2N}h$ of the metric already gives such an order of vanishing for any smooth V, this is equivalent to

$$V\rho \in \rho^N \mathcal{C}^\infty(X) \Longleftrightarrow V \in \mathcal{V}_{\mathrm{Nc}}(X). \tag{4}$$

Locally, in adapted coordinates, in which $x=\rho$ must always be an *admissible* defining function, i.e. one for which (4) holds, this Lie algebra and $\mathcal{C}^\infty(X)$ module is spanned by

$$x^N\partial_x,\ \partial_{y_j}. \tag{5}$$

It follows that it is the space of all smooth sections of a vector bundle, ${}^{\mathrm{Nc}}TX$, for which (5) gives a local basis. In the case $N=2$ I introduced this Lie algebra long ago; it depends on the choice of ρ as a trivialization of the normal bundle to the boundary, but nothing more. For $N \ge 3$ it only depends on the choice of ρ modulo terms $O(\rho^N)$ as is clear from (4).

Exercise 22. Can you give a 'geometric' description of an N-cusp structure on a compact manifold with boundary, analogous to the trivialization of the normal bundle in case $N=2$?

Similarly we define the N-cusp cotangent, and form, bundles based on $\mathcal{C}^\infty$ combinations of the forms

$$dx,\ \rho^N dy_j \tag{6}$$

I will denote these bundles ${}^{\mathrm{Nc}}T^*X$ and ${}^{\mathrm{Nc}}\Lambda X$; note that they depend on more than N!

Since ${}^{\mathrm{Nc}}T^*X$ is, by definition, the dual of ${}^{\mathrm{Nc}}TX$, a smooth section of the latter, $V \in \mathcal{V}_{\mathrm{Nc}}(X)$, defines a smooth function on the former which is linear on the fibres; we normalize this by defining $\sigma(V)$ to be iV thought of as a linear function. A function $f \in \mathcal{C}^\infty(X)$ similarly defines a smooth function on ${}^{\mathrm{Nc}}T^*X$ which is constant on the fibres (and we do not put an i in the identification of $\sigma(f)$ with f in this sense). Let $\mathrm{Diff}^k_{\mathrm{Nc}}(X)$ be the space of N-cusp differential operators of order k (at most). Thus $P \in \mathrm{Diff}^k_{\mathrm{Nc}}(X)$ is an operator, for example on $\mathcal{C}^\infty(X)$, which can be written as a finite sum of up to k fold products of elements of $\mathcal{V}_{\mathrm{Nc}}(X)$; this one can think of as the enveloping algebra of $\mathcal{V}_{\mathrm{Nc}}(X)$ as a Lie algebra and $\mathcal{C}^\infty(X)$ module (in particular $k=0$ factors means the action of $f \in \mathcal{C}^\infty(X)$ by multiplication); the $\operatorname{Diff}^k_{\mathrm{Nc}}(X)$ clearly form a (n order-)filtered algebra. Moreover, from the fact that $\mathcal{V}_{\mathrm{Nc}}(X)$ is a Lie algebra

$$[\operatorname{Diff}^k_{\mathrm{Nc}}(X), \operatorname{Diff}^l_{\mathrm{Nc}}(X)] \subset \operatorname{Diff}^{k+l-1}_{\mathrm{Nc}}(X) \tag{7}$$

we see that the commutative product in $\mathcal{C}^\infty({}^{\mathrm{Nc}}T^*X)$ leads to a short exact sequence

$$\operatorname{Diff}^{k-1}_{\mathrm{Nc}}(X) \hookrightarrow \operatorname{Diff}^k_{\mathrm{Nc}}(X) \xrightarrow{\sigma_k} \mathcal{P}^k({}^{\mathrm{Nc}}T^*X). \tag{8}$$

Here the quotient space is the space of smooth functions on ${}^{\mathrm{Nc}}T^*X$ which are homogeneous (polynomials) of degree k on the fibres. The symbol map is *determined* by the fact that it is multiplicative and our earlier normalization on $\mathcal{C}^\infty(X) = \operatorname{Diff}^0_{\mathrm{Nc}}(X)$ and $\mathcal{V}_{\mathrm{Nc}}(X)$.

We can extend these definitions to sections of vector bundles without pain. Either localize everything, which is a bit painful, or interpret the tensor product in

$$\operatorname{Diff}^k_{\mathrm{Nc}}(X; E, F) = \operatorname{Diff}^k_{\mathrm{Nc}}(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X; \hom(E, F)). \tag{9}$$

Of course it is important that this defines a space of operators $\mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F)$.

