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18.158: Sept 9, 2003

I. Manifolds and related spaces

In these lectures I want to touch upon various aspects of local and global analysis on compact manifolds, especially relating to boundaries and corners. This is rather a big subject! My hope is to explain some topics in reasonable detail but really to convey some idea of what I think is important. Exactly what I will manage to cover is not really clear to me at the start, but here is a partial list:—

- Manifolds and related spaces
- Dirac and other differential operators
- Pseudodifferential operators

- Hodge theory
- Boundary conditions
- Cones
- K-theory
- Gerbes

If that isn't enough for 20 lectures or so, there are plenty of other things to look at.

My plan is actually to produce a document based on these lectures. In fact I hope you will all participate in this. Before each class I will write up some sort of notes on which the lecture will be based — this of course is the first one. These may not be as complete or polished as you or I might hope but that is your opportunity. I encourage you to give me something based on each lecture. This could be almost anything:-

- Corrections & legible rewriting
- Expansion of my notes based on the lecture
- Additions
- Examples worked out
- Problems solved
- Questions
- References and cross-references

These can be in any form you choose - verbal (dangerous), hand written, email, TeX document etc. I reserve the right to use any of this material in the 'document', however things may turn out – of course I will try to keep a grasp on who did what. This is all somewhat experimental!

So I hope to have:

- Rough notes before each lecture
- Modified version to have keyed by the next lecture
- TeX version by one week later
- Continuing subsequent revisions.

I will try to arrange that you can see this, and participate, at all stages

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The basic spaces I want to 'work on' are manifolds of various types. These consist, typically, of a space, X, with some basic topological structure, additional structure given by some space of functions on X and perhaps further global restrictions. I do not want to formalize this, but let me just consider examples.

I.1 Manifolds. A C^∞ manifold is a Hausdorff, ~~topol~~ topological space, X, with a space of 'smooth' functions $\mathcal{F} \subset C^0(X)$. Here $C^0(X)$ is the space of continuous real- (or in context complex-) valued functions $f : X \longrightarrow \mathbb{R}$. If $U \subset X$ is open we can <u>localize</u> any $\mathcal{F} \subset C^0(X)$ to U by defining

$$\mathcal{F}(U) = \{ f \in C^0(U);\ \forall\, p \in U\ \exists\ V \subset U \text{ open, with } p \in V \text{ and } g \in \mathcal{F} \text{ s.t. } f = g \text{ on } V \}.$$

If $\mathcal{F}$ is linear then so is $\mathcal{F}(U) \subset C^0(U)$.

The additional conditions we place on

$\mathcal{Y}$ to make it a C^∞ structure of dimension n are that

(1) For any $p \in X$ $\exists$ $f_1, \dots, f_n \in \mathcal{Y}$ and an open set $U \ni p$ s.t $F = (f_1, \dots, f_n) : U \to U' \subset \mathbb{R}^n$ * and

$$g \in \mathcal{Y}(U) \iff \exists\, h \in C^\infty(U') \text{ with } g = h \circ F$$

[* is a homeomorphism of open sets and]

(2) $\mathcal{Y}$ is local, i.e.

$$\mathcal{Y} = \{ f \in C^0(X) ;\ \text{each } p \in X \text{ has an open neighbourhood } U \subset X \text{ s.t. } f|_U \in \mathcal{Y}(U) \}.$$

We also demand that X be <u>second countable</u> and then say X is a C^∞ manifold with $C^\infty(X) = \mathcal{Y}$.

<u>Exercise 1</u> Show that this, slightly odd, definition is equivalent to others that you know or can point to.

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The point of saying things this way is only that we can see pretty clearly the role of the 'model space', in this case $\mathbb{R}^n$, or really open sets in $\mathbb{R}^n$, and the model function space, $C^\infty(\mathbb{R}^n)$.

Exercise 2 Show that it suffices to demand that $U' = \mathring{B}^n = \{x \in \mathbb{R}^n ; |x| < 1\}$ in the definition above.

