9. Lecture IX, October 7, 2003

Let me start today by doing a little piece of analysis using the Mellin transform.

Lemma 1. *If $u \in x^{-t}L^2_{\mathrm{b}}([0,\infty))$ for some $t > 0$ has support in $x < 1$ and is such that $x\frac{d}{dx}u \in L^2_{\mathrm{b}}([0,\infty))$ then there exists $u_j \in \dot{\mathcal{C}}^\infty([0,\infty))$ such that $u_j \to u$ in $x^{-t}L^2_{\mathrm{b}}([0,\infty)$ and $x\frac{d}{dx}u_j \to x\frac{d}{dx}u_j$ in $L^2_{\mathrm{b}}([0,\infty)$.*

Exercise 1. If you are so inclined, find a proof which does not use the Mellin transform!

Proof. First note that if $u \in L_{\mathrm{b}}(X)$ then $x^\epsilon u \to u$ in $L^2_{\mathrm{b}}(X)$ as $\epsilon \downarrow 0$ for any compact manifold with boundary.

Exercise 2. Write out a careful proof of this.

Since we know that $x\frac{d}{dx} : H^1_{\mathrm{b}}([0,\infty) \longrightarrow L^2_{\mathrm{b}}([0,\infty))$ is continuous and that $\dot{\mathcal{C}}^\infty([0,\infty))$ is dense in $H^1_{\mathrm{b}}([0,\infty))$ it suffices to show that the sequence can be chosen in this space. So, the obvious way to get such a sequence is to take $u_j = x^{\epsilon_j}u$, with $\epsilon_j \downarrow 0$. From the Paley-Wiener theorem, the Mellin transform of u is holomorphic in $\operatorname{Im} s < -t$ and is square integrable on real lines in this half space with uniformly bounded L^2 norm. On the other hand,

$$(x\frac{d}{dx}u)_M = -isu_M \tag{1}$$

must be similarly holomorphic and L^2 in $\operatorname{Im} s < 0$. Thus certainly, $u_j \to u$ in $x^{-\delta}L^2_{\mathrm{b}}([0,\infty)$ for any fixed $\delta > 0$ (and in particular u lies in this space.) Now

$$x\frac{d}{dx}(x^\epsilon u) = x^\epsilon(x\frac{d}{dx}u) + \epsilon x^\epsilon u$$

and as already noted, the first term converges in L^2_{b} to $x\frac{d}{dx}u$ as $\epsilon \downarrow 0$ so it suffices to show that the second term converges to 0 in this space. The square of the L^2 norm of its Mellin transform may be estimated as follows:

$$\epsilon^2 \int_{\mathbb{R}} |u(s-i\epsilon)|^2 ds \\ \le \epsilon \int_{|s|\ge\epsilon^{\frac12}} |(s-i\epsilon)u(s-i\epsilon)|^2 ds + \int_{|s|\le\epsilon^{\frac12}} |(s-i\epsilon)u(s-i\epsilon)|^2 ds$$

where in the first term the estimate $|s-i\epsilon|^2 \ge \epsilon$ is used and in the second $|s-i\epsilon| \ge \epsilon^2$. By the assumed square-integrability of $x\frac{d}{dx}u$ both terms tend to 0 with ϵ. $\square$

Using this and some related analysis I next want to write down the domains of $d+\delta$ that we have been discussing. First, we always have

$$x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) \subset D_A \cap D_R. \tag{2}$$

This is a direct result of the fact (discussed further below) that

$$\dot{\mathcal{C}}^\infty(X;\Lambda^*) \subset x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) \tag{3}$$

is dense with respect to the natural Sobolev norm and

$$d,\delta : x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) \longrightarrow L^2_{\mathrm{c}}(X : {}^{\mathcal{C}}\Lambda^*) \tag{4}$$

are continuous. Thus for an element $u \in x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$ there is an approximating sequence $\phi_j \in \dot{\mathcal{C}}^\infty(X;\Lambda^*)$, with $\phi_j \to u$, $d\phi_j \to du$ and $\delta\phi_j \to \delta u$ all in $L^2_g(X;\Lambda^*)$.