Exercise 23. Check that there are no surprises and the symbol extends in the obvious way and gives rise to a short exact sequence as in (8) but with bundles inserted appropriately.

As usual, ellipticity means precisely that $\sigma_k(P)$ is invertible off the zero section of ${}^{\mathrm{Nc}}T^*X$.

Now we look at $d+\delta$ from this point of view.

Lemma 4. *For an N-cusp metric* (1), $d + \delta \in \rho^{-N} \operatorname{Diff}^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*)$ *is elliptic in this sense.*

Proof. To check that $d \in \rho^{-N} \operatorname{Diff}^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*)$ just work out its action on the local basis of 1-forms (6):

$$\begin{aligned} d(adx + \sum_{k=1}^{n-1} b_k x^N dy_k) &= x^{-N}(\sum_{j=1}^{n-1} (\partial_{y_j} a) x^N dy_j \wedge dx \\ &+ \sum_{k=1}^{n-1} (x^N \partial_x b_k + N x^{N-1} b_k) dx \wedge x^N dy_k + \sum_{l,k=1}^{n-1} (\partial_{y_l} b_k) x^N dy_l \wedge x^N dy_k \end{aligned} \tag{10}$$

Remark 2. It is precisely at this point that we see a simplification arising in the cases $N \geq 2$ relative to the conic case, $N = 1$. Namely in the middle, 'cross', term here the term of order 0, which arises from the x-differentiation of x^N in the basis of forms, vanishes at $x = 0$ if $N > 1$. This means that this term will not show up in the 'model' operator we later consider, as we considered the model cone earlier. For this reason the non-zero eigenvalue problems that appeared for the cone, and caused most of the computation work, do not show up at all for $N > 1$. I did say the cone was the hardest case earlier! It also means that the 'model operator' for $d + \delta$, when we try to look at what happens to leading order at the boundary is not, or at least should not be thought of, as the operator for the model problem.

For the latter this x^{N-1} term would appear, but it is irrelevant to the analysis and is better dropped. Make of that what you will.

Then we can either check, exactly as before, that Hodge $\star$ is an isomorphism of ${}^{\mathrm{Nc}}\Lambda^* X$, which is essentially immediate from the definition, or else that the adjoints with respect to a measure such as

$$dg = \rho^{N(n-1)}\nu,\ 0 < \nu \in \mathcal{C}^\infty(X;\Omega), \tag{11}$$

and non-degenerate fibres inner products, of elements of $\mathrm{Diff}^k_{\mathrm{Nc}}(X;E,F)$ are in $\mathrm{Diff}^k_{\mathrm{Nc}}(X;F,E)$. Anyway, we easily conclude that $\delta \in \mathrm{Diff}^1_{\mathrm{Nc}}(X;{}^{\mathrm{Nc}}\Lambda^*)$.

The argument for ellipticity is the same as before. We can see from (10) that the symbol of d at (p,ξ), $\xi \in {}^{\mathrm{Nc}}T^*_pX$, is $i\xi\wedge$ acting on ${}^{\mathrm{Nc}}\Lambda^*_pX$. The symbol of the adjoint is the adjoint of the symbol (or use $\star$) so the symbol of δ is $-ic_\xi$ in terms of metric contraction. The result is a Clifford action of ${}^{\mathrm{Nc}}T^*X$ on ${}^{\mathrm{Nc}}\Lambda^*X$, and in any case is elliptic since its square is diagonal and given by multiplication by the metric (remember, this is a course on Dirac operators, except I have only talked about one so far!) □

We want uniform elliptic estimates (and more, we really want a way to write down the inverse of $d+\delta$). To get these we will work on a double space which resolves $\mathcal{V}_{\mathrm{Nc}}(X)$. This is supposed to be obtained through iterated blow-up, $\beta : X^2_{\mathrm{Nc}} \longrightarrow X^2$ and be such that we can lift $\mathcal{V}_{\mathrm{Nc}}$ smoothly from either factor of X and the resulting smooth vector fields are transversal to the lifted diagonal. We also want the stretched projections $\pi_{H,\mathrm{Nc}} = \pi_H \circ \beta$, $H = L, R$, to be b-fibrations which are transversal to the lifted diagonal (among other things this means that the lifted diagonal is diffeomorphic to X). Let's try to do it; maybe just sticking to $N = 2$ would be wise, but I will go on and outline the general case.