We can modify this general set up in many ways. One fundamental ~~sec~~ class of spaces defined this way are manifolds with corners. Here we take as model spaces relatively open subsets of

$$\mathbb{R}^{n,k} = [0,\infty)^k \times \mathbb{R}^{n-k},$$

with $\quad C^\infty(\mathbb{R}^{n,k}) = C^\infty(\mathbb{R}^n)\big|_{\mathbb{R}^{n,k}}.$

There are a lot of such open subsets; if you prefer to have only finitely many local models consider only (for a fixed dimension) the products

$$[0,1)^k \times (-1,1)^{n-k} = I^{n,k}.$$

<u>Definition</u> A manifold with fixed corners is
(connected)
a second countable Hausdorff topological space X with $\mathcal{F} \subset C^0(X)$ given such that

(1) For each $p \in X$ there are n elements $f_1, \dots, f_n \in \mathcal{F}$ for which $F: U \ni q \mapsto (f_1(q), \dots, f_n(q)) \in I^{n,k}$ is a homeomorphism from some neighbourhood of p and

$$g \in \mathcal{F}(U) \Leftrightarrow g = h \circ F \text{ for some } h \in C^\infty(I^{n,k})$$

(2) $\mathcal{F}$ is local.

<u>Exercise 3</u> Show that n is fixed (since X is assumed connected) ~~but~~ and that k is determined by p.

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The most familiar case is when k is restricted to $k \leq 1$; X is then called a manifold with boundary.

Note the weird 'tied' in the name. This is not standard ~~notation~~ nomenclature but is inserted here because I like a ~~[illegible]~~ manifold with corners to have an additional property. Namely consider

$$\partial_1 X = \{ p \in X ; \ k = 1 \}$$

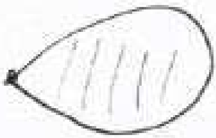
the 'codimension one' part of the boundary of X. This may have several, in fact from our definition even infinitely many, components. Take a component $\mathring{H} \subset \partial_1 X$ and let H be its closure in X. We demand that

(i) $\mathcal{F}/_H$ makes H into a manifold with tied corners.

If this is true for all components then X is a manifold with corners.

<u>Exercise 4</u> Show that this is not automatic I/10

by considering a 2-dimensional example as pictured.



<u>Exercise 5</u> Show that the closure of each component of $\partial_1 X$, when X is a manifold with corners is itself a manifold with corners.

The closures of the components of $\partial_1 X$ are the boundary hypersurfaces of X. An additional condition $(\cdot)$ is that these be <u>embedded</u>.

We denote by $M_1(X)$ the set of these boundary hypersurfaces.

<u>Exercise</u> Show that each component of an intersection $H_1 \cap \cdots \cap H_N$, $H_i \in M_1(X)$ is a manifold with corners.

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Other examples of spaces obtained by variants of this definition include real-analytic manifolds (with corners) and for instance vector bundles.

<u>Example</u> (Silly) Suppose we take as model spaces the convex open subsets of a vector space and model functions to be the restrictions of linear functions to these sets; ~~Show that we~~ what spaces do we get?

A less pointless example of this type is a vector bundle. On the total space, W, we consider two function spaces

$$\mathcal{L}_0, \mathcal{L}_1 \subset C^0(W)$$

with the localization property as with respect to $I^{n,k} \times \mathbb{R}^N$ with the space $C^\infty(I^{n,k})$ and linear C^∞ functions in the second variable

$$I^{n,k} \times \mathbb{R}^N \to \mathbb{R}.$$

Exercise 7 Write out the definition of a vector bundle structure on W in detail in both the real and complex (fibre) cases and show that it reduces to the usual definition.

Exercise 8 Do the same for a fibration of compact manifolds with corners by identifying ~~it as~~ a pair of function spaces on the this rather with total space with the model being $I^{n,k} \times Z$, where Z is a fixed compact manifold with corners. Don't forget the global embeddedness condition.

There are other, more complicated, constructions of this type which are worth thinking about, at least a little. For

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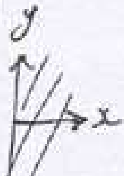
instance, suppose X is a compact manifold with boundary. Instead of the whole of $C^\infty(X)$ we can consider the cone structure on X, namely just

(*) $\quad \mathcal{G} = \{ u \in C^\infty(X);\ u\big|_{\partial X} \text{ is constant} \}.$

Certainly this has the effect that $\mathcal{G}$ no longer separates boundary points. You might like to consider why $\mathcal{G}$ should represent a cone structure. One reason is that the vector fields on X, V, such that $V \cdot \mathcal{G} \subset C^\infty(X)$ are locally spanned by



$$\partial_x \text{ and } \frac{1}{x}\partial_y.$$

I will come back to examples like this later.

