

V/1 Pseudodifferential operators.
As I said on Tuesday, pseudodifferential operators as a special class of operators between sections of vector bundles. It did me to prove the ellipticity theorem I wrote down, at least to prove the Hörmander. Today I want to

1) Talk briefly about the Schwartz kernel theorem, which describes general operators

2) Talk about smoothing operators

3) Distribute the papers of pseudodifferential operators, as time permits, show how non-commutativity from the ellipticity theorem can be used to prove the ellipticity theorem.

Last time I describe distributions,

$$C^{-\infty}(X) = \left\{ u : C^{\infty}(X; \mathbb{R}) \rightarrow \mathbb{C} \right\}$$

without giving many examples, or even defining

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to topology a $C^\infty(X; \mathbb{D})$. At least I want to
study the first form, if not the second.

However, first let me generalize by defining
the 'distributable sections' of a vector bundle.

If E is a vector bundle and E^*
is its dual bundle then we have a pairing

$$E_f \times E_f^* \rightarrow \mathbb{C} \quad \forall f \in X$$

and here

$$C^\infty(X; E) \times C^\infty(X; E^*) \ni (u, v)$$

$$\mapsto f \in C^\infty(X)$$

$$f(\phi) = u(\phi) \cdot v(\phi) \in \mathbb{C}.$$

The generalization of F is another bundle in \mathcal{F}

$$C^\infty(X; E) \times C^\infty(X; E^* \otimes F) \rightarrow C^\infty(X; F)$$

In particular if we apply $\alpha \in F = \mathcal{S}X$ IV/3

$$C^\infty(X; E) \times C^\infty(X; E^* \otimes \mathcal{N}) \rightarrow C^\infty(X; \mathcal{S})$$

gives a non-degenerate pairing (if linear, no
separability) i.e.

$$u \in C^\infty(X; E), \quad \int u \cdot v = 0 \text{ for } v \in C^\infty(X; E^* \otimes \mathcal{N})$$

$$\Leftrightarrow u = 0$$

This generalizes the case $E = \mathcal{F}$ discussed
last time. In particular we define

$$C^{-\infty}(X; E) = \left\{ u : C^\infty(X; E^* \otimes \mathcal{N}) \rightarrow \mathbb{C} \text{ (its linear maps)} \right\}$$

as before we get an injection

$$C^\infty(X; E) \hookrightarrow C^{-\infty}(X; E)$$

which we regard as an identification.

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Now suppose we consider the manifold
 (for the moment we put without boundary)
 x, y with red bar \bar{x}, \bar{y} over them. Over
 the present $X^{x,y}$ we define the bundle
 $\text{Hom}(F, E)$ by

$$\text{Hom}_{(\#)}(F, E) = \text{hom}(F_p, E_q)$$

$$\cong E_{\bar{q}} \otimes F_p^* \quad \forall (\#)(X^{x,y}).$$

Exam check that there is ~~to take~~ a C^∞
 red bar on $X^{x,y}$.

Now suppose $K \in C^{-\infty}(X^{x,y}; \text{Hom}(E, F))$
 $\otimes \mathcal{L}Y$)

we say as the density bundle on Y

$$\Omega_{(\#)} Y = \Omega_2 Y.$$

but as a bundle on $X \times Y$.

Proposition If $u \in C^\infty(Y; F)$ and

$K \in C^\infty(X \times Y; \text{Hom}(E, F) \otimes \mathcal{R}Y)$ then

$$K \cdot u \in C^\infty(X \times Y; E \otimes \mathcal{R}Y)$$

is well-defined as is the Y -action

$$\int_K K \cdot u \in C^\infty(X; E).$$

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The map so defined

$$C^\infty(Y; F) \rightarrow C^\infty(X; E)$$

is continuous as long as every

such continuous linear map corresponds

to a unique $K \in C^\infty(X \times Y; \text{Hom}(E, F) \otimes \mathcal{R}Y)$

in this way.

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As usual I do not plan to prove this, because I am not really going to use it. The justification of operators with distributional sections of H_{operator} is in the Schwartz kernel theorem. The main difficulty (not really so hard) is the construction of the kernel K for the operator. The uniqueness is not so hard, but is the fact that the kernel defines an operator.