Locally our vector fields are $x^N\partial_x$ and ∂_{y_j}. Just as in the conic case we do not want, in fact cannot, do anything to the tangential ∂_{y_j} vector fields, since they are already non-degenerate. Basically this resolution problem is again 1-dimensional, or since we are in the double space, 2-dimensional, just involving x and x'. An obvious thing to look at is the lift of $x^N\partial_x$ to the space X^2_{b}, which resolves $\mathcal{V}_{\mathrm{b}}(X)$. In the coordinates $s = x/x'\ x'$, y_j and y'_j near a point on the lifted diagonal $s = 1$, $y = y'$ in X^2_{b} we know that $x\partial_x = s\partial_s$. (So that the lift of $\mathcal{V}_{\mathrm{b}}(X)$ is everwhere transversal to the diagonal). So of course, $x^N\partial_x$ lifts to $(x')^{N-1}s^N\partial_s$. Since $s = 1$ on the diagonal, this vanishes exactly at $x' = 0$ on the lifted diagonal, which is to say at its boundary. However, we cannot blow-up the boundary of the diagonal, since the ∂_{y_j} are not tangent to it! The smallest reasonable thing to blow up is

$$B_2 = \{x' = 0, s = 1\} \subset \mathrm{ff}(X^2_{\mathrm{b}}). \tag{12}$$

Exercise 24. Check that this is actually a well-defined submanifold of X^2_{b} which depends (only) on the choice of cusp structure, i.e. the defining function ρ. Note that it is a boundary p-submanifold, i.e. is an interior p-submanifold of the boundary hypersurface $\mathrm{ff}(X^2_{\mathrm{b}}$.

Even though the notation is not quite defined, we consider

$$X^2_{\mathrm{cu}} = [X^2_{\mathrm{b}}; B_2]. \tag{13}$$

I have not defined the blow up of a boundary p-submanifold such as B but it is a straightforward generalization of the blow up of a boundary face. We get a new boundary face ff and b-map as blow-down map. The main (big) difference is that this map is not a b-submersion, if fact it is a b-submersion exactly when B is a boundary face, which it is not here. This is replaced by the fact that

Lemma 5. *The Lie algebra $\mathcal{V}_{\mathrm{b}}(X;Y)$ of vector fields tangent to the boundary faces of X and to the p-submanifold Y lifts smoothly to $[X;Y]$ to span $\mathcal{V}_{\mathrm{b}}([X;Y])$ as a module over $\mathcal{C}^\infty([X;Y])$.*

Thus we do in fact know the range of β_*.

Exercise 25. Check this in the particular case of interest here, namely for $B_2 \subset X^2_{\mathrm{b}}$.

Proposition 6. *The cusp algebra $\mathcal{V}_{\mathrm{cu}}(X)$ defined by a choice of boundary defining function $\rho \in \mathcal{C}^\infty(X)$ on a compact manifold with boundary lifts, from the right or left factor, to a space of smooth vector fields on X^2_{cu} (defined of course by the same choice of ρ) to be transversal to the lifted diagonal, which is an interior p-submanifold $\mathrm{Diag}_{\mathrm{cu}} \subset X^2_{\mathrm{cu}}$. The left and right stretched projections, $\pi_{O,\mathrm{cu}} = \pi_O \circ \beta_{\mathrm{cu}}$, $O = L, R$, are b-fibrations which are transversal to $\mathrm{Diag}_{\mathrm{cu}}$.*

Proof. That the cusp algebra lifts, we know from Lemma 5. In any case it is a rather straightforward computation which I will do! We can ignore ∂_{y_j} throughout and we have only to deal with the one vector field, which starts off as $x^2\partial_x$. After we lift it to X^2_{b} it is $x's^2\partial_s$ in terms of the coordinates $x', s = x/x'$ which are valid near the boundary of the diagonal. We can drop the s^2 since it is non-vanishing and switch from s to $t = s-1$ which has the virtue of vanishing at $B_2 = \{x' = 0, t = 0\}$ and our vector field is a non-vanishing smooth multiple of $x'\partial_t$. The lifted diagonal is $y = y'$ $t = 0$ and near it we can use the singular coordinates x' and $t_2 = t/x'$ (together with y and y'). In terms of these our vector field has become ∂_{t_2}, so with ∂_{y_j} we do indeed get a set of smooth vector fields transversal to the interior p-submanifold $\mathrm{Diag}_{\mathrm{cu}}$.