<u>Problem 1</u> Suppose X is a compact manifold with boundary and $\varphi : \partial X \to Y$ is a fibration;

what are the vector fields associated in this way to

$$\mathcal{G}_\varphi = \{u \in C^\infty(X);\ u|_{\partial X} \in \varphi^* C^\infty(Y)\}?$$

What might this structure define?

NB Problems are supposed to be more open-ended than Exercises!

The spaces of functions described above are C^∞ algebras. This is if $\mathcal{I}$ consists of real-valued functions, $f_1, \dots, f_k \in \mathcal{I}$ and $h \in C^\infty(\mathbb{R}^k)$ then $h(f_1, \dots, f_k) \in \mathcal{I}$. This is not always the case but even if it is there may be problems!

Problem 2. ~~Exercise 4~~ If $\tilde X$ is a compact manifold (say without boundary) and $Y \subset \tilde X$ is a compact embedded submanifold (i.e. $C^\infty(\tilde X)|_Y = C^\infty(Y)$) show that

$$\mathcal{I}' = \{u \in C^\infty(\tilde X);\ u|_Y \text{ is constant}\}$$

is C^∞ algebra. Show how to identify $\tilde X \setminus Y$ with the interior of a compact manifold with boundary X' so that $\mathcal{I}' \subset \mathcal{G}$ given by (*)

· Dirac and other differential operators

Let me quickly review the basic definitions of forms and vector fields, starting from a C^∞ manifold. As before, the idea is doing this so to help with subsequent generalizations.

A C^∞ manifold, X, for the moment without corners if you wish, comes equipped with a space of smooth functions, $\mathcal{C}^\infty(X)$. A point $p \in X$ can be recovered from its defining ideal

$$\mathcal{J}_p = \{u \in C^\infty(X);\ u(p) = 0\}.$$

If we let $\mathcal{J}_p^2 \subset C^\infty(X)$ be the finite span of products of elements of $\mathcal{J}_p$

$$\mathcal{J}_p^2 = \{u \in C^\infty(X);\ u = \sum_{i=1}^{N} f_i g_i,\ f_i, g_i \in \mathcal{J}_p\}$$

Then

$$T_p^* X = \mathcal{J}_p / \mathcal{J}_p^2$$

is the cotangent fibre at p. If $u \in C^\infty(X)$ then $u - u(p) \in \mathcal{I}_p$ so there is a well-defined element

(1) $$du(p) \in T_p^* X.$$

<u>Exercise</u>! Show that if $z_1, \dots, z_n$ are local coordinates at p then $dz_1, \dots, dz_n$ is a basis for $T_p^* X$; conclude that

$$T^* X = \bigcup_{p \in X} T_p^* X$$

is a (real) vector bundle over X.

Note that the local coordinates in $T^* U = \bigcup_{p \in U} T_p^* X$ induced by local coordinates $z_1, \dots, z_n$ in $U \subset X$ are given by $(z_1, \dots, z_n, \zeta_1, \dots, \zeta_n)$ where

$$\zeta = \sum_{j=1}^{n} \zeta_j \, dz_j \, , \quad \zeta \in T_p^* X.$$

The C^∞ structure on $T^* X$ is consistent with (1), namely

(2) $$d : C^\infty(X) \longrightarrow C^\infty(X; T^* X)$$

where on the right we use the notation for smooth section of T^*X. This is the basic example of a (geometric) differential operator.