From the behaviour of solutions to $du = 0$ and $\delta u = 0$ in the model case we can add a few more pieces to the domains. These are all determined by the eigenfunctions of the limiting metric, h_0, on the boundary. Choose $\chi \in \mathcal{C}^\infty(X)$, a cut-off function supported very near the boundary and identically equal to 1 in some neighbourhood of it. Then set, for n odd

$$
\begin{aligned}
E_A &= \chi \cdot H^{\frac{n}{2}-\frac{1}{2}}_{\mathrm{Ho}(h_0)}(\partial X),\\
E_R &= \chi \cdot dx \wedge H^{\frac{n}{2}-\frac{1}{2}}_{\mathrm{Ho}(h_0)}(\partial X)
\end{aligned} \tag{5}
$$

in terms of the Hodge cohomology, i.e. harmonic forms, on the boundary for the metric h_0.

Similarly we fix spaces associated to non-harmonic eigenforms of the tangential Laplacian. If λ is such an eigenvalue for exact k forms on the boundary, so there is a non-trivial

$$
0 \neq e_\lambda \in \mathcal{C}^\infty(\partial X; \Lambda^k),\ e_\lambda = de'_\lambda,\ d\delta e_\lambda = \lambda e_\lambda \tag{6}
$$

we consider

$$
\begin{gathered}
f_\lambda = x^{-is_\lambda}(x^k(-is_\lambda + k)e_\lambda + x^{k-1}dx \wedge e'_\lambda),\ \text{where}\\
-is_\lambda = -\frac{n}{2} + |\frac{n}{2} - k| + e_\lambda,\ e_\lambda = \sqrt{(\frac{n}{2}-k)^2 + \lambda} - |\frac{n}{2} - k|
\end{gathered} \tag{7}
$$

and then let

$$
\begin{aligned}
G^{\frac{n}{2}-\frac{1}{2}} &= \sum_{0<e_\lambda<\frac{1}{2},\ k=\frac{n}{2}-\frac{1}{2}} \mathbb{C}\chi f_\lambda,\\
G^{\frac{n}{2}} &= \sum_{0<e_\lambda<1,\ k=\frac{n}{2}} \mathbb{C}\chi f_\lambda,\\
G^{\frac{n}{2}+\frac{1}{2}} &= \sum_{0<e_\lambda<\frac{1}{2},\ k=\frac{n}{2}+\frac{1}{2}} \mathbb{C}\chi f_\lambda
\end{aligned} \tag{8}
$$

be the corresponding finite dimensional subspace of k-forms on X. Here, each eigenvalue of the boundary Laplacian on exact k forms, with e_λ in the indicated range, is repeated with its (finite) multiplicity, as the e_λ run over a basis. Of course the first and third spaces only make sense when n is odd, and the second when n is even.

Observe that each of these spaces is contained in $L^2_g(X; \Lambda^*)$ but intersects the smaller space $xL^2_g(X; \Lambda)$ in 0. The point here is that

$$
du,\ \delta_0 u \in \dot{\mathcal{C}}^\infty(X; \Lambda^*),\ u \in G^k \tag{9}
$$

provided δ_0 corresponds to a product-type conic metric, equal to $dx^2 + x^2 h_0$ near the boundary.

Exercise 3. Check (9) carefully! It follows from the formulæ for d and δ and the fact that the 2-vector implicit in (7) is a null vector of the 2×2 matrix implicit in the computation of the joint (formal) null space of d and δ_0 above

$$
\begin{pmatrix} -1 & -is_\lambda + k \\ is_\lambda - (n-k) & \lambda \end{pmatrix} \begin{pmatrix} -is_\lambda + k \\ 1 \end{pmatrix} = 0. \tag{10}
$$

Here of course s_λ has been chosen so the matrix has rank 1.

