11. Lecture XI: October 14

Last time I defined (again) the space $\Psi_{\mathrm{b}}^{-\infty}(X)$ and showed that its elements are operators on $\dot{\mathcal{C}}^\infty(X)$. Today I want to prove that they form an algebra and discuss some of its properties, relating them in general to geometric properties of X_{b}^2 ('the b-double space'). As a warm-up exercise, that turns out to be close to the proof of the composition theorem, let me discuss

Proposition 4. *The elements of* $\Psi_{\mathrm{b}}^{-\infty}(X)$ *act on* $\mathcal{C}^\infty(X)$.

Of course this is also important in its own right, as a further justification that the elements of $\Psi_{\mathrm{b}}^{-\infty}(X)$ act on 'almost everything'.

Proof. We are supposed to get this action in the same way as the action on $\dot{\mathcal{C}}^\infty(X)$. Notice that $\pi_{\mathrm{b},R}^* : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X_{\mathrm{b}}^2)$ so the big difference is that we do not have vanishing at the preimage of the boundary, hence not at $\mathrm{ff}(X_{\mathrm{b}}^2)$. We can summarize the operations in the little diagram

(1)
$$\begin{tikzcd} & X_{\mathrm{b}}^2 \arrow[dl, "\pi_{L,\mathrm{b}}"'] \arrow[dr, "\pi_{R,\mathrm{b}}"] & \\ X & & X \end{tikzcd}$$

In fact it is clear from the definition that

(2) $$\Psi_{\mathrm{b}}^{-\infty}(X) \text{ is a } \mathcal{C}^\infty(X_{\mathrm{b}}^2)\text{-module.}$$

Thus, in trying to show that when we integrate $A\pi_{\mathrm{b},R}^* f$ over the right factor of X we get an element of $\mathcal{C}^\infty(X)$, we might as well forget about f and just integrate a general A. Thus we are trying to show that the push-forward map to the left factor gives

(3) $$(\pi_{\mathrm{b},L})_* : \Psi_{\mathrm{b}}^{-\infty}(X) \longrightarrow \mathcal{C}^\infty(X).$$

Note that if the map in question was a fibration, as the left projection from X^2 is, then this is a version of Fubini's theorem. However $\pi_{\mathrm{b},L}$ is *NOT* (quite) a fibration. If it were a fibration then (3) would be true without the vanishing conditions on the 'old boundary faces' which are inherent in the definition of $\Psi_{\mathrm{b}}^{-\infty}(X)$ and $\mathcal{C}^\infty(X_{\mathrm{b}}^2; \pi_{\mathrm{b},R}^*\Omega_{\mathrm{b}})$ itself would push-forward into $\mathcal{C}^\infty(X)$; it does not, so we have to make use of this vanishing.

So, to business. To prove (3) we can work locally on X_{b}^2. Indeed using a partition of unity we can cut the kernel, A, up into small pieces and assume that it has support in the preimage of the product of coordinate neighbourhoods in the two factors of X. If we are away from the front face of X_{b}^2, so away from the corner 'downstairs' in X^2, then (3) is obvious – the map is locally a fibration and in any case we are back to the previous result and the image is actually in $\dot{\mathcal{C}}^\infty(X)$.

Thus, we can assume that A has its support in a 'polar coordinate' neighbourhood $[0,\epsilon)_r \times [-1,1]_t \times U \times U'$ where U, U' are open neighbourhoods of $0 \in \mathbb{R}^{n-1}$ with coordinates y, y' and $r = x+x'$, $t = (x-x')/r$ are projective polar coordinates. Then

(4) $$A = a\frac{dx'}{x'}dy',\ a \in (1-t)^k(1+t)^{k'}\mathcal{C}_c^\infty([0,\epsilon) \times [-1,1] \times U \times U')\ \forall\ k, k'.$$