The usual tangent bundle TX can be defined either as the dual of T^*X or directly in terms of derivations. That is

$$T_pX = \{v : C^\infty(X) \to \mathbb{R};\ v \text{ is linear and}$$
$$v(fg) = f(p)\,v(g) + v(f)\,g(p)\}.$$

Exercise Show that $T_pX \equiv (T_p^*X)^*$ with the identification being given by a pairing

$$T_pX \times T_p^*X \to \quad (v, \xi) \longmapsto v(f), \quad f \in \mathcal{I}_p,\ [f] = \xi \text{ in } T_p^*X.$$

A section $V \in C^\infty(X; TX)$ then defines a linear map

$$V : C^\infty(X) \to C^\infty(X), \quad Vf(p) = V_pf.$$

By definition a linear differential operator, with smooth coefficients acting on functions is just a combination of such vector fields

$$P : C^\infty(X) \longrightarrow C^\infty(X),$$

$$(3) \qquad Pu = \sum_{\substack{\text{finite} \\ a \in A}} V_{1,a} \cdots V_{k_a,a}\, u, \quad (X \text{ compact}).$$

The first order case, $k_a \le 1$, is just

$$P = V + f, \quad V \in C^\infty(X; TX),\ f \in C^\infty(X).$$

We are more interested in differential operators acting on (complex) vector bundles. Let me give a couple of equivalent definitions.

First, we can 'reset' to local coordinates. The basis of T_pX induced by local coordinates $z_1, \dots, z_n$ is $\partial/\partial z_1, \dots, \partial/\partial z_n$ where

$$\partial/\partial z_j \cdot dz_k = \delta_{jk}.$$

The local coordinate form of (3) is that

$$(4) \qquad Pu = \sum_{|\alpha| \le m} p_\alpha(z) D^\alpha u, \quad p_\alpha \ C^\infty$$

using multi-index notation, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, II/5

$|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial z}\right)^\alpha = i^{-|\alpha|} \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$.

Thus a differential operator P is just a linear map $P: C^\infty(X) \to C^\infty(X)$ which takes the form (4) in any local coordinates (it suffices for this to be true in a covering of X by charts).

A vector bundle E has local trivializations, i.e. X is covered by coordinate patches U_i over each of which E has a basis of smooth sections $e_1, \dots, e_L$. If F is another vector bundle then we can find a covering by coordinate charts over which both E and F have local bases, i.e. are trivial. Then

$$P: C^\infty(X; E) \longrightarrow C^\infty(X; F)$$

a linear map, is a differential operator of order (at most) m if

$$(5) \quad P\left(\sum_{l=1}^{L} u_l e_l\right) = \sum_{l'=1}^{L'} (P_{l'l}\, u_l) f_{l'}$$

when the $P_{l',l}$ are of the form (4). An interesting, although essentially useless, result is

<u>Theorem</u> (Peetre) $P: C^\infty(X;E) \to C^\infty(X;F)$ a linear map is a differential operator if and only if

$$u=0 \text{ on } U\subset X \Rightarrow Pu=0 \text{ on } U$$

$$\forall\, U\subset X \text{ open.}$$

<u>Exercise</u> Let $\mathrm{Diff}^m(X;E,F)$ denote the space of linear differential operators between sections of bundles E and F (of order at most m). Show that composition of operators defines a product

(6) $\mathrm{Diff}^{m_2}(X;E_2,E_3)\cdot \mathrm{Diff}^{m_1}(X;E_1,E_2) \subset \mathrm{Diff}^{m_1+m_2}(X;E_1,E_3)$.

As I said before, we are mostly interested in <u>first order</u> operators. Let us continue a little in the general case however.

Suppose $f \in C^\infty(X)$ and $u \in C^\infty(X;E)$ then for $\lambda \in \mathbb{C}$,

$$(7) \qquad u_\lambda = e^{i\lambda f} u \in C^\infty(X;E).$$

Leibniz' formula shows us how to distribute differentiation over such a product, namely

$$D_z^\alpha (uv) = \sum_{\beta \le \alpha} \binom{\beta}{\alpha} D_z^{\alpha-\beta} u \cdot D_z^\beta v.$$

If we think what this means for P, i.e. local coordinates looking like (4) we see that

$$(8) \quad P(e^{i\lambda f} u) = e^{i\lambda f} P_{\lambda,f}\, u.$$

Here $P_{\lambda,f}$ is again a differential operator of the same order, now with coefficients depending on λ and f. In fact

$$P_{\lambda,f} = \sum_{s=0}^{m} \lambda^s P_{(s),f}$$

is a polynomial of degree at most m in λ. The powers of λ come from ~~the~~ derivatives 'hitting' $e^{i\lambda f}$ so it follows that

$$P_{(s),f} \in \mathrm{Diff}^{m-s}(X; E, F).$$

Now, for $s = m$, $\mathrm{Diff}^0(X; E, F) = C^\infty(X; \hom(E, F))$ is just the space of bundle maps from E to F. Of course P_s still depends on f but we can see that

$P_{(m),f}$ is a polynomial in df of degree (at most) m.