This takes care, as we shall see, of all the possible poles we discovered within the 'critical strip' $-\frac{n}{2} < \operatorname{Im} s < -\frac{n}{2}+1$ for the Mellin transform of a form annihilated by d and δ_0. We need also to consider the poles on the line $\operatorname{Im} s = -\frac{n}{2}+1$. To handle these we consider an infinite-dimensional space of functions on the line

$$\mathcal{L} = \left\{ h \in x^{-\epsilon} L^2_{\mathrm{b}}([0,\infty)), \epsilon > 0; h = 0 \text{ in } x > 1,\ x\frac{d}{dx} h \in L^2_{\mathrm{b}}([0,\infty)) \right\}. \tag{11}$$

Notice that Lemma 1 applies to elements of this space and shows in particular that it is independent of the choice of ϵ. With these functions as coefficients we consider spaces related to those in (5) and determined by the harmonic $\frac{n}{2}-1$ forms on the boundary with respect to h_0 :

$$\begin{aligned} E_{\mathcal{L}}^{\frac{n}{2}-1} &= \mathcal{L}(x) \cdot H^{\frac{n}{2}-1}_{\mathrm{Ho}(h_0)}(\partial X), \\ E_{\mathcal{L}}^{\frac{n}{2}+1} &= x^2 \mathcal{L}(x) \cdot dx \wedge H^{\frac{n}{2}-1}_{\mathrm{Ho}(h_0)}(\partial X). \end{aligned} \tag{12}$$

Exercise 4. Again you should do the little computation to see that if n is even then

$$E_{\mathcal{L}}^{\frac{n}{2}\pm 1} \subset L^2_g(X;\Lambda^*) \text{ and } u \in E_{\mathcal{L}}^{\frac{n}{2}\pm 1} \Longrightarrow du,\ \delta_0 u \in L^2_g(X;\Lambda^*). \tag{13}$$

Similarly we consider spaces closely related to those in (8) involving the form (7) corresponding to an exact boundary k-form which is an eigenform for the boundary Laplacian:

$$\begin{aligned} G_{\mathcal{L}}^{\frac{n}{2}-\frac{1}{2}} &= \sum_{e_\lambda = \frac{1}{2},\ k = \frac{n}{2}-\frac{1}{2}} \mathcal{L} \cdot f_\lambda, \\ G_{\mathcal{L}}^{\frac{n}{2}} &= \sum_{e_\lambda = 1,\ k = \frac{n}{2}} \mathcal{L} \cdot f_\lambda, \\ G_{\mathcal{L}}^{\frac{n}{2}+\frac{1}{2}} &= \sum_{e_\lambda = \frac{1}{2},\ k = \frac{n}{2}+\frac{1}{2}} \mathcal{L} \cdot f_\lambda. \end{aligned} \tag{14}$$

Notice that the non-triviality of these spaces corresponds to an 'accident' in which there is a positive eigenvalue for which e_λ takes on a specific value.

Exercise 5. If you haven't thought about this already, given an example of a function which is in $\mathcal{L}$ but is not in $L^2_{\mathrm{b}}([0,\infty))$.

Finally we get to an explicit description of the domains.

Proposition 1. *For a conic metric on a compact manifold with boundary*

$$\begin{aligned} D &= \left\{ u \in L^2_g(X;\Lambda^*); du, \delta u \in L^2_g(X;\Lambda^*) \right\} \\ &= x^{-\frac{n}{2}+1} H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) + E^*_A + E^*_R + G^* + E^*_{\mathcal{L}} + G^*_{\mathcal{L}}; \end{aligned} \tag{15}$$

and D_A and D_R are the same without the summands G^*_R and G^*_A respectively.

Remark 1. a) Before proceeding to the proof of this, note that the difference between D_A and D_R amounts to the repalacement of a finite dimensional subspace of the domain by another, of the same dimension – because by Poincaré duality $H^{\frac{n}{2}\pm\frac{1}{2}}(\partial X)$ have the same dimension.

b) The 'complicated' (in particular infinite-dimensional) extra terms in (15), $E^*_{\mathcal{L}}$ and $G^*_{\mathcal{L}}$, are really rather insignificant. As follows from the discussion below, if we give D the obvious norm

$$\|u\|_D^2 = \|u\|^2_{L^2_g} + \|du\|^2_{L^2_g} + \|\delta u\|^2_{L^2_g} \tag{16}$$

then $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*) + E^*_{\mathcal{L}} + G^*_{\mathcal{L}}$ is the closure of $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*)$ (and hence also of $\dot{\mathcal{C}}^\infty(X;\Lambda^*)$) in D.

c) In particular this means that the quotient of D by D_0, the closure of $\dot{\mathcal{C}}^\infty(X;\Lambda^*)$ in D, is finite dimensional.

