

III/1 Ellipticity and pseudodifferential operators.

I want to start today with a 'triviality' about integration and densities. Recall that on \mathbb{R}^n we can integrate functions which are compactly supported and reasonably smooth. The certainly includes functions in $C_c^\infty(\overline{\mathbb{R}^n})$ - smooth functions which vanish outside a large ball.

$$C_c^\infty(\overline{\mathbb{R}^n}) \ni u \mapsto \int_{\mathbb{R}^n} u \, dx.$$

The problem with the integral is that it does not behave well under differentiation. For instance if $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a global diffeomorphism and $u \in C_c^\infty(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} F^* u \, dx = \int_{\mathbb{R}^n} u \circ F \, dx \stackrel{(in \\ general)}{\neq} \int_{\mathbb{R}^n} u \, dx$$

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$$\int_{\mathbb{R}^n} u \, dx = \int_{\mathbb{R}^n} F^* u \cdot |F| \, dx$$

where $F = dt \frac{\partial F_i}{\partial x_k}$. Thus to integrate invariantly we need something which transforms with a factor $|F|$. The object in question is a density.

Recall that the n -form bundle on X , where $\dim X = n$, is the totally antisymmetric part of the n -fold tensor product:

$$\Lambda_p^n X = \{ \lambda^n(T_p^* X) \subset (T_p^* X)^{\otimes n} \}.$$

Thus an element of $(T_p^* X)^{\otimes n}$ is a multilinear form

$$h: T_p^* X \times \dots \times T_p^* X \rightarrow \mathbb{R} \text{ (or } \mathbb{C})$$

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μ is $\lambda_p^n \times C(T_p^* X)^{\otimes n}$ consists of the totally antisymmetric elements

$$\mu(\dots, v_i, v_{i+1}, \dots)$$

$$= -\mu(\dots, v_{i+1}, v_i, \dots) \quad \forall i = 1, \dots, n-1.$$

We can also define the fibre $\lambda_p^n(T_p^* X)$ & the 'multi-edges' bundle as the totally antisymmetric multilinear forms

$$T_p^* X \times T_p^* X \times \dots \times T_p^* X \rightarrow \mathbb{R}.$$

\leftarrow n-fibre \rightarrow

The reason for considering this is:

Exami check that $\lambda_p^n X = \lambda^n(T_p^* X)$ is canonically isomorphic with the dual of $\lambda^n(T_p X)$.

This $\lambda_p^n X$ is just the space of linear forms (as a 1-D vector space)

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$$\mu_p: \Lambda_p^*(T_p X) \longrightarrow \mathbb{R}.$$

We 'know' that this form transfers with a factor of $|J_F|$, the determinant of the Jacobian under coordinate changes (this is one way to define the determinant). We want to get $|J_F|$ into the transformation law. To do so we define

$$I_p X = \{ v: \Lambda_p^*(T_p X) \setminus 0 \rightarrow \mathbb{R}$$

(*) absolutely homogeneous of degree 1,
 $v(s\gamma) = |s| v(\gamma) \quad \forall \gamma \in \Lambda_p^*(T_p X) \setminus 0$,
 $s \in \mathbb{R} \setminus \{0\}$.

Exercise check that this is a $1-D$ vector space and that if $\omega \in \Lambda_p^* X$ the $|\omega| \in I_p X$ is a basis element.

Exercise show that $I_p X$ is a well-defined linear bundle over X what is trivial is

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be a global non-vanishing (positive) smooth section. Show in particular that if g is a Riemannian metric on X then $\|dg\|$ (that I would denote dg) is a well-defined global positive section of $\Omega^1 X$.

As already noted, the part about density, which are sections of $\Omega^1 X$, is trivial. They can be invariantly integrated. Thus, there is a global measure.

$$(I) \int_X : C^\infty(X; \mathbb{R}) \rightarrow \mathbb{R}$$

see that if $u \in C^\infty(X; \mathbb{R})$ has support in a coordinate patch, $u = u(x)/|dx|$, $|dx|$ being Lebesgue measure,

$$\int_X u = \int_{\mathbb{R}^k} u(x) dx.$$

Exercises Check that all concepts used are p.o.i.