Here the factors of $1-t$ and $1+t$ reflect the assumed rapid vanishing at the old boundaries, which are $t=\pm1$. In fact, because we want to discuss the map back to x,y variables, it is convenient to introduce the *singular* projective coordinates

$$s = x'/x, x, y, y' \text{ so } t = \frac{1-s}{1+s},\ r=(s+1)x. \tag{5}$$

These are valid coordinates in some region $0 \le r \le \epsilon$, $t \in (-1,1]$ with $t=1,-1$ corresponding to $s=0,\infty$. My claim is that, despite the singularity of these coordinates, we can translate the conditions on a to imply

$$a'(s,x,y,y') = a(r,t,y,y') \Longrightarrow a' \in \mathcal{S}([0,\infty); \mathcal{C}^\infty_c([0,\delta)\times U \times U')). \tag{6}$$

By this I just mean that a' is $\mathcal{C}^\infty$ has support contained in $[0,\infty)\times K$ for some compact $K \subset [0,\delta)\times U\times U'$ and all derivatives (meaning in s, x, y and y' of all orders) vanish rapidly as $s\to\infty$.

This is clear in $0\le s\le S$ where the coordinates are legitimate. In $s \ge S > 0$ for any fixed S, we can introduce $s' = 1/s$ $(= x/x')$ taking values in $(0,1/S)$. We still do not quite get legitimate coordinates since $t=\frac{s'-1}{s'+1}$ is fine, but $r=(1+s')x/s'$ is not smooth. Since $x = x's'$, s', x' *are* legitimate coordinates in this region, with $r=(1+s')x'$ so we do get a smooth function, b, of s', x', y, y' which vanishes to infinite order at $s'=0$ and has bounded support in x'. Notice that such a function can indeed be written as a smooth function of s', x, y, y' :

$$b'(s',x,y,y') = b(s', \frac{x}{s'}, y, y') \tag{7}$$

because the singularities in the second variable, as $s'\to 0$ are swamped by the rapid decay in s'. For instance we can write

$$b = (s')^N b_N(s', \frac{x}{s'}, y, y'),\ b_N\ \mathcal{C}^\infty,$$

from which it follows that the first $N-1$ derivatives in x are continuous down to $s=0$. Now, for a function to be smooth and vanish to infinite order at $s'=0$ is equivalent to its being 'Schwartz' in the variable $s=1/s'$ near $s=\infty$. Thus we do really have (6).

Exercise 16. Prove the converse to (6) that this (with the correct support constraints) does actually characterize the kernels of elements in $\Psi_{\mathrm{b}}^{-\infty}(X)$.

Finally then we can write our push-forward integral as

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} a(\frac{x'}{x}, x, y, y')\frac{dx'}{x'}dy' \tag{8}$$

where the supports in x' and y' are actually bounded. Changing variable from x' to $s=x'/x$ this becomes

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} a(s,x,y,y')\frac{ds}{s}dy' \tag{9}$$

Note that the measure has 'miraculously' become regular except at $s=0$, ∞ where we have corresponding rapid vanishing (or decay) in the integrand. Thus the integral (9) converges absolutely and uniformly to a smooth function of x and y. This is what we need to prove. □

This proof is a bit hands-on for my taste! For later purposes I will generalize this result and make it more geometric. The results I will formulate next will first (as usual you might say) be used to prove something worthwhile, in this case the composition theorem, and then later it will be proved. The proof can be based on computations in singular coordinates just like that above, but there are other approaches too.

First think about the properties of smooth maps between compact manifolds with corners. We know what smoothness means already, but we need to add some conditions as to how boundaries are mapped. Recall that the boundary hypersurfaces each have defining functions (if you like these are simply generators of the $\mathcal{C}^\infty$-module of functions which vanish on the boundary hypersurface in question), ρ_H for each $H \in M_1(X)$.

Definition 2. A smooth map $F : X \longrightarrow X'$ is a *b-map* if each boundary defining function $\rho'_{H'}$, $H' \in M_1(X')$ pulls back to a product of boundary defining functions for X :

$$f^*\rho'_{H'} = a_{H'} \prod_{H \in M_1(X)} \rho_H^{e(H,H')},\ a < a_{H'} \in \mathcal{C}^\infty(X). \tag{10}$$

It is an interior b-map if it maps the interior of X into the interior of X'. It is a *simple* b-map if it is a b-map and in addition the exponents $e(H, H')$ take only the values $0, 1$. It is a *b-normal* map if it is a b-map and in addition for each $H \in M_1(X)$ there is at most one $H' \in M_1(X')$ such that $e(H, H') \neq 0$.