Exercise 6. Check the statement following (16); the discussion below shows that this set is contained in the closure; the converse amounts to the exclusion of the other sets E^*_A, E^*_R and G^*. For the first two this is done below and a similar argument also works for the third.

Exercise 7. Show that the bilinear form

$$W : D \times D \ni (u,v) \longmapsto \int_X ((du+\delta u, v) - (u, dv + \delta v))\, dg \tag{17}$$

is antisymmetric (if we are dealing with real forms and the real pairing) and vanishes on D_0. Let $D_m = D_A \cap D_R$ be the subspace of D consisting of the elements have approximating sequences $u_j \in \dot{\mathcal{C}}^\infty(X;\Lambda^*)$ such that $u_j \to u$ and $du_j \to du$ in L^2_g and also are approximable in L^2_g by a possibly different sequence v_j for which δv_j converges in L^2_g. Show that W vanishes on D_m and that that D/D_m is a symplectic vector space in which D_A/D_0 and D_R/D_0 are complementary Lagrangian subspaces.

Proof. From elliptic regularity (which I still have to prove) we 'know' that

$$u \in L^2_g(X;{}^{\mathcal{C}}\Lambda^*),\ du, \delta u \in L^2_g(X;{}^{\mathcal{C}}\Lambda^*) \Longrightarrow u \in x^{-\frac{n}{2}}H^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*). \tag{18}$$

Thus, we start off with one factor of x less than we need to get into the first term in the putative expansion of D.

Exercise 8. Check again that you know why all the terms in (15) are in $L^2_g(X;{}^{\mathcal{C}}\Lambda^*)$.

Now, we are dealing with a conic metric which is not necessarily of product type near the boundary. On the other hand, the result we are looking for only depends on the limiting metric h_0 and not the higher perturbations

$$g = dx^2 + x^2h(x,y,dy,dx) = g_0 + xq(x,y,xdy,dx),\ g_0 = x^2 + x^2h_0(y,dy). \tag{19}$$

To see this directly observe that the Hodge star operator has a similar property

$$\star_g = \star_{g_0} + xA \tag{20}$$

where A is a smooth homomorphism of ${}^{\mathcal{C}}\Lambda^*$.

Exercise 9. See if you can do this reasonably neatly!

This in turn implies that

$$\delta_g = \delta_{g_0} + B,\ B \in \mathrm{Diff}^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*). \tag{21}$$

Thus B has no $1/x$ factor. Now,

$$D \subset x^{\frac{n}{2}}H^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*) \xrightarrow{B} L^2_g(X;{}^{\mathcal{C}}\Lambda^*) \tag{22}$$

from which it follows that D, D_A and D_R for the metric g are the same as they are for a product metric g_0 with the same limiting metric h_0. Thus we are reduced to the case of a product-type metric for which we were able to do computations using the Mellin transform.

All the terms in the expansion of D, apart from the first, correspond to the poles we discovered in examining the condition $du = \delta_0 u = 0$. We are now working with weaker regularity, namely that $du,\ \delta_0 u \in L^2_g$. Thus, writing out $u \in D$ in terms of its normal and tangential parts as before tangential parts

$$\begin{gathered}\begin{pmatrix}-d_t & x\partial_x + k\\ 0 & d_t\end{pmatrix}\begin{pmatrix}u_n\\ u_t\end{pmatrix} \in x^{-\frac{n}{2}+1}L^2_{\mathrm{b}}([0,\infty);L^2(\partial X))\\ \begin{pmatrix}-\delta_t & 0\\ -x\partial_x-(n-k) & \delta_t\end{pmatrix}\begin{pmatrix}u_n\\ u_t\end{pmatrix} \in x^{-\frac{n}{2}+1}L^2_{\mathrm{b}}([0,\infty);L^2(\partial X))\end{gathered} \tag{23}$$