So, X^2_{cu} does resolve $\mathcal{V}_{\mathrm{cu}}(X)$. Now we need to check that we haven't gone too far somehow. So, $\pi_{R,\mathrm{cu}}$ is a well-defined b-map. Why is it a b-submersion? Consider the vector field $x\partial_x + x'\partial_{x'}$. This is in $\mathcal{V}_{\mathrm{b}}(X^2)$ and so lifts to X^2_{b} to be smooth. Near B_2 it has become

$$(t-1)\partial_t + (x'\partial_{x'} - (t-1)\partial_t) = x'\partial_{x'}.$$

in terms of the coordinates $t = s-1 = (x-x')/x'$ and x'. Here the first term is the lift of $x\partial_x$ and the second is the lift of $x'\partial'_x = -s\partial_s + x'\partial_{x'}$ in the new coordinates. Thus it is certainly tangent to B_2 and so lifts to be smooth on X^2_{cu} by Lemma 5. But this means that the vector field to which it lifts pushes forward under $\pi_{L,\mathrm{cu}}$ (or $\pi_{R,\mathrm{cu}}$ for that matter) to $x\partial_x$ on X. So in fact $(\pi_{L,\mathrm{cu}})_* : {}^{\mathrm{b}}T_pX^2_{\mathrm{cu}} \longrightarrow {}^{\mathrm{b}}T_{p'}X$ must always be surjective! Since the image manifold is a manifold with boundary the additional condition of b-normality is void. Thus $\pi_{L,\mathrm{cu}}$ is a b-fibration. That this b-fibration is transversal to $\mathrm{Diag}_{\mathrm{cu}}$ is the statement that the null space of the (ordinary or b-) differential $(\pi_{R,\mathrm{cu}})_*$ contains a complement to the tangent space of $\mathrm{Diag}_{\mathrm{cu}}$ at each point. This we already know, since the lifts from the left factor of elements of $\mathcal{V}_{\mathrm{cu}}(X)$ must be killed by $\pi_{R,\mathrm{cu}}$, and this lift spans such a complement at each point. □

Now, having done this in the cusp case, $N = 2$, I may as well go on into the higher cusp cases. First try $N = 3$. Then we have $x^3\partial_x$ in place of $x^2\partial_x$. So, when we lift it up to X^2_{cu} from the left, we can see from the computation above that in the coordinates t_2, x' (and always y, y') that we get a smooth positive multiple of $x'\partial_{t_2}$. So, we have to blow up $B_3 = \{t_2 = 0, x' = 0\} \subset X^2_{\mathrm{cu}}$. Thus we conclude that

$$\mathcal{V}_{3\mathrm{c}}(X) \text{ is resolved on } X^2_{3\mathrm{c}} = [X^2_{\mathrm{cu}}; B_3]. \tag{14}$$

In fact the same argument clearly works for any N. Proceeding by induction we can claim that the functions

$$t_N = \frac{x - x'}{(x')^N}, x', y, y' \tag{15}$$

lift to X^2_{Nc} to give coordinates near the lifted diagonal in which it becomes $t_N = 0$, $y = y'$ and such that $x^{N+1}\partial_x$ lifts from the left factor to be a smooth positive multiple of $x'\partial_{t_N}$. Then we can define $B_{N+1} = \{x' = 0, t_N = 0\}$ and define the next space as

$$X^2_{(\mathrm{N}+1)\mathrm{c}} = [X^2_{\mathrm{Nc}}; B_N]. \tag{16}$$

Proposition 7. *Proposition 6 carries over to the N-cusp algebra with X^2_{cu} replaced by X^2_{Nc}.*

With this behind us, we can define

$$\Psi^k_{\mathrm{Nc}}(X) = \{A \in I^k(X^2_{\mathrm{Nc}}, \mathrm{Diag}_{\mathrm{Nc}}; \pi^*_{R,\mathrm{Nc}}\Omega_{\mathrm{Nc}}); A \equiv 0 \text{ at } \partial X^2_{\mathrm{Nc}} \setminus \mathrm{ff}_{\mathrm{Nc}}\}, \tag{17}$$

where $\Omega_{\mathrm{Nc}} = \rho^{-Nn+1}\Omega_{\mathrm{b}} = \rho^{-Nn}\Omega$.

