

• Dirac and other differential operators

Let me quickly review the basic definitions. A forms and vector fields, starting from a C^∞ manifold. As before, the idea is doing this to help with subsequent generalizations.

A C^∞ manifold, X , for the moment without corners if you wish, comes equipped with a space of smooth functions, $C^\infty(X)$. A point $p \in X$ can be recovered from its defining ideal

$$\mathcal{J}_p = \{u \in C^\infty(X); u(p) = 0\}.$$

If we let $\mathcal{J}_p^2 \subset C^\infty(X)$ be the finite span of products of elements of \mathcal{J}_p

$$\mathcal{J}_p^2 = \left\{ u \in C^\infty(X); u = \sum_{i=1}^n f_i g_i, f_i, g_i \in \mathcal{J}_p \right\}$$

then

$$T_p^* X = \mathcal{J}_p / \mathcal{J}_p^2$$

II/2

as the cotangent fibre at p . If $u \in C^\infty(X)$ then
 $m-u(p) \in T_p^*X$ so there is a well-defined element

$$(1) \quad du(p) \in T_p^*X.$$

Exercise Show that if g_1, \dots, g_n are local coordinates at p then dg_1, \dots, dg_n is a basis for T_p^*X ; conclude that

$$T^*X = \bigcup_{p \in X} T_p^*X$$

is a (real) vector bundle over X .

Note that the local coordinates in $T^U = \bigcup_{p \in U} T_p^*X$ induce by local coordinates $\tilde{g}_1, \dots, \tilde{g}_n$ in $U \cap X$ are given by $(g_1, \dots, g_n, \tilde{g}_1, \dots, \tilde{g}_n)$ where

$$\xi = \sum_j \xi_j \, dg_j, \quad \xi \in T_p^*X.$$

The C^∞ structure on T^*X is consistent with (1), namely

$$(2) \quad d: C^\infty(X) \rightarrow C^\infty(X; T^*X)$$

where on the right we use the notation for smooth

sections of T^*X , this is the basic example of a (gradient) differential operator.

The usual tangent bundle TX can be defined either as the dual of T^*X or directly in terms of derivations. That is

$$T_p X = \{v: C^\infty(X) \rightarrow \mathbb{R}; v \text{ is linear and } v(fg) = f(p)v(g) + v(f)g(p)\}.$$

Exercise Show that $T_p X \cong (T_p^* X)^*$ with the identifications being given by a pairing

$$T_p X \times T_p^* X \xrightarrow{\text{pairing}} (v, \xi) \mapsto v(\xi), \quad \xi \in T_p^* X,$$

A section $V \in C^\infty(X; TX)$ then defines a linear map

$$V: C^\infty(X) \rightarrow C^\infty(X), \quad Vf(p) = V_p f.$$

By definition a linear differential operator, with smooth coefficients acting a functions is just a combination of such vector fields

$P: C^\infty(X) \rightarrow C^\infty(X)$,

$$(3) \quad P_u = \sum_{\substack{\text{finite} \\ a \in A}} V_{1,a} \dots V_{k_a, a} u, \quad (X \text{ compact}).$$

The first add on, $k_a \leq 1$, is just

$$P = V + f, \quad V \in C^\infty(X; TX), \quad f \in C^\infty(X).$$

We are more interested in differential operators acting on (complex) vector bundles. Let me give a couple of equivalent definitions.

First, we can 'reset' to local coordinates.

The basis of $T_p X$ induces by local coordinate

β_1, \dots, β_n in $\partial/\partial_{\beta_1}, \dots, \partial/\partial_{\beta_n}$ where

$$\partial/\partial_{\beta_j} \cdot d\beta_k = \delta_{jk}.$$

The local coordinate form of (3) is then

$$(4) \quad P_u = \sum_{|\alpha| \leq m} p_\alpha(\beta) D^\alpha u, \quad p_\alpha \in C^\infty$$

using multiindex notat., $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$,

115

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad D^\alpha = \left(\frac{1}{i} \frac{\partial}{\partial z_j} \right)^\alpha = i^{-|\alpha|} \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdot \dots \cdot \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}},$$

This a differential operator P as get a / w/
here

P: $C^\infty(X) \rightarrow C^*(X)$ what takes the form (4)

in any local coordinate (it suffices for this to be fine w.r.t \mathcal{X} by charts).

A vector bundle E has local trivialization,

π : X is covered by coordinate patches U_i on each of which E has a basis of smooth sections e_1, \dots, e_n .

If F is another vector such that we can

If F is another vector such that we can find a convex by coordinate charts or not but

E at F have bird baths, etc. on trail. The

a linear map, or a differentiable operator of order
(at most) n if

$$(5) \quad \Phi\left(\sum_{k=1}^L u_k e_k\right) = \sum_{l'=1}^{L'} (P_{ll'} u_{e_l}) f_{e_l}$$

16

when the $P_{i,j}$ are of the form (4). An interesting, although essentially weaker, result is

Theorem (Preste) $P: C^*(X; E) \rightarrow C^*(X; F)$ a linear map is a differential operator if and only if

$$u=0 \text{ on } U(X) \Rightarrow P_u=0 \text{ a } U$$

$\forall U \subset X$ open.

Exercise Let $\text{Diff}^{(m)}(X; E, F)$ denote the space of linear differential operators between sections of bundles E and F . Show that composition of operators (of order at most m)

defines a product

$$(6) \quad \text{Diff}^{(m_2)}(X; E_2, E_3) \cdot \text{Diff}^{(n)}(X; E_1, E_2) \subset \text{Diff}^{(m_1+n)}(X; E_1, E_3).$$

As I said before, we can mostly reduce a first order operator. Let me continue a little in the general case however.

