10. Lecture X; in part

So, I had some difficulties in this lecture! Zhang Zhou pointed out subsequently that the proof of self-adjointness of $d+\delta$ with domain D_R is inadequate. This is a case of me trying to avoid work earlier, only to cause trouble later. The difficulty is that from the definition of D_R^*,

$$D_R^* = \big\{u \in L^2_g(X;\Lambda^*); D_R \ni \phi \longmapsto \langle (d+\delta)\phi, u\rangle \text{ extends by continuity to } L^2_g(X;\Lambda^*)\big\} \tag{1}$$

it does not follow directly that the domain is order-graded. That is, we certainly deduce that $(d+\delta)u \in L^2_g$ for the distributional action of the differential operator but we do not know that du, δu are separately in L^2_g. So, we are forced to go back to the earlier analysis and work harder to derive not just the structure of D but the structure of the, in general larger, space

$$D_{\max}(d+\delta) = \big\{u \in L^2_g(X;\Lambda^*); (d+\delta)u \in L^2_g(X;\Lambda^*)\big\}\,. \tag{2}$$

As I said, I should have done this directly but maybe it is better to postpone it to this point where we have the experience to do it relatively easily.

Proposition 2. *The maximal domain $D_{\max}(d+\delta) = D + U'$ where U' is the finite dimensional vector space which for even n is*

$$U' = \chi(x)\,\mathrm{sp}\{x^{-is_\sigma+k}((is-(n-k-2))\psi_k - dx\wedge d\omega_k)\tag{3}$$

where the linear span is over k and coexact k-forms ψ_k which are eigenforms of the boundary Laplacian, $\Delta\psi_k = \sigma\psi_k$ with

$$-is = -\frac{n}{2} + 1 - \sqrt{(\frac{n}{2}-1-k)^2+\sigma} > -\frac{n}{2}. \tag{4}$$

It follows immediately that such forms can occur only in for dimensions $k = \frac{n}{2}-1$ if n is even of $k = \frac{n}{2}-1\pm\frac{1}{2}$ if n is odd. The forms in (3) can never be degree-graded, unless zero of course.

Proof. Elliptic regularity applies as before to show that if $u \in D_{\max}(d+\delta)$ for a conic metric then $u \in x^{-\frac{n}{2}}H^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*)$. Thus, exactly as with the discussion of D, the space $D_{\max}$ is the same for any two metrics with the same boundary metric h_0. We can therefore work with a product-type metric and analyse the conditions under which

$$v = \chi\sum_k (x^k u_{t,k}(x) + x^{k-1}dx\wedge u_{n,k-1}(x)) \tag{5}$$

is such that $(d+\delta)u \in L^2_g$, given that u itself is in L^2_g, which is just the condition

$$u_{n,*}, u_{t,*} \in x^{-\frac{n}{2}}L^2_{\mathrm{b}}(X;\Lambda^*(\partial X)). \tag{6}$$

The Hodge decomposition on the boundary allows these tangential and normal parts to be divided. Namely we set

$$L^2(\partial X;\Lambda^*) + L^2(\partial X;\Lambda^*) = H + G + U \tag{7}$$

where H is the harmonic part in all degrees, G is the part we discussed extensively before

$$u \in G \Longleftrightarrow u_n \in dH^1(\partial X;\Lambda^*),\ u_t \in \delta H^1(\partial X;\Lambda^*) \tag{8}$$

and U is the remaining part

$$u \in U \Longleftrightarrow u_n \in \delta H^1(\partial X; \Lambda^*),\ u_t \in dH^1(\partial X; \Lambda^*). \tag{9}$$

The harmonic part is finite dimensional, given by a smoothing operator applied to (u_t, u_n) whereas the components in G and U are given by the action of pseudodifferential projections of order 0. This means that, as necessary, we can track their regularity in Sobolev spaces.

The point of the decomposition (7) is that it is directly related to the action of $d+\delta$, in the product-conic case. Thus, $d_t + \delta_t$ maps exact to coexact forms and conversely so the components of (u_n, u_t) in H, G and U are mapped, respectively, into H, U and G so must separately take values in L^2_g. The H and G components were analysed earlier, so consider the component in U. From the form of d and δ (and taking care to get the value of k right for the action in a given form degree) we arrive at the condition

$$-\delta_0 u_{n,k+1} + (x\partial_x + k)u_{t,k} \in xL^2_g(-x\partial_x - (n-k-2))u_{n,k+1} + du_{t,k} \in xL^2_g \tag{10}$$

as conditions between the tangential component in degree k and the normal component in degree $k+1$ (as opposed to $k-1$ for G.)