The analysis of the (truncated) Mellin transform proceeds very much as before except that the right side in (23) leads only to a holomorphic Mellin transform in $\operatorname{Im} s < -\frac{n}{2}+1$ with L^2 integral on real lines in this set uniformly bounded. Moreover, the invertibility of the full matrix

$$\begin{pmatrix}-d_t-\delta_t & x\partial_x + k\\ -x\partial_x-(n-k) & d_t+\delta_t\end{pmatrix} \tag{24}$$

off the imaginary axis follows as before and it only has a finite number of poles in $-\frac{n}{2} < \operatorname{Im} s < -\frac{n}{2}+1$ of finite multiplicity. Thus we conclude that u_M is meromorpic as a function in $\operatorname{Im} s < -\frac{n}{2}+1$ with values in $H^1(\partial X; {}^{\mathcal{C}}\Lambda^*)$ and su_M is square-integrable, with values in L^2, on real lines except possibly near $\operatorname{Re} s = 0$.

Writing out the steps in the argument we find

(1) From the tangential part of the first condition and the normal part of the second, the coexact part of u_t and the exact part of u_n, in terms of the Hodge decomposition with respect to h_0, must be the Mellin transforms of functions in $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(\partial X; {}^{\mathcal{C}}\Lambda^*)$.
(2) The harmonic parts must be such that $(-is+k)u^H_{n,M}$ and $(is-n+k)u^H_{t,M}$ are the Mellin transforms of functions in $x^{-\frac{n}{2}+1}([0,\infty))$ with values in this vector space.
(3) For the exact part of $u_{n,M}$ and the coexact part of $u_{t,M}$ the projection onto the span of the eigenforms with eigenvalues larger than some R are necessarily in $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}$. Each of the components corresponding to an eigenvalue λ satisfy the same equation as before with an error in $x^{\frac{n}{2}+1}H^1([0,\infty))$.

So the poles in $\operatorname{Im} s < -\frac{n}{2}+1$ of the Mellin transform of $u \in D$ are therefore precisely the same as thos of the solutions of $du = \delta_0 u = 0$ as analysed before. The terms in the spaces G^*_A, $G^*_R$and E^* have exactly these poles, with arbitrary coefficients of the appropriate type. Thus, subtracting them we may arrange that u_M has no poles below $\operatorname{Im} s = -\frac{n}{2}+1$. However the result may still not be the Mellin transform of a function in $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$. However, a similar argument for the poles lying on $\operatorname{Im} s = -\frac{n}{2}+1$ gives rise to terms in $G^*_{\mathcal{L}}$ and $E^*_{\mathcal{L}}$. After subtracting these terms the result is a form in $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$ which shows that D is indeed given by (15).

To shows that D_R is as indicated, we need to show that all terms apart from E_A are contained within it, and that $E_A \cap D_R = \{0\}$. The first requires the construction of approximations $u_j \in \dot{\mathcal{C}}^\infty(X;\Lambda^*)$ such that $u_j \to u$and $du_j \to du$ in L^2_g. For terms in E_R^* a simple cut-off suffices. For terms in $E_{\mathcal{L}}^*$ and $G_{\mathcal{L}}^*$ approximability follows from Lemma 1. For the terms in G^*, which are of the form

$$\chi x^{-is_\lambda}(x^k(-is_\lambda + k)e_\lambda + x^{k-1}dx \wedge e'_\lambda)$$

we first approximate the normal term using a simple cut-off setting

$$u_{j,n} = x^{-is_\lambda}(1 - \chi(x/\epsilon))(-is_\lambda + k)dx \wedge e'_\lambda$$

and then fix the tangential part by solving

$$u_{j,t} = (-is_\lambda + k)\chi(x)x^{-k}\int_0^x t^{k-1-is_\lambda}(1 - \chi(t/\epsilon)dte_\lambda. \tag{25}$$

That $\square$