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Exercise Show that the definition (\star) of the density
bundle can be modified to yield a denser structure as
 $\alpha \in \mathbb{R}$:

$$v \in \Omega_p^\alpha X \Leftrightarrow u: \Lambda_p^n(T_p X) \setminus 0 \rightarrow \mathbb{R},$$

$$u(s\gamma) = |s|^\alpha \gamma \quad \forall s \in \mathbb{R} \setminus \{0\},$$

$$\gamma \in \Lambda_p^n(T_p X) \setminus 0.$$

Show that this always gives trivial line bundles
over X . Show that there is a canonical iso-
morphism

$$\Omega^\alpha X \cong X \times \mathbb{R}$$

as well for any α, β . This is a canonical
isomorphism

$$\Omega^\alpha X \otimes \Omega^\beta X \cong \Omega^{\alpha+\beta} X.$$

Use this to show that Ω^α is a complex
pre-Hilbert norm space a $C^\infty(X; \mathbb{C}^{n_2})$

$$\text{inner product } \langle u, v \rangle = \int_X \bar{u}v.$$

Using the integral as the observation
that if $f \in C^\infty(X; \mathbb{R})$ and $u \in C^\infty(X)$ then
 $f u \in C^\infty(X; \mathbb{R})$ we define a bilinear
map

$$(C^\infty(X) \times C^\infty(X; \mathbb{R})) \rightarrow \mathbb{R} \quad (\text{or } \mathbb{R})$$

$$(f, u) \mapsto \int_X f u.$$

Exercise: think for a minute (no longer)
about the topology on $C^\infty(X; \mathbb{R})$ and $C^\infty(X)$
of 'uniform convergence of all derivatives'
on compact subsets of coordinate patches.

Thus if $f \in C^\infty(X)$ then

$$M_f : (C^\infty(X; \mathbb{R})) \rightarrow \mathbb{R}$$

$$u \mapsto \int_X f u$$

is a continuous linear map.

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Exercise Show that $M_f = 0$ (i.e.

$M_f(u) = 0 \forall u \in C^\infty(X; \mathbb{R})$) implies $f = 0$ in $C^\infty(X)$.

Distributions τ in the compact support
sense below are defined to be exact
and continuous linear maps

$$C^{-\infty}(X) = \left\{ \tau: C^\infty(X; \mathbb{R}) \rightarrow \mathbb{C} \text{ linear and continuous} \right\}.$$

The exercise above shows that

$$C^\infty(X) \ni f \longmapsto M_f \in C^{-\infty}(X)$$

is an injection. We regard this as
an identification and write

$$C^\infty(X) \subset C^{-\infty}(X).$$

The reason this is useful to do, to regard $C^\infty(X)$ as a subset of $C^{-\infty}(X)$, is that many (but quite all) operations in $C^\infty(X)$ extend naturally to $C^{-\infty}(X)$. The most important examples of this are

1. Multiplication by $C^\infty(X)$: If $v \in C^\infty(X)$ and $f \in C^\infty(X)$ then

$$M_{vf}(u) = \int_X v f u = \int_X f(vu)$$

$$= M_f(vu) \quad \forall u \in C^\infty(X; \mathbb{R})$$

For $w \in C^{-\infty}(X)$ we define $vw \in C^{-\infty}(X)$, $v \in C^\infty(X)$ being given, by

$$vw(u) = w(vu) \quad \forall u \in C^\infty(X; \mathbb{R})$$

This defines a bilinear map

$$C^\infty(X) \times C^{-\infty}(X) \rightarrow C^{-\infty}(X).$$

2. Action of differential operators. If

$P \in \text{Diff}^m(X)$, so $P: C^\infty(X) \rightarrow C^\infty(X)$,
then we can extend P to an op-

$$P: C^{-\infty}(X) \rightarrow C^{-\infty}(X).$$

First we define $P^t \in \text{Diff}^m(X; \Omega)$ by
the formula

$$(t) \quad \int_X f(P^t u) = \int_X P f \cdot u \quad \forall f \in C^\infty(X).$$

Exercise Show that (t) does define a
differential operator $P^t \in \text{Diff}^m(X; \Omega)$. First
suppose that f has support in a coordinate
patch as we integrated by parts to
show that if $u = u(x)/dx$ at

$$P = \sum_{|\alpha| \leq m} f_\alpha(x) D_x^\alpha$$

then

$$P^t u = \sum_{|\alpha| \leq m} (-D_x)^\alpha f_\alpha(x) u.$$

Use a fact of ours to show that P^t exists, satisfying (†) as then we can take exercise to show that it must be unique.