Despite the fact that it is a trifle unfortunate we write the operator, compactly as $K \in C^\infty(X \times Y; \mathbb{F} \otimes \mathbb{F}) \otimes \mathcal{L}(Y)$

$$Ku = \int_Y K(x, y) \cdot u(y) dy$$

(even generally using the same notation for kernel

at first

Example The identity operator

$$C^\infty(x) \ni u \mapsto u + C^\infty(x)$$

has kernel $K \in C^\infty(X^2; \mathbb{R})$ (where \mathbb{R} is in the right factor) where in any local coordinates is

$$(Id) \quad K = Id = \delta(x-x')/|x'|$$

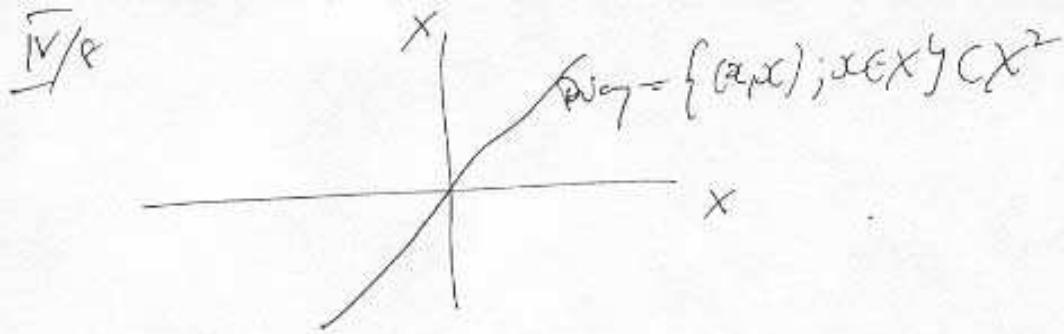
Ex 2 Show that under coordinate change
 (Id) takes the same form. (Note, its components are the same in both factors)

It is very important to observe that

$$Id = 0 \text{ except at } \cup C^\infty \text{ open}$$

$$\cup \cap \text{Diag} = \emptyset$$

$$\therefore \text{supp}(Id) \subset \text{Diag}.$$



Now, it follows easily from the various definitions that if $P \in \text{Diff}^m(X; E, F)$ and $K: C^\infty(X; G) \rightarrow C^\infty(X; E)$ then $P \cdot K: C^\infty(X; G) \rightarrow C^\infty(X; F)$ is given by

$$P \cdot K \in C^\infty(X \times Y; \text{Hom}(F; G) \otimes \mathcal{L}Y).$$

Here $P \notin P$ may be non-foliate - an element

$$P \in \text{Diff}^m(X \times Y; \text{Hom}(E; G) \otimes \mathcal{L}Y;$$

$$\text{Hom}(F; G) \otimes \mathcal{L}Y)$$

such that α acts in X' of $F = E \otimes F$.

Exam Try to make this claim in some
concrete way.

we apply this in the case $K = I\mathbb{L}$. Then we find that the kernel of P itself is just, in local coordinates

$$P(x, D) \cdot \delta(x - x')$$

- since of course supports lie in the diagonal.

Now let us consider another extreme case of Schwartz kernel theorem. Namely the elements $\otimes K \in C^*(X \times Y; \text{Hom}(E, F) \otimes \mathcal{D}Y)$ not define operator

$$K: C^\infty(Y; E) \rightarrow C^\infty(X; F)$$

in fact they define much better operator

$$K: C^\infty(Y; F) \rightarrow C^*(X; F)$$

which is called 'smoothing operator'.

$\mathbb{N}/10$ thus we start off with a definition

$$\Psi^{\infty}(X; E, F) \longleftrightarrow C^{\infty}(\mathbb{A}^2; \text{Hom}(E, F) \otimes \mathbb{R})$$

smoothly periodic.

- If $A \in \Psi^{\infty}(X; E, F)$ & $B \in \Psi^{\infty}(X; F, G)$ then
 $BA \in \Psi^{\infty}(X; E, G)$
- If $K \in C^{\infty}(X; \mathbb{A}^2, \text{Hom}(E, F))$ is any cont. func
periodic then $A \in \Psi^{\infty}(X; \tilde{E}, E)$, $B \in \Psi^{\infty}(X; F, \tilde{F}) \Rightarrow$
 $BKA \in \Psi^{\infty}(X; \tilde{E}, \tilde{F})$
- If $P \in \text{Diff}^m(X; E, F)$, $A \in \Psi^{\infty}(X; \tilde{E}, E) \cong$
 $B \in \Psi^{\infty}(X; F, \tilde{F})$ then $PA \in \Psi^{\infty}(X; \tilde{E}, P) \cong$
 $BP \in \Psi^{\infty}(X; E, \tilde{F})$.
- If $A \in \Psi^{\infty}(X; E)$ then
 $N(I_d - A) \subset C^{\infty}(X; E)$ is f.d. in $C^{\infty}(X; E)$
 $R(I_d - A) \in C^{\infty}(X; E)$ called a left
complement $N(I_d - A^*)$.