$\sigma(P)(df) =$

Proposition $P_{(m),f} = \lim\limits_{\lambda \to \infty} \lambda^{-m} e^{-i\lambda f} P e^{i\lambda f}$ is a well-defined polynomial, homogeneous of degree m, on the fibres of T^*X with values in $\hom(E, F)$:

$$\sigma(P) \in P^m(T^*X; \hom(E, F)).$$

In local coordinates, (5), (4)

$$P\left(\sum_{l=1}^{L} u_l e_l\right) = \sum_{|\alpha|\le m} \sum_{l'=1}^{L'} \left(p_{\alpha,l',l}(z) D^\alpha u_l\right) f_{l'},$$

( )

$$\sigma(P)(z,\zeta) = \sum_{|\alpha|=m} \sum_{l'=1}^{L'} \left(p_{\alpha,l',l}(z)\, \zeta^\alpha u_l\right) f_{l'}.$$

Of course, we could just use this as a definition, but then we would have to check coordinate independence! The abstract-nonsense definition makes it easy to check that

$$\sigma_{m+m'}(PQ) = \sigma_m(P)\cdot\sigma_{m'}(Q)$$

( )

$$P \in \mathrm{Diff}^m(X; E_2, E_3),\quad Q \in \mathrm{Diff}^{m'}(X; E_1, E_2).$$

This "σ" kills the differential non-commutativity, and just keeps the ~~basic~~ bundle non-commutativity.

The real importance of these 'symbol' maps

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so that we get a short exact sequence

$$\mathrm{Diff}^{m-1}(X;E,F) \hookrightarrow \mathrm{Diff}^m(X;E,F) \xrightarrow{\sigma_m} P^m(T^*X;\hom(E,F)).$$

Exercise Prove this!

Let me recall what a metric on a (real) bundle is. It is simply a positive-definite inner product on each fibre, varying smoothly. Thus a fibre on each fibre, varying smoothly. Thus each fibre E_p of $E \to X$ has metric $\langle\ ,\ \rangle_p$ on the fibre E_p of E and

to be such that $\langle e, e' \rangle$ is smooth $\forall\, e, e' \in C^\infty(X;E)$.

A metric on TX is a Riemann metric. Now, the length-squared function for the dual metric

$$|\ |_p^2 : T_p^*X \longrightarrow \mathbb{R}$$

is a homogeneous polynomial of degree two.

A *generalized Dirac operator* on a bundle E (complex) is a first order differential operator $\eth \in \mathrm{Diff}^1(X; E)$ s.t.

(ii) $(\sigma_1(\eth))^2 = |\cdot|^2 \times \mathrm{Id}$

for a metric on $\underline{\underline{X}}$ (not E!)

What does this mean? Consider, for $p \in X$, the symbol

$$\sigma_1(\eth)(\zeta) = a(\zeta) \in \hom(E_p).$$

This defines a *linear* map

$$T_p^*X \ni \zeta \longmapsto a(\zeta) \in \hom(E_p).$$

Let's see the effect of (ii). If $\zeta_1, \zeta_2 \in T_p^*X$ then

$$\left(a(\zeta_1+\zeta_2)\right)^2 = |\zeta_1+\zeta_2|^2$$

$$= |\xi_1|^2 + 2\langle \xi_1, \xi_2 \rangle + |\xi_2|^2$$

$$= \left(cl(\xi_1)\right)^2 + cl(\xi_1)\cdot cl(\xi_2) + cl(\xi_2)\, cl(\xi_1) + \left(cl(\xi_2)\right)^2$$

$$\Rightarrow \quad cl(\xi_1)\cdot cl(\xi_2) + cl(\xi_2)\cdot cl(\xi_1)$$

$$= 2\langle \xi_1, \xi_2 \rangle .$$

<u>Def$^{\underline{n}}$</u>: If V $(= T_pX)$ is a real vector space with Euclidean inner product $\langle\ ,\ \rangle$ the algebra

$$\sum_{k=0}^{\infty} V^{\otimes k} \Big/ \left(\xi_1\otimes\xi_2 + \xi_2\otimes\xi_1 - 2\langle \xi_1, \xi_2\rangle\right)$$

is called the <u>Clifford algebra</u> of V.
As a vector space it is isomorphic to $\Lambda^* V$.