We define $P: C^{-\infty}(X) \rightarrow C^{-\infty}(X)$

by

$$Pv(u) = v(P^t u) \quad \forall u \in C(X),$$

Explain how P restricts to the original
messing on $C(X) \subset C^{-\infty}(X)$.

I will not go into the theory of general distribution here, it is much convenient to have a space that contains 'everything' we are interested in. Recall however, the basic

Theorem (Schwartz representation). If

$u \in C^{-\infty}(x)$ then $\exists \overset{m}{\underset{\leftarrow}{\mathbb{N}_0}}$ s.t.

$v \in L^2(x)$ and $P \in \text{Diff}^m(x)$ s.t.

$$(n) \quad u = Pv.$$

We work

$$H^{-m}(x) = \left\{ u \in C^{-\infty}(x) \text{ of the form } \underset{(n)}{\left[\begin{array}{c} \\ v \\ P \end{array} \right]} \right\}, \quad n \in \mathbb{N}_0.$$

Then $H^0(x) = L^2(x)$. We also set

$$H^k(x) = \left\{ u \in C^{-\infty}(x); \exists v \in L^2(x) \text{ and } P \in \text{Diff}^m(x) \text{ s.t. } u = Pv \right\}.$$

The stricter term has shows (for ^{the}
a compact manifold without boundary) that

$$C^{-\infty}(X) = \bigcup_{m \in \mathbb{Z}} H^m(X).$$

A (somewhat simple) regularity theorem
says

$$C^\infty(X) = \bigcap_{m \in \mathbb{Z}} H^m(X).$$

I will define and describe classes
of distributions with growing supports,
called converged distributions. They are
closely related to pseudodifferential operators,
which I will describe first (but not
straight away). Pseudodifferential operators

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acting from sections of one bundle to another, $\Psi^m(X; E, F)$, as a space of operators

$$P: C^\infty(X; E) \rightarrow C^\infty(X; F)$$

indeed by, but considerably generalizing, the differential operators I will start with a fundamental property which motivates their usefulness.

Theorem Suppose X is a compact manifold without boundary and $P \in \text{Diff}^m(X; E, F)$ is elliptic. Then $P: C^\infty(X; E) \rightarrow C^\infty(X; F)$ is Fredholm in the sense that

① $N(P) = \{u \in C^\infty(X, E); Pu = 0\}$ is finite dimensional

② $R(P) = P \cdot C^\infty(X, E) \subset C^\infty(X, F)$

is closed

③ $R(P)$ is a finite dimensional complement, i.e. $\exists g_1, \dots, g_N \in C^\infty(X, F)$
s.t.

$$(C) C^\infty(X, F) = P \cdot C^\infty(X, E) + \text{span}(g_1, \dots, g_N)$$

For the moment I will not discuss 1/4 part of this. In (C) we can always assume that the g_i are independent modulo $P \cdot C^\infty(X, E)$. If f has drifts for ~~has~~
① - ③ then P be a generator with

$(Q): C^\infty(X, F) \rightarrow C^\infty(X, F)$
what is a continuous linear map

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Show that

$$Q \cdot P u = u + u' , \quad u' \in N(P)$$

$$\forall u \in C^{\infty}(X, E)$$

and

$$PQf = f + \sum_{k=1}^N c_k g_k \quad \forall f \in C^{\infty}(X, F).$$

Exercise Construct such a Q by choosing a complement to $N(P)$, $D \subset C^{\infty}(X, E)$ s.t.

$$C^{\infty}(X, E) = D + N(P)$$

(i.e. $D \cap N(P) = \{0\}$), show that

$P: D \rightarrow R(P)$ is an isomorphism

and the $Q = P_D^{-1} \cap R(D)$, $\oplus_{k=1}^N$

such that $Qg_j = 0 \quad \forall j = 1, \dots, N$ defines such a generalized inverse.