We are going to assert that

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$$\bigcap_{n \in \mathbb{N}} \Psi^n(X; E, F) = \Psi^\infty(X; E, F)$$

- For $A \in \Psi^n(X; E, F)$, A (its kernel) is singular at $\text{Dreg}(X) \subset X$.

Let's look at the kernel of $P \in \text{Diff}^n(X)$ we had earlier again. It is

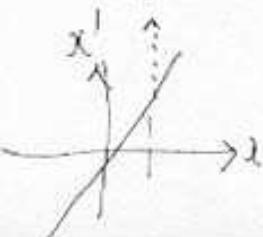
$$P(x, D) = \sum_{k=0}^{\infty} f_k(x) D_x^k \delta(x - x').$$

If we take the Fourier transform in x' in the sign we get $\sum_{k=0}^{\infty} f_k(x) e^{ikx'}$

$$P(x, D) e^{ikx'} = n(k) e^{ikx'}$$

$$n = (2\pi)^{-1} \int e^{i\alpha \cdot \vec{x}} \tilde{n}(\vec{x})$$

$$P(x, D)_n = (2\pi)^{-1} \int e^{i\alpha \cdot \vec{x}} \underbrace{f(\alpha, \vec{x})}_{\text{IFT } \rightarrow P(x, D)} \tilde{n}(\vec{x}) d\vec{x}$$



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One way to think of pseudo-differential operators is that we replace 'polynomial' functions $p(x, \xi)$ on T^*X by more general "symbolic" functions. Indeed, it is easy to introduce them! Namely, a polynomial in ξ (that is C^∞ w.r.t. x)
of degree m
satisfies the estimate (on compact x -sets)

$$(S) \quad |D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{\alpha\beta} (1+|\xi|)^{m-|\beta|}.$$

In fact it can not vanish identically if $|\beta| > m$. Hence for $m \in \mathbb{R}$ we take (S) as the definition

$$S_m^m(\mathbb{R}^n; \mathbb{R}^k) = \left\{ p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^k) \text{ satisfies } (S) \forall \alpha, \beta \right\}.$$

Exercise: If $L(x, \xi)$: $\mathbb{R}^n \rightarrow \mathbb{R}^k$ is an invertible linear map, then L is C^∞ in $x \in \mathbb{R}^n$ with all derivatives bounded.

$\phi \in S_\alpha^m(\mathbb{R}^k, \mathbb{R}^l)$

N(13)

$\Leftrightarrow \phi(x, L(x, \tau)) \in S_\alpha^{m'}(\mathbb{R}^k; \mathbb{R}^l)$

as smdles f' gret diff'ren $\in \mathbb{R}^k$,

$x_i = X_i(\tau)$ s.t. $|D^\alpha x_i| \leq C_\alpha$ + ($\alpha \geq 1$)
wth x_i satisfy the same condition.

The mean tht we can define

$$S^m(\mathbb{E}) \subset C^\infty(\mathbb{E})$$

for any red vdo, bth \mathbb{E} .

So has let me left you part 1

$\Psi^m(X; E, F)$, $m \in \mathbb{R}$. As already note
this are open

$$C^\infty(X; E) \rightarrow C^\infty(X; F).$$

① $\Psi^m(X; E, F) \subset \Psi^{m'}(X; E, F)$ $m' \geq m$

② $\Psi^m(X; F, G) \circ \Psi^{m'}(X; E, F)$

$$\subset \Psi^{m+m'}(X; E, G)$$

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③ $\text{Diff}^m(X; E, F) \subset \bar{\Psi}^m(X; E, F) \quad m \in \mathbb{N}$.