As before we analyse the degree to which the condition (10) does *not* imply that $u \in x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$ by using the Mellin transform to find any possible poles in the strip $-\frac{n}{2} < \operatorname{Im} s \le -\frac{n}{2}+1$. Such poles must satisfy

$$-\delta\phi_{k+1} + (-is+k)\psi_k = 0,\ (is - (n-k-2))\phi_{k+1} + d\psi_k = 0 \tag{11}$$

where ϕ_{k+1} is exact and ψ_k is coexact. Eliminating between the equations as before gives

$$\delta\delta\psi_k + (is - (n-k-2))(-is+k)\psi_k = 0 \tag{12}$$

which is to say that $(is - n + k + 2)(is - k) = \sigma$ must be a positive eigenvalue of Δ acting on coexact k-forms. Completing the square we find

$$-is = -\frac{n}{2} + 1 \pm \sqrt{(\frac{n}{2} - 1 - k)^2 + \sigma}. \tag{13}$$

This of course is pure imaginary and can lie in the 'critical strip' only when the sign is $-$ and then only when $k = \frac{n}{2} - \frac{3}{2}$, $k = \frac{n}{2} - 1$ or $k - \frac{n}{2} - \frac{1}{2}$ and only for correspondingly small eigenvalues, namely $\sigma_{\frac{n}{2}\frac{3}{2}} < \frac{3}{4}$, $\sigma_{\frac{n}{2}1} < 1$ and $\sigma_{\frac{n}{2}\frac{1}{2}} < \frac{3}{4}$. In particular there are never such 'accidental poles' on the line $-is = -\frac{n}{2} + 1$. These poles can be removed by subtracting a term as in (3). $\square$

Now, the defect form Q is defined on the whole of $D_{\max}$:

$$Q(u,v) = \int_X (\langle (d+\delta)u, v\rangle - \langle u, (d+\delta)v\rangle)\, dg. \tag{14}$$

Moreover by the approximability conditions already discussed it vanishes if either factor is in

$$\begin{aligned} D_{\min}(d+\delta) = \big\{&u \in L^2_g(X; \Lambda^*); \\ &\exists u_n \to u \text{ in } L^2_g,\ u_n \in \dot{\mathcal{C}}^\infty(X; \lambda^*),\ (d+\delta)u_n \to (d+\delta)u \text{ in } L^2_g\big\}. \end{aligned} \tag{15}$$

For the moment we know at least that $D_{\min}$ contains all put the finite dimensional parts E^*_A, E^*_R, G^* in D and U^* in $D_{\max}$. There remains a little computation to do:

Lemma 2. *The defect (or boundary) pairing Q defines non-degenerate pairings between E_A^* and E_R^* and also between G^* and U^* and vanishes on all other pairings.*

Proof. Well, we already know the first part. The second part follows by a similar integration-by-parts argument and the eigendecomposition of boundary forms. □

Exercise 10. Carry through the argument here!

So, with this extra work we can see why

$$D_R^* = D_R. \tag{16}$$

Namely, $u \in D_R^*$ certainly implies that $u \in D_{\max}(d+\delta)$. Then the definition implies that Q must vanish on $D_R \times D_R^*$. The fact that $G^* \subset D_R$ and the lemma above then shows that $U^* \cap D_R^* = \{0\}$, which is to say $D_R^* \subset D$ where the previous argument, just the pairing argument for E_*^* takes over and shows that $D_R^* \subset D_R$, and hence they are equal.

Exercise 11. Use the same argument to decide on the exact identity of $D_{\min}(d+\delta)$. Show that an self-adjoint operator $\eth_B$ which is given by $d+\delta$ acting on some domain D_B with $D_{\min} \subset B \subset D_{\max}$ corresponds to a maximal subspace of $E^* + G^* + U^*$ on which Q vanishes.

Since my handwritten lecture notes for today are at best misleading on blow-up of a boundary face of a manifold with corners I have typed up something closer to what I actually said.

