

18.158: Sept 9, 2003

I. Manifolds and related spaces

In these lectures I want to touch upon various aspects of local and global analysis on compact manifolds, especially relating to boundaries and corners. This is rather a big subject! My hope is to explain some topics in reasonable detail but really to convey some idea of what I think is important. Exactly what I will manage to cover is not really clear to me at the start, but here is a partial list:-

- Manifolds and related spaces
- Dirac and other differential operators
- Pseudodifferential operators

- Hodge theory
- Boundary conditions
- Caves
- K-theory
- Curves

If that isn't enough for 20 lectures or so, there are plenty of other things to look at.

My plan is actually to produce a document based on these lectures. In fact I hope you will all participate in this. Before each class I will write up some sort of notes on what the lecture will be based — this of course is the first one. This may not be as complete or polished as you or I might hope

but that is your opportunity. I encourage you to give me something based on each lecture. This could be almost anything:-

- Corrections & legible rewriting
- Expansion of my notes based on the lectures
- Additions
- Examples worked out
- Problems solved
- Questions
- References and cross-references

These can be in any form you choose - verbal (dangerous), hand written, email, Tex document etc. I reserve the right to use any of this material in the document,

however this may turn out - of course I will try to keep a group on who did what. This is all somewhat experimental!

so I hope to have:

- Rough notes before each lecture

- Modified version to have typed by its next lecture

- TeX version by one week later

- Continuing subsequent revisions.

I will try to arrange that you can see this, and participate, at all stages

The basic spaces I want to 'work on' are manifolds of various types. These consist, typically, of a space, X , with

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some basic topological structure, additional structure given by some space of functions on X and perhaps further global restrictions. I do not want to formalize this, but let me just consider examples.

I.1 Manifolds. A C^∞ manifold is a Hausdorff, ~~topological~~ topological space, X , with a space of ('smooth') functions $\mathcal{F} \subset C^0(X)$. Here $C^0(X)$ is the space of continuous real- (or in context complex-) valued functions $f: X \rightarrow \mathbb{R}$. If $U \subset X$ is open we can restrict any $f \in C^0(X)$ to U by defining

$$\mathcal{F}(U) = \left\{ f \in C^0(U); \forall p \in U \exists V \subset U \text{ open, with } p \in V \text{ and } g \in \mathcal{F} \text{ s.t. } f = g \text{ on } V \right\}.$$

If \mathcal{F} is linear then so is $\mathcal{F}(U) \subset C^0(U)$. The additional conditions we place on

of dimension n

\mathcal{F} to make it a C^∞ structure/ce that

- (1) For any $p \in X$ $f_1, \dots, f_n \in \mathcal{F}$ and
an open set $U \ni p$ s.t. $F = (f_1, \dots, f_n) : U \rightarrow U' \subset \mathbb{R}^n$ /*
 $g \in \mathcal{F}(U) \Leftrightarrow \exists h \in C^\infty(U')$ with
 $g = h \circ F$
[h is a homeomorphism of open sets and].

(2) \mathcal{F} is local, i.e.

$$\mathcal{F} = \left\{ f \in C^\infty(X); \text{ each } p \in X \text{ has an open neighborhood } U \subset X \text{ s.t. } f|_U \in \mathcal{F}(U) \right\}.$$

We also demand $|U| \times \mathcal{F}$ to be second countable
and then say X is a C^∞ manifold with
 $C^\infty(X) = \mathcal{F}$.

Exercise! Show that this, slightly odd,
definition is equivalent to others that you know
or can point to.

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The point of saying things the way is
only that we can see pretty clearly the role of
the 'model space', in the case \mathbb{R}^n , a really
open sets in \mathbb{R}^n , and the model function space,
 $C^\infty(\mathbb{R}^n)$.

Exercise 2. Show that it suffices to demand
that $U = \overset{\circ}{B} = \{x \in \mathbb{R}^n; |x| < 1\}$ is in the
definition above.

We can modify this general set up in
many ways. One fundamental ~~one~~ class of
spaces defined this way are manifolds with
corners. Here we take as model spaces
relatively open subsets of

$$\mathbb{R}^{n,k} = [0, \infty)^k \times \mathbb{R}^{n-k},$$

with $C^\infty(\mathbb{R}^{n,k}) = C^\infty(\mathbb{R}^n) /_{\mathbb{R}^{n,k}}$.

There are a lot of such open subsets; if you prefer to have only finitely many local charts consider only (for a fixed dimension) the product

$$[0,1)^k \times (-1,1)^{n-k} = I^{n,k}.$$

Definition A manifold with fixed corners is a (connected) second countable Hausdorff topological space X with $\mathcal{F} \subset C^\infty(X)$ giving such that

- (1) $\forall p \in X$ there are elements $f_1, \dots, f_n \in \mathcal{F}$ such that $F: U \ni q \mapsto (f_1(q), \dots, f_n(q)) \in I^{n,k}$ is a homeomorphism from some neighbourhood of p and

$$g \in \mathcal{L}(U) \Leftrightarrow g = h \circ F \text{ for some } h \in C^\infty(I^{n,k})$$

- (2) \mathcal{F} is a local.