Exercise 26. I leave it to you to show how to define the operators on sections of vector bundles.

There are lots of things to say about these operators, and I will say at least some of them. The first thing is to see that $\mathrm{Diff}^k_{\mathrm{Nc}}(X) \subset \Psi^k_{\mathrm{Nc}}(X)$. The place to start here is the identity operator! In local coordinates it can be written

$$\mathrm{Id}\, u(x, y) = \int \delta(x - x')\delta(y - y')u(x', y')|dx'dy'|. \tag{18}$$

To lift the kernel up to X^2_{b} we need to introduce say $s = x/x'$ as variable in place of x. Since it is essentially a parameter we can use the fact that the delta 'function' is homogeneous of degree -1, so

$$\delta(x - x') = (x')^{-1}\delta(s - 1) = (x')^{-1}\delta(t). \tag{19}$$

But the factor of x' just turns dx' into dx'/x' and we get a coefficient b-density:

$$\delta(t)\delta(y - y')|\frac{dx'}{x'}dy'| \in \Psi^0_{\mathrm{b}}(X). \tag{20}$$

We can continue this way up to X^2_{Nc} to see that

$$\mathrm{Id} = \delta(t_n)\delta(y - y')|\frac{dx'}{(x')^N}dy'| \in \Psi^0_{\mathrm{Nc}}(X). \tag{21}$$

Clearly it is elliptic, since it has symbol 1.

Exercise 27. Now use the fact that $\mathcal{V}_{\mathrm{Nc}}(X)$ lifts from the left fact to be smooth vector fields inn $\mathcal{V}_{\mathrm{b}}(X^2_{\mathrm{Nc}})$ to show that $\mathrm{Diff}^k_{\mathrm{Nc}}(X) \subset \Psi^k_{\mathrm{Nc}}(X)$.

Proposition 8. *The elements of $\Psi^m_{\mathrm{Nc}}(X)$, for any $m \in \mathbb{R}$, define continuous linear operators on $\mathcal{C}^\infty(X)$.*

Proof. This is an application of the push-forward theorem using the fact that the stretched projections are b-fibrations; it is also necessary to sort out the behaviour of the density factors. □

Proposition 9. *The $\Psi^k_{\mathrm{Nc}}(X)$ form an order-filtered asymptotically complete *-algebra of operators on $\mathcal{C}^\infty(X)$ with multiplicative symbol map giving a short exact sequence*

$$\Psi^{m-1}_{\mathrm{Nc}}(X) \hookrightarrow \Psi^m_{\mathrm{Nc}}(X) \longrightarrow (S^m/S^{m-1})({}^{\mathrm{Nc}}T^*X) \tag{22}$$

and each $A \in \Psi^0_{\mathrm{Nc}}(X)$ is bounded on $L^2(X)$.

Proof. So, I have left a bit of a hole in the preparation for the product formula – in particular I don't quite have the machinery in place to prove even the composition formula in the boundaryless case, so I will have to talk about that too. So this whole proof will take a little while – maybe you should bypass it as I will do in the lecture!

First we consider the composition formula for operators of order $-\infty$, which of course is part of the claim. The idea here is exactly the same as before. We want to find a triple product with appropriate properties. This will involve a bit of an effort. Let's start with the triple b-product X^3_{b}. We already know that X^3_{b} maps back under $\pi O, \mathrm{b}$ to X^2_{b}. Now, inside X^2_{b} we have the submanifold B_2 that we need to blow up to turn X^2_{b} into X^2_{cu}. So, we consider the inverse image $\pi^{-1}_{O,\mathrm{b}}(B_2) = B_{2,O}$ for $O = F, S, C$. Two of the boundary faces of X^3_{b} are mapped into the front face of X^2_{b} under each of the stretched projections so we actually get two, intersecting, boundary p-submanifolds as the preimage of B_2 from each of the projections. Somewhere there are some pictures of what is going on!

□

In particular our elliptic construction from the boundaryless case carries over unchanged.