Rather than just define $X_b^2 = [X^2; (\partial X)^2]$, which we need for the definition of the algebra $\Psi_b^{-\infty}(X)$, I will give the general definition of $[Z, F]$ where F is a boundary face of a compact manifold with corners, Z. By definition (and this is what makes this easier than the general case of an appropriately embedded submanifold) there are global defining functions for F. Namely, if H_i are the boundary hypersurfaces of Z which contain F and x_i are defining functions for the H_i, $i = 1, \dots, k$ then the simultaneous vanishing of the x_i defines F, at least locally near F (there may be other components of the intersection of these k hypersurfaces).

Now to define a manifold it suffices to give the space of smooth functions on it. We can set

$$\begin{aligned} &\mathcal{C}^\infty([Z; F]) = \\ &\{u \in \mathcal{C}^\infty(Z \setminus F); u \text{ is smooth in any normal polar coordinates at a point of } F\}. \end{aligned} \tag{17}$$

Here, by normal polar coordinates, I mean polar coordinates in the defining functions x_i. Thus the local coordinates are x_i, y_j, $j = 1, \dots, n-k$. By polar coordinates I will, for the moment, mean 'projective' polar coordinates. These are the k functions

$$r = x_1 + \cdots + x_k,\ t_i = \frac{x_i}{r},\ i = 1, \dots, k-1,\ y_j. \tag{18}$$

Notice that (as always locally near F) $r = 0$ only at F. It is only because it is a 'corner' of Z that we can do this. We can replace any one of the 'angular' variables t_i by $t_k = \frac{x_k}{r}$, or more generally take any $k-1$ of these k variables as coordinates. To see that the definition (17) really makes sense, we need to show that it does not actually depend on the choices of the x_i and y_j, although the latter is pretty obvious.

Lemma 3. *If $F : U, 0 \longrightarrow U', 0$ is a diffeomorphism of (relatively) open subsets of $\mathbb{R}^{n,k}$ with $F(0) = 0$ then the pull-back under F of any $\mathcal{C}^\infty$ function of the polar coordinates r, $t_1, \dots, t_{k-1}$, y_j is also a $\mathcal{C}^\infty$ function of these variables.*

Proof. It suffices to show that the pull-pack functions F^*r, F^*t_i and F^*y_j are $\mathcal{C}^\infty$ functions of r, t_i and y_j, since then the same is true for any smooth function of these variables. It is clear than any relabelling of the x_i has this property, so we can assume that F maps each of the boundary hypersurfaces x_i into itself (rather than permuting them). Thus $F^*x_i = a_i x_i$ with $0 < a_i$ a $\mathcal{C}^\infty$ functions near 0 on $\mathbb{R}^{n,k}$. Since $x_i = t_i r$, it follows that

$$F^*r = \sum_{i=1}^{k} F^*x_i = (\sum_{i=1}^{k} a_i t_i) r = \alpha r,\ 0 < \alpha. \tag{19}$$

Here we use the fact that the t_i, $1 = 1, \dots, k-1$ take values in the standard simplex in $\mathbb{R}^{k-1}$, i.e. $0 \le t_i \le 1$ and $t_1 + \dots + t_{k-1} \le 1$. Since $t_k = 1 - t_1 - \dots - t_{k-1}$ and the a_i in (19) are smooth and positive, it is indeed the case that the coefficient α is positive and a smooth function of the polar variables. From this the rest follows easily, since for instance

$$F^*t_i = \frac{F^*x_i}{F^*r} = \alpha^{-1} a_i t_i. \tag{20}$$

□

Exercise 12. Check that these projective coordinates are equivalent to polar coordinates in the usual sense. That is, show that the functions $R = (x_1^2 + \dots + x_k^2)^{\frac{1}{2}}$ and $\omega_i = x_i/R$ are smooth functions of r and the t_i and conversely. Notice that the ω_i are the coordinates of a vector in $\mathbb{S}^{k-1,k-1} = \mathbb{S}^{k-1} \cap \mathbb{R}^{k,k}$, and that any $k-1$ of the ω_i can be used as coordinates at a point on the sphere, except if there is one which takes the value 1 at the point, in which case only the others form a coordinate system.