Exercise 3 Show that n is fixed (since X is assumed connected) but as that k is dependent of p .

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The most familiar one is when k is restricted
 $\Rightarrow k \leq 1$; X is then called a manifold with
 boundary.

Note the weird 'fied' in the name.
 This is not standard notation nomenclature
 but is inserted here because I like a
~~top~~ manifold with corners to have an
 additional property. Namely consider

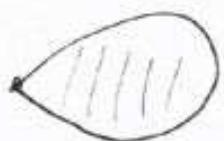
$$\partial_1 X = \{ \phi x; k=1 \}$$

The 'cylinder-side on' part of the boundary of X .
 This may have several, in fact from an definition
 even infinitely many, components. Take a
 component $H \subset \partial_1 X$ and let H be its
 closure in X . We demand that

- (1) \mathcal{D}_H make H into a manifold with ~~fied~~
 corners.

If this is a true for all manifolds then X is a manifold with corners.

Exercise 4 Show that this is not automatic $\frac{1}{10}$
by considering a 2-dimensional example as
picture.



Exercise 5 Show that the closure of each component of $\partial_1 X$, where X is a manifold with corners is itself a manifold with corners.

The closures of the components of $\partial_1 X$ are the boundary hypersurfaces of X . A additional condition is that (\cdot) can not then be embedded. We denote by $H_1(X)$ the set of these boundary hypersurfaces.

Exercise Show that each component of an intersection $H_1 \cap \dots \cap H_N$, $H_i \in \mathcal{H}_1(X)$ is a manifold with corners.

Other examples of spaces obtained by variants of this definition include real-analytic manifolds (with corners) as far as stone vector bundles.

Example (Silly) Suppose we take a usual space E consists open subsets of a vector space and usual functions to be the restrictions of linear functions to these sets, show that what spaces do we get?

A less pointless example of the type is a vector bundle. On the total space, W , we consider two function spaces

$$\mathcal{Z}, \mathcal{Z}_1, C^{\infty}(W)$$

then the localization property is not respect to $I^{n,k} \times \mathbb{R}^N$ with the space $C^{\infty}(I^{n,k})$ and unital C^{∞} function in the second variable

$$I^{n,k} \times \mathbb{R}^N \rightarrow \mathbb{R}.$$

Exercise 7 Write out the definition of a vector bundle structure on W in detail in both the real and complex (fiber) cases and show that it reduces to the usual definition.

Exercise 8 Do the same for a fibration of compact manifolds with corner by identifying it as a pair of functor spaces on the then reduce with total space with the model being $I^{n+k} \times Z$, where Z is a fixed compact manifold with corners. Don't forget the global embedded boundary condition.

There are other, more complicated, constructions of this type which are worth thinking about, at least a little. For

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instance, suppose X is a compact manifold with boundary. Instead of the whole of $C^\infty(X)$ we can consider the conic structure at X , namely just

$$(*) \quad g = \{ u \in C^\infty(X) ; \frac{\partial u}{\partial x} \text{ is constant} \}.$$

Letting this be the effect that g no longer separates boundary points. You might like to consider why g should represent a conic structure. One reason is that the vector fields uX, V , such that $V \cdot g \in C^\infty(X)$ are linearly spanned by

$$\frac{\partial}{\partial x} \quad x \times \frac{1}{2} dy_j.$$

I will come back to examples like this later.

Problem 1 Suppose X is a compact manifold with boundary and $\varphi: \partial X \rightarrow Y$ is a fibration;

What are the vector fields associated in the way
to

$$\mathcal{L}_\varphi = \{u \in C^\infty(X); u|_{\partial X} \in \varphi^* C^\infty(Y)\}?$$

What might this structure define?

N.B. Patterns are supposed to be more open-ended
than Exercises!

The spaces of functions described above are
 C^∞ algebras. This if \mathbb{Y} consists of real-valued
functions, f_1, \dots, f_n and $h \in C^\infty(\mathbb{R}^n)$. Note
 $h(f_1, \dots, f_n) \in \mathbb{Y}$. This is not always the
case but even if it is this may be patterns!

Pattern 2 If X is a compact manifold (say
Exercise 4) If X is compact
without boundary) and $Y \subset X$ is an embedded
submanifold ($\iota: C^\infty(X)/\iota Y \cong C^\infty(Y)$) show that

$$\mathbb{Z}' = \{u \in C^\infty(X); u|_Y \text{ is constant}\}$$

is C^∞ algebra. Show how to identify $\mathbb{X} \otimes \mathbb{Y}$ with
the interior of a compact manifold with
boundary so that $\mathbb{Z}' \subset \mathbb{Y}$ given by (*)