Proposition 10. *If $P \in \Psi^k_{\mathrm{Nc}}(X; E, F)$ is elliptic then there exists $Q \in \Psi^{-k}_{\mathrm{Nc}}(X; F, E)$ such that $P \circ Q - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{Nc}}(X; F)$ and $Q \circ P - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{Nc}}(X; E)$.*

One important point is that, as in the conic case, $(x/x')^s$ is a multiplier on the space $\Psi^k_{\mathrm{Nc}}(X; E, F)$, although as we shall see, much more is true for $N > 1$. Anyway, by conjugating and using ellipticity it follows that

$$u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*),\ \rho^N(d+\delta)u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*) \Longrightarrow u \in \rho^{-\frac{Nn}{2}} H^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*) \tag{23}$$

Here I have put in the weight that comes from the metric. The Sobolev space on the right is just that based on $\mathcal{V}_{\mathrm{Nc}}(X)$, so $u \in H^p_{\mathrm{Nc}}(X)$ just means that $\mathcal{V}_{\mathrm{Nc}}(X)^j u \subset L^2_{\mathrm{Nc}}(X)$ for all $j \le p$. This of course applies to the maximal, ungraded, domain

$$D_{\max} = \left\{u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*); (d+\delta)u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*)\right\} \subset \rho^{-\frac{Nn}{2}} H^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*) \tag{24}$$

So, we then want to work out more precisely what this domain, and the various smaller ones we have defined, D, D_A, D_R and $D_{\min}$ are. This turns out to be fairly straightforward when $N \ge 2$ using the finer conjugation property anticipated above. Namely, for any real τ,

$$\exp(i\tau(\frac{1}{x} - \frac{1}{x'}) \text{ is a multiplier on } \Psi^k_{\mathrm{cu}}(X). \tag{25}$$

More generally

$$\exp(i\tau(x^{-N} - x'^{-N}) \text{ is a multiplier on } \Psi^k_{\mathrm{Nc}}(X). \tag{26}$$

Exercise 28. Check this by seeing what happens to the function when lifted to X^2_{Nc}; i.e. show that it is smooth except at the part of the boundary where the kernels are assumed to vanish rapidly where it only has a singularity of finite order.

Now, the maximal graded domain

$$D = \left\{u \in L^2_g(X;\Lambda^*); du, \delta u \in L^2_g(X;\Lambda^*)\right\} \tag{27}$$

with norm $\|u\|^2_D = \|u\|^2_{L^2} + \|du\|_{L^2} + \|\delta u\|_{L^2}$ and the relative and absolute domains

$$D_R = \{u \in D; \exists \dot{\mathcal{C}}^\infty(X;\Lambda^*) \ni u_n \to u \text{ in } L^2_g(X;\Lambda^*) \text{ s.t. } du_n \to du \text{ in } L^2_g(X;\Lambda^*)\},\ D_A = *D_R. \tag{28}$$

Lemma 6. *If $k = \frac{n-1}{2}$ (so n is odd) and $U : H^k_{\mathrm{Ho},h_0}(\partial X;\Lambda^k) \longrightarrow D$, and $V : H^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X) \longrightarrow D$ such that*

$$U\phi = \phi + \rho\mathcal{C}^\infty(X;\Lambda^k),\ V\phi = d\rho \wedge \phi + \rho\mathcal{C}^\infty(X;\Lambda^{k+1}). \tag{29}$$

Proof. For $\phi \in H^k_{\mathrm{Ho},h_0}(X)$, let $\phi(x)$ be the representative harmonic with respect to the varying metric $h(x,y,dy,0)$. The termss in dx can be suppressed since these are already $O(x^2)$ with respect to the metric. It then follows that $U\phi = \chi\phi(x)$, for an appropriate cut-off χ, is in D and then we can take $V\phi = *U\phi$. □

Basically there is nothing else!

Theorem 2. *For a cusp metric as in* (1) *the graded L^2 domain*

$$D = \overline{x^{-\frac{n}{2}+1}H^1_{\mathrm{cu}}(X;{}^{\mathrm{cu}}\Lambda^*)} + UH^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X) + VUH^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X), \tag{30}$$

where the closure is with respect to $\|\cdot\|_D$, and the relative domain

$$D_R = \overline{x^{-\frac{n}{2}+1}H^1_{\mathrm{cu}}(X;{}^{\mathrm{cu}}\Lambda^*)} + UH^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X) \tag{31}$$

and with this domain, $d+\delta$ is a self-adjoint Fredholm operator with consequent Hodge decomposition

$$L^2(X;{}^{\mathrm{cu}}\Lambda^*) = H^*_g(X) \oplus dD_R \oplus \delta D_R \tag{32}$$

and null space canonically isomorphic to the L^2 deRham cohomology.

Proof. This involves computations similar to, but easier than, those in the conic case. □