Suppose $f \in C^*(X)$ and $u \in C^*(X; E)$ then
for $\lambda \in \mathbb{C}$,

$$(7) \quad u_\lambda = e^{i\lambda f} u \in C^*(X; E).$$

Leibniz' formula shows us how to distribute differentiation over such a product, namely

$$D_g^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\beta}{\alpha} D_g^{\alpha-\beta} u \cdot D_g^\beta v.$$

If we think it has sense for P , "i"
and consider looking like (6) we see that

$$(8) \quad P(e^{i\lambda f} u) = e^{i\lambda f} P_{\lambda, f} u.$$

Here $P_{\lambda, f}$ is again a differential operator of the
same order, now with coefficients depending on λ

as f . In fact

$$\lambda P_{\lambda, f} = \sum_{s=0}^m \lambda^s P_{(s), f}$$

II/8

a polynomial of degree at most $n - l$.
 The powers of λ come from the derivative 'hitting'
 e^{idt} so it follows that

$$P_{\text{rel},f} \in \text{Diff}^{m-s}(X, E, F).$$

Now, if $s = m$, $\text{Diff}^0(X, E, F) = C^0(X, \text{hom}(E, F))$
 is just the space of bundle maps from E to F . If
 $s < m$ then P_s still depends on f but we can see that

$P_{m,f}$ is a polynomial in df of degree
 (of most $\int |df|$.)

$$\sigma(P) =$$

$$\text{Definition } P_{\text{can},f} = \lim_{\lambda \rightarrow \infty} \lambda^{-m} e^{-idt} P e^{idt}$$

is a well-defined polynomial, homogeneous of
 degree m , on the fibers of T^*X and values
 in $\text{hom}(E, F)$:

$$\sigma(P) \in \mathcal{O}^m(T^*X; \text{hom}(E, F)).$$

In local coordinates, (5), (4)

$$(1) \quad P\left(\sum_{l=1}^L u_l e_l\right) = \sum_{|\alpha| \leq m} \sum_{l'=1}^{L'} \left(p_{\alpha, l', l} {}^{(3)}D^{\alpha}_l \right) f_{l'},$$

$$\sigma(P)(3,5) = \sum_{|\alpha|=m} \sum_{l'=1}^{L'} \left(p_{\alpha, l', l} {}^{(3)}S^{\alpha}_l \right) f_{l'}.$$

Of course, we could just use this as a definition, but then we would have to check coordinate independence! The abstract - no seen definition makes it easy to check (4)

$$(2) \quad \sigma(PQ) = \sigma_m(P) \cdot \sigma_{m'}(Q)$$

$$P \in \text{Diff}^m(X; E_1, E_2), \quad Q \in \text{Diff}^{m'}(X; F_1, F_2).$$

This " σ " tells the different non-commutations, and just like the basic non-commutativity, but

The real importance of this 'symbol' is

J/10
so we get a short exact sequence

$$\text{Diff}^{n-1}(X, E, F) \hookrightarrow \text{Diff}^n(X, \tilde{F}, F) \xrightarrow{\sigma_n} P^n(T^*x; \text{ker}(E, F)).$$

Exers Done this!

Let me recall what a metric on a (real) bundle is. It is simply a positive-definite inner product on each fibre, varying smoothly. Then a metric $\langle \cdot, \cdot \rangle_p$ on the fibers E_p of $E \downarrow X$

to be said that $\langle e, e' \rangle$ is short $\forall e, e' \in C^\infty(X, E)$.

A metric on TX is a Riemann metric. Now, the length-square function for the line metric

$$| |_p^2 : T_p^*X \rightarrow \mathbb{R}$$

is a homogeneous polynomial of degree two.

II/11

A generalized Dirac form on a bundle
 E (complex) is a first order differential form
 $\sigma \in \text{Diff}^1(X; E)$ s.t.

$$(1) \quad (\sigma(\partial))^2 = 1 \cdot 1^2 \times \text{Id}$$

for a matrix $a \in \mathbb{X}$ (not $\in!$)

What does this mean? Consider, for ex.,
the symbol

$$\sigma_1(\partial)(s) = d(s) \in \text{Hom}(E).$$

This defines a linear map

$$T_p^* X \ni s \mapsto d(s) \in \text{Hom}(E).$$

Let's see the effect of (1). If $s_1, s_2 \in T_p^* X$

then

$$\sigma_1(d(s_1 + s_2))^2 = |s_1 + s_2|^2$$

\mathbb{H}/\mathbb{K}

$$= |\zeta_1|^2 + 2\langle \zeta_1, \zeta_2 \rangle + |\zeta_2|^2$$

$$= (\ell(\zeta_1))^2 + \ell(\zeta_1) \cdot \ell(\zeta_2) + \ell(\zeta_2) \ell(\zeta_1)$$

$$+ (\ell(\zeta_2))^2$$

$$\Rightarrow \ell(\zeta_1) \cdot \ell(\zeta_2) + \ell(\zeta_2) \cdot \ell(\zeta_1)$$

$$= 2 \langle \zeta_1, \zeta_2 \rangle.$$

Def: If $V (= T_p X)$ is a real vector space with Endomorphism \langle , \rangle the algebra

$$\sum_{k=0}^{\infty} V^{\otimes k} / (\zeta_1 \otimes \zeta_2 + \zeta_2 \otimes \zeta_1 - 2\langle \zeta_1, \zeta_2 \rangle)$$

is called the clifford algebra of V .
 As a vector space it is isomorphic to $\wedge^* V$.