Having shown that the definition (17) does actually make sense independent of coordinates, we need to check that the space of functions so defined is indeed the space of all $\mathcal{C}^\infty$ functions on a compact manifold with corners. To make this space concrete we can use the chosen defining functions x_i to identify it as

$$[Z, F] = (Z \setminus F) \cup \mathit{\Delta} \times F,\ \mathit{\Delta} = \{t \in \mathbb{R}^{k-1,k-1}; t_1 + \dots + t_{k-1} \le 1\}. \tag{21}$$

Proposition 3. *Once the defining functions x_i are fixed, the space in* (17) *gives* $[Z; F]$ *in* (21)*, for F a boundary face of a compact manifold with corners Z, a natural structure as a compact manifold with corners.*

Proof. Already proved really. We can identify a neighbourhood of F in Z with the product $F \times U$ where U is a neighbourhood of 0 in $\mathbb{R}^{k,k}$ of the form $x_1 + \dots + x_{<} \epsilon$, $\epsilon > 0$. Then the functions r and $t_i = x_i/r$ allows us to identify the part of the union in (21) consisting of $\setminus F$ and $\mathit{\Delta} \times F$ with $F \times \mathit{\Delta} \times [0, \epsilon)_r$. This is consistent with the definition of $\mathcal{C}^\infty([Z; F])$ and so gives the space a $\mathcal{C}^\infty$ structure. □

Exercise 13. If you want to define $[Z; F]$ as a set, canonically and not as in (21) by reference to some particular collection of defining functions, it is not hard to do; so do it! The usual way is to introduce the normal bundle to F. This is the quotient of the tangent bundle to Z over F, $T_F Z$, by the tangent bundle to F. Thus $N_p F = T_p Z / T_p F$ for all $p \in F$. It is a bundle of rank k over F and has a positive 'quadrant' bundle, namely the image of the tangent vectors which satisfy $Vx_i \geq 0$ for all i. This condition is independent of the choice of defining functions x_i. If we let N^+F denote this 'quadrant bundle' we can pass to the corresponding 'fractional sphere bundle' – really a bundle of simplices – given by the quotient by the fibre $\mathbb{R}^+$ action, $SN^+F = (N^+F \setminus 0)/\mathbb{R}^+$. After all this we can set, as a set,

$$[Z; F] = (Z \setminus F) \cup SN^+F. \tag{22}$$

Check that the choice of defining functions x_i gives a natural identification $SN^+F = F \times \Delta$ and the $\mathcal{C}^\infty$ structure induced on $[X; F]$ in (22) by this choice is actually independent of the choice.

Now, the blown up space comes with a smooth map back to the original

$$\beta : [X; F] \longrightarrow Z,\ \beta(r, t, y) = (rt, y), \tag{23}$$

which is independent of any choices, since it is just the canonical identification on the first part of (21), or (22), and the projection onto F on the second part. Under this map, the 'new' boundary face $r = 0$ is identified with F; sometimes I call $r = 0$ the 'front face' of the blow up, or if a sudden algebraic wave overcomes me, the (exceptional) divisor. The lifts (proper transforms) of the old boundary faces $x_i = 0$ are the $t_i = 0$, which are mapped smoothly onto them; I will often use the notation $\beta^*(H)$ for the lift of a boundary hypersurface and $\mathrm{ff}(\beta)$ for the front face. Notice however that

$$\beta^{-1}(\{x_i = 0\}) = \{r = 0\} \cup \{t_i = 0\} = \mathrm{ff}(\beta) \cup \beta^*\{x_i = 0\}, \tag{24}$$

so the preimage of a boundary hypersurface containing F is the union of its lift (proper transform) and the front face (divisor).

Exercise 14. Make sure you see that in this real setting, blowing up a boundary hypersurface does absolutely nothing.

Now, back to the matter at hand. We want do identify the space of 'order $-\infty$ b-pseudodifferential operators' on X with a space of smooth kernels on $X^2_{\mathrm{b}} = [X^2, (\partial X)^2]$. To do so, we should be careful and include the obligator right density factors. Since we are in this 'b-category' it is natural (and wise) to take the density to be a b-density.