④ There is a short exact sequence

$$\bar{\Psi}^m(X; E, F) \hookrightarrow \bar{\Psi}^{m+1}(X; E, F) \xrightarrow{\sigma_m}$$

$$\frac{S^m(T^*X; \text{hom}(F, F))}{S^{m-1}(T^*X; \text{hom}(E, F))}$$

⑤ $\sigma_{\text{mult}}(AB) = \sigma_n(A) \circ \sigma_{n-1}(B)$

if $A \in \bar{\Psi}^m(X; F, G)$, $B \in \bar{\Psi}^{m+1}(X; E, F)$

⑥ $\sigma_n(P)$ is constant for $P \in \text{Diff}^m(X; E, F)$.

⑦ If $A_j \in \bar{\Psi}^{m-j}(X; E, F)$ is any sequence then $\exists A \in \bar{\Psi}^m(X; E, F)$ s.t.

$$A - \sum_{j=0}^N A_j \in \bar{\Psi}^{m-N+1}(X; E, F)$$

⑧ $A \in \bar{\Psi}^m(X; E, F) \Rightarrow A^* \in \bar{\Psi}^m(K; F, E), \quad \sigma(A^*) = \sigma(A)^*$
Let's try to make the same for the extended a generalized over $P \in \text{Diff}^m(X; E, P)$

who is after.

Step 1. $\sigma_m(P) \in \tilde{\Psi}^m(TX; \text{ker}(E, F))$ is
nilpotent for $g \in TX \setminus 0$. Then

$$\alpha = \sigma_m(P)^{-1} \cdot (1 - P(g, T_1)) \in S^{-m}(TX; \text{ker}(F(E)))$$

$P \in C^\infty(TX)$, $P \equiv 1$ near 0.

Step 2 Choose $A_0 \in \tilde{\Psi}^{-m}(X; F, E)$,

$$\sigma_m(A_0) = \alpha_0. \text{ Then } A_0 P \in \tilde{\Psi}^0(X; E),$$

$$\sigma_0(A_0 P) = \alpha_0 \cdot \sigma_m(P) = \text{Id} + \rho = \text{Id} \text{ in}$$

$$S^0/S^{-1}. \text{ So } A_0 P - \text{Id} = B_1 \in \tilde{\Psi}^{-1}(X; E).$$

Step 3 Choose A_1 with $\sigma_{m-1}(A_1) = -\sigma(B_1) \alpha_0$.

$$\tilde{\Psi}^{m-1}(X; F, E)$$

$$\Rightarrow (A_0 + A_1) P = \text{Id} + B_1 + A_1 P = \text{Id} + B_1,$$

$$B_1 \in \tilde{\Psi}^{-2}(X; E)$$

Step 4 Do like $\forall j$, $A_j \in \tilde{\Psi}^{-m-j}(X; E)$ s.t.

$$(A_0 + \dots + A_j) \cdot P = \text{Id} + B_{j+1}, B_{j+1} \in \tilde{\Psi}^{-j-1}(X; E)$$

by induction (same as step 3).

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Step 5 Now choose $A \sim \sum_{i=0}^{\infty} A_i$; it follows that

$$AP - ID \in \bigcap_i \psi^{-1}(X; \mathbb{F}) = \psi^{-\infty}(X; \mathbb{F}).$$

$$\Rightarrow AP = ID + B, \quad B \in \psi^{-\infty}(X; \mathbb{F}).$$

Step 6 We could do exactly the same thing and
this by choosing \tilde{A} s.t. $PA = ID + \tilde{B}, \tilde{B} \in \psi^{-\infty}(X; \mathbb{F})$.
Hence we

$$\begin{aligned} A &= A(ID + \tilde{B}) - A\tilde{B} \\ &= AP\tilde{A} - A\tilde{B} \\ &= (ID + B)\tilde{A} - A\tilde{B} \\ &= \tilde{A} + B\tilde{A} - A\tilde{B} \end{aligned}$$

so $\tilde{A} - A \in \psi^{-\infty}(X; \mathbb{F}; \mathbb{F}) \Rightarrow A$ absorbs

$$PA = ID + B', \quad B' \in \psi^{-\infty}(X; \mathbb{F})$$

Step 7 • $N(P) \subset N(ID + B)$ is f.d.

• $R(P) \supset R(ID + B')$ is dense in f.d.
complement.

\Rightarrow Follows from Step 8 (one can show
 $\psi^{-\infty}(X; \mathbb{F}; \mathbb{F})$).