Such a generalized norm $\|\cdot\|$ is a 'typical' (well, except for the fact it is elliptic) element of $\Psi^k(X; F, E)$. In fact it is much better to consider $\|\cdot\|$ first, using definitions of the Ψ^k 's and from this deduce the theorem.

Let me apply this theorem to get a basic result in differential topology/analysis. Recall that we have defined

$$J: C^\infty(X; \wedge^k) \rightarrow C^\infty(X; \wedge^{k+1})$$

$\forall k$ by stating for $k=0$ and extending to the general case using

$$(1) \quad J(\alpha \wedge \nu) = d\alpha \wedge \nu + (-1)^k \alpha \wedge d\nu$$

$\alpha \in C^\infty(X; \wedge^k), \nu \in C^\infty(X; \wedge^{k+1})$.

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It follows directly from the definition (that this is one of the important reasons for that definition) that $d^2 = 0$. This

$$(d) C^\infty(X) \xrightarrow{d} C^\infty(X; \wedge^1) \xrightarrow{d} C^\infty(X; \wedge^2) \rightarrow \dots$$

is complex. We can 'roll up' this complex to get

$$d: C^\infty(X; \wedge^*) \rightarrow C^\infty(X; \wedge^*)$$

$$d\varphi = \sum_{j=0}^n \bigoplus \wedge^j X.$$

Let me compute the symbol of d . Recall that to do so we look at $d(e^{i\theta f} u), f \in C^\infty(X)$, $u \in C^\infty(X; \wedge^*)$. Using (A) we find

$$d(e^{i\theta f} u) = e^{i\theta f} (i\theta df \wedge u + u).$$

The symbol is the coefficient λ , so at

$$g = df(p) \in T_p^* X$$

$$(5) \quad \sigma_g(d) \lrcorner u = i_g \wedge u$$

as a harmonic for a $\lambda_p^* X$. Check its
is nt electric, surface with $g_1 g = 0$.

To get an electric field we con
follow Hodge and define

$$\delta \in \text{Diff}^1(X, \lambda^*), \quad \delta = d^*$$

using the Riemannian inner product

$\langle , \rangle_{\alpha} : \Lambda_p^k X$ of Riemannian

$$\text{derivative } dg, \quad (u, v)_p = \langle u, \bar{v} \rangle_p,$$

$$(6) \quad \int_X \langle v, du \rangle dg = \int_X \langle \delta v, u \rangle dg$$

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$$\forall u \in C^\infty(X; \Lambda^k), v \in C^\infty(X; \Lambda^{k+1}).$$

Exercice Generalize the early exercise (t)

to show that δ is well-defined by this formula, $\delta \in \text{Diff}^+(X; \Lambda^{k+1}, \Lambda^k) \forall t$.

Show that $\delta^2 = 0$ at that

$$(\sigma^*) \quad \sigma_\xi(\delta) = (\sigma_\xi(\delta))^*$$

Now we can compute the form of $\sigma_\xi(\delta)$ directly from (σ) . If $\xi \in T_p^* X$ let $w \in T_p X$ be its image under the natural isomorphism $G: T_p^* X \rightarrow T_p X$

$$\langle G\xi, w \rangle = \xi(w) \quad \forall w \in T_p X.$$

then defi

$$r_{\xi} : \Lambda_p^{k+1} X \rightarrow \Lambda_p^k X$$

$$\text{by } r_{\xi} \cdot u(v_1, \dots, v_k)$$

$$= u(G\xi, v_1, \dots, v_k).$$

If filter for (σ^*) the

$$o_{\xi}(\delta) = -i r_{\xi} \quad \text{of } \xi \in T_p^* X.$$

Let us write this out even more explicitly, assuming $H \models \xi \neq 0$. Set $\hat{\xi} = \xi/\|\xi\|$
as defini $T_p^* X = R \cdot \hat{\xi} + \hat{\xi}^\perp$. Then

$$\Lambda_p^k X = \Lambda_p^{k-1} \hat{\xi}^\perp \oplus \Lambda^k \hat{\xi}^\perp.$$

That is,

$$u = \hat{\xi} \wedge u_1 + u_2$$

$$\text{and } u_1 \in \Lambda_p^{k-1} V, u_2 \in \Lambda^k V, \xi \wedge u_1 = 0.$$

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In lens of the decomposition

$$\sigma_{\tilde{s}}(d) \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = i|\xi| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\sigma_{\tilde{s}}(d) \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -i|\xi| \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\Rightarrow \sigma_{\tilde{s}}(d+\delta) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = i|\xi| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

When $d+\delta \in \text{Diff}^1(X; \lambda^*)$.