To do so, let me introduct another little bit of notation. Since we will need to talk about the projections of X^2 onto the factors, set

$$\pi_R : X^2 \longrightarrow X,\ \pi_R(x, x') = x',\ \pi_L : X^2 \longrightarrow X,\ \pi_L(x, x') = x. \tag{25}$$

Then we need the corresponding 'stretched' maps from X^2_{b} :

$$\pi_{\mathrm{b},R} : X^2_{\mathrm{b}} \longrightarrow X,\ \pi_{\mathrm{b},R} = \pi_R \circ \beta,\ \pi_{\mathrm{b},L} : X^2_{\mathrm{b}} \longrightarrow X,\ \pi_{\mathrm{b},L} = \pi_L \circ \beta. \tag{26}$$

Definition 1. On any compact manifold with boundary we set

$$\Psi^{-\infty}_{\mathrm{b}}(X) = \{A \in \mathcal{C}^\infty(X^2_{\mathrm{b}}; \pi^*_{\mathrm{b},R}\Omega_{\mathrm{b}}); A \equiv 0 \text{ at } \beta^*(\partial X \times X) \cup \beta^*(X \times \partial X)\}. \tag{27}$$

recall that $u \equiv 0$ at H for a smooth function u indicates that it vanishes with its Taylor series at each point of H, so all derivatives vanish there.

Now, as we shall see, these are operators and form an algebra.

Let me show first that the act on the smallest reasonable space, $\dot{\mathcal{C}}^\infty(X)$. It is easy to see that the Schwartz kernel theorem applies here and shows that $A \in \Psi^{-\infty}_{\mathrm{b}}(X)$ does define an operator from $\dot{\mathcal{C}}^\infty(X)$ to $\mathcal{C}^{-\infty}(X)$, the space of extendible distributions (the dual of $\dot{\mathcal{C}}^\infty(X;\Omega)$. This is pretty unimpressive, and would not allow us to compose the operators. To do so, observe how the action should go. We want to 'define'

$$Af = \int_X A(z,z')f(z') \tag{28}$$

where $f \in \dot{\mathcal{C}}^\infty(X)$ and I have formally written z, z' for the two 'variables' in X. Notice that the kernel A is supposed to carry within it the density factor needed to carry out the integral.

Trying to interpret (28) rigourously, we have to think of A as a smooth function on X^2_{b}. The functin f is on X but in (28) this is clearly supposed to be interpreted as the right factor of X^2. So we can consider the product

$$A \cdot \pi^*_{\mathrm{b},R} f \in \dot{\mathcal{C}}^\infty(X^2_{\mathrm{b}}; \pi^*_{\mathrm{b},R}\Omega_{\mathrm{b}}). \tag{29}$$

Here we are using the obvious fact that $\pi^*_{\mathrm{b},R} f \in \mathcal{C}^\infty(X^2_{\mathrm{b}})$ which is just the smoothness of the map, but also the fact that it vanishes to infinite order at $\mathrm{ff} \cap \beta^*(X \times \partial X) = \pi^{-1}_{\mathrm{b},R}(\partial X)$, simply because f, by assumption, vanishes to all orders at the boundary. Now, A already vanishes to infinite order at the old boundaries so the product in (29) does, as claimed, vanish to infinite order at all boundaries.

Now, one elementary property of the blow-up procedure is that it induces an isomorphism on functions that are smooth and vanish to infinite order at all boundaries

$$\beta^* : \dot{\mathcal{C}}^\infty(Z) \longleftrightarrow \dot{\mathcal{C}}^\infty([Z;F]) \tag{30}$$

for the blow-up of any boundary face. This allows us to interpret the product in (29) as a section of the b-density bundle

$$Af \in \dot{\mathcal{C}}^\infty(X^2; \pi^*_R\Omega_{\mathrm{b}}) = \dot{\mathcal{C}}^\infty(X^2; \pi_R\Omega) \tag{31}$$

where we use the fact that the sections of Ω_{b} which vanish to infinite order at the boundary are the same, naturally, as the sections of Ω, the ordinary density bundle. Finally then we see that

$$\Psi^{-\infty}_{\mathrm{b}}(X) \times \dot{\mathcal{C}}^\infty(X) \ni (A,f) \longmapsto \int_X Af \in \dot{\mathcal{C}}^\infty(X) \tag{32}$$

is actually a continuous bilinear map. In particular we get the desired operator interpretation

$$A : \dot{\mathcal{C}}^\infty(X) \longrightarrow \dot{\mathcal{C}}^\infty(X),\ A \in \Psi^{-\infty}_{\mathrm{b}}(X). \tag{33}$$

Exercise 15. Check that this action is faithful, i.e. if A vanishes as an operator (33) then it vanishes as an element of the space $\Psi^{-\infty}_{\mathrm{b}}(X)$.