A simple b-normal map is one which is simple and b-normal, etc, duh.

Exercise 17. Translate these definitions into statements about the behaviour of the ideals corresponding to boundary faces.

Now recall that the b-cotangent bundle ${}^{\mathrm{b}}T^*X$ is the ordinary cotangent bundle in the interior, but near a boundary face has as local basis the 'logarithmic differentials' dx_i/x_i and dy_j in terms of our usual adapted coordiantes. The b-tangent bundle, its (pre-)dual, has corresponding basis $x_i\partial_{x_i}$, ∂_{y_j}.

Proposition 5. *Any interior b-map the usual differential on the interior extends by continuity to a 'b-differential' and its dual*

$$f^{*\,\mathrm{b}} : {}^{\mathrm{b}}T^*_{f(p)}X' \longrightarrow {}^{\mathrm{b}}T^*_pX,\ f_{*\,\mathrm{b}} : {}^{\mathrm{b}}T_pX' \longrightarrow {}^{\mathrm{b}}T_{f(p)}X, forall p \in X. \tag{11}$$

Note that despite some danger of confusion, I will generally denote this 'new' differential by f^* or f_*, just like the usual one.

Exercise 18. See if you can carry the proof through.

Definition 3. An interior b-map $f : X \longmapsto X'$ is said to be a *b-submersion* if it is surjective and $f_{*\,\mathrm{b}} = f_* : T_pX \longrightarrow T_{f(p)}X'$ is surjective for each $p \in X$. A b-submersion which is also b-normal is said to be a *b-fibration*

Exercise 19. Check that these definitions are not at all vacuous!

(1) Show that the blow-down map $\beta : [X, F] \longrightarrow X$ for F a boundary face of a manifold with corners is always a b-submersion but not a submersion in the usual sense unless F is a boundary hypersurface (in which case it is the identity map).

(2) Show that this blow-down map is never b-normal, and hence is not a b-fibration, unless H is a boundary hypersurface.
(3) Show that the 'stretched projection' $\pi_{L,\mathrm{b}} : X^2_{\mathrm{b}} \longrightarrow X$ is a b-fibration but is not a fibration in the usual sense.

I will discuss the general structure of b-fibrations, and so on, later. For the moment I will just quote a push-forward result

Theorem 1. *For a simple b-fibration f, suppose for each $H' \in M_1(X')$ for which $e(H,H') \neq 0$ for some $H \in M_1(X)$ a particular such $H = p_f(H')$ is chosen, then push-forward (fibre-integration) gives a map*

$$f_* : \big\{u \in \mathcal{C}^\infty(X;\Omega_{\mathrm{b}}); u \equiv 0 \text{ at } H \in M_1(X) \\ \text{unless } H = p_f(H') \text{ for some } H' \in M_1(X')\big\} \longrightarrow \mathcal{C}^\infty(X';\Omega_{\mathrm{b}}). \tag{12}$$

Of course you are very welcome to try to prove this, but it is easier when we have a little more machinery at our disposal. For the moment I suggest

Exercise 20. Show that this theorem does imply Proposition 4 in the form (3). Hint: Since the theorem deals with b-densities and (3) is about 'partial' b-densities, something has to be done! First show that there is a natural isomorphism

$$(\pi_{L,\mathrm{b}})^*\Omega_{\mathrm{b}}(X) \otimes (\pi_{R,\mathrm{b}})^*\Omega_{\mathrm{b}}(X) \equiv \Omega_{\mathrm{b}}(X^2_{\mathrm{b}}) \tag{13}$$

(Hint-within-a-hint, the corresponding statement on X^2 is true). Now to get (3), choose a positive b-density $0 < \nu_{\mathrm{b}} \in \mathcal{C}^\infty(X;\Omega_{\mathrm{b}})$ and show that Theorem 1 can be applied to $\Psi^{-\infty}_{\mathrm{b}}(X) \cdot \pi^*_{L,\mathrm{b}}\nu_{\mathrm{b}}$. Check that the result is independent of the choice of ν_{b}.

Despite appearances there is something going on here to do with b-densities as opposed to ordinary densities.