Proposition $d+\delta$ is a (generalized) direct summand.

$$\begin{aligned} \text{Rif } \sigma_{\tilde{s}}((d+\delta)^2) &= (\sigma_{\tilde{s}}(d+\delta))^2 \\ &= -|\xi|^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = |\xi|^2 \cdot \text{Id}. \end{aligned}$$

In fact also, $d+\delta$ is elliptic. Thus,
 $\|u\|_{d+\delta}$ Then other applies. In fact
 $d+\delta$ is (formally) self-adjoint :-

$$(sa) \quad \int_X (u, (d+\delta)v) = \int_X ((d+\delta)u, v) \quad \forall u, v \in C^\infty(X, \Lambda^*)$$

as follows from (3). This means that (3)
of L₂ theorem becomes

$$(H_1) \quad C^\infty(X, \Lambda^*) = (d+\delta) C^\infty(X, \Lambda^*) + N(P).$$

Cutting we can choose the g's in (3) so
that indeed $u \in N(P)$ w/p, not (sa) L₂

$$(1) \quad \int_X (u, (d+\delta)v) = 0 \quad \forall v \in C^\infty(X, \Lambda^*)$$

$$\text{so } u \in N(P) \cap (d+\delta) C^\infty(X, \Lambda^*) \Rightarrow u=0.$$

$$\text{Consequently } (d+\delta)u=0 \Rightarrow u \in N(P).$$

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If we replace \mathcal{A}^* by a normed set $N(\mathcal{P})$ we get (H1) means $\exists g \in C^\infty(X; \lambda^*)$,
 $g \perp N(d\delta)$, $g \notin (d\delta)^* C^\infty(X; \lambda^*)$ [Njupte yf.]

Now (H2) can form the Hahn decomposition

$$(H2) \quad C^\infty(X; \lambda^*) = dC^\infty(X; \lambda^*) + \delta C^\infty(X; \lambda^*) + H_{\text{fr}}^*(X)$$

as $H_{\text{fr}}^*(X) = N(d\delta)$ by definition.

Then (H3) there is a natural compatibility

$$H_{\text{fr}}^k(X) = \left\{ u \in C^\infty(X; \lambda^k) ; d^k u = 0 \right\} / dC^\infty(X; \lambda^k)$$

$$\leftrightarrow H_{\text{fr}}^k(X) = H_{\text{fr}}^*(X) \cap C^\infty(X; \lambda^k)$$

given by (H2).

Proof. First observe that $u \in H_{\text{fr}}^*(X) = N(d\delta)$

subject to $\|du\|_2 = 0$. Then

$$0 = \int_X (du, (d+\delta)u) = \int_X (du, du) + \int_X (\delta^2 u, u)$$

$$= \|du\|_2^2 \Rightarrow \|du\|_2 = 0.$$

For the st. follo. to

$$H_{H_0}^k(x) = \bigoplus_{j=0}^n H_{H_0}^{k-j}(x)$$

is given, since

Exercice del. this.

If we apply (H_2) to $u \in C^\infty(X, \mathbb{R})$

$$u = dv_1 + \delta v_2 + h, \quad h \in H_{H_0}^k(x)$$

and the substra. $du = 0$ we have

$$0 = d^2 v_1 + d\delta v_2 + dh_1 = d\delta v_2$$

$$\Rightarrow 0 = \int_X (v_2, d\delta v_2) = \int_X (\delta v_2, du) = \|du\|_2^2$$

$$\Leftrightarrow \delta v_2 = 0 \Rightarrow u = dv_1 + h,$$

Th4 Furthermore, this decomposes as very, such
 $h \perp d\chi$, so define the map we want

$$H_{dR}^k(X) \rightarrow H_{\text{top}}^k(X), \quad u \mapsto h.$$

It is also clear an iso in fiber.

L

Replace Eilenberg Mac by (3)

$$C^\infty(X; F) = PC^\infty(X; E) + N(P)$$

as direct sum. etc.