III/1 Ellipticity and pseudodifferential operators.

I want to start today with a 'trivialty' about integration and densities. Recall that on $\mathbb{R}^n$ we can integrate functions which are compactly supported and reasonably smooth. This certainly includes functions in $C_c^\infty(\mathbb{R}^n)$ – smooth functions which vanish outside a large ball.

$$C_c^\infty(\mathbb{R}^n) \ni u \longmapsto \int_{\mathbb{R}^n} u\,dx.$$

The problem with the integral is that it does not behave well under diffeomorphisms. For instance if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a global diffeomorphism and $u \in C^\infty(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} F^* u\,dx = \int_{\mathbb{R}^n} u \circ F\,dx \underset{\substack{\text{(in}\\\text{general)}}}{\neq} \int_{\mathbb{R}^n} u\,dx$$

III/2 The reason is that

$$\int_{\mathbb{R}^n} u \, dx = \int_{\mathbb{R}^n} F^* u \cdot |J_F| \, dx$$

where $J_F = \det \dfrac{\partial F_j}{\partial x_k}$. Thus to integrate invariantly we need something ~~rather~~ which transforms with a factor $|J_F|$. The object in question is a density.

Recall that the n-form bundle on X, where $\dim X = n$, is the totally antisymmetric part of the n-fold tensor product:

$$\Lambda^n_p X = \Lambda^n (T_p^* X) \subset (T_p^* X)^{\otimes n}.$$

Thus an element of $(T_p^* X)^{\otimes n}$ is a multilinear form

$$\mu : T_p X \times \cdots \times T_p X \longrightarrow \mathbb{R} \ (\text{or } \mathbb{C})$$

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as $\Lambda^n_p X \subset (T_p^* X)^{\otimes n}$ consists of the totally antisymmetric elements

$$\mu(\dots, v_i, v_{i+1} \dots) = -\mu(\dots, v_{i+1}, v_i \dots) \quad \forall i = 1, \dots n-1.$$

We can also define the fibre $\Lambda^n_p(T_p X)$ of the 'multivector' bundle as the totally antisymmetric multilinear forms

$$\underbrace{T_p^* X \times T_p^* X \times \dots \times T_p^* X}_{n\text{-factors}} \to \mathbb{R}.$$

The reason for considering this is:

Exercise check that $\Lambda^n_p X = \Lambda^n(T_p^* X)$ is canonically identified with the dual of $\Lambda^n(T_p X)$.

Thus $\Lambda^n_p X$ is just the space of linear forms (so a 1-D vector space)

$$\mu_p : \Lambda^n_p(T_pX) \longrightarrow \mathbb{R}.$$

We 'know' that these forms transform with a factor of J_F, the determinant of the Jacobian, under coordinate changes (this is one way to *define* the determinant). We want to get $|J_F|$ into the transformation laws. To do so *define*

$$\Omega_p X = \{ \nu : \Lambda^n_p(T_pX) \setminus 0 \longrightarrow \mathbb{R}$$

(*) absolutely homogeneous of degree 1,

$$\nu(s\gamma) = |s|\,\nu(\gamma) \quad \forall \gamma \in \Lambda^n_p(T_pX)\setminus 0,$$

$$s \in \mathbb{R}\setminus 0 \}.$$

Exercise Check that this is a 1-D vector space and that if $0 \neq \mu \in \Lambda^n_p X$ then $|\mu| \in \Omega_p X$ is a basis element.

Exercise Show that ΩX is a well-defined line bundle on X which is *trivial* i.e.

be a global non-vanishing (positive) smooth section. Show in particular that if g is a Riemannian metric on X then $|dg|$ (which I would denote dg) is a well-defined global positive section of ΩX.

As already noted, the point about densities, which are sections of ΩX, is that they can be invariantly integrated. Thus, there is a global linear map

$$(\mathrm{I}) \quad \int_X : C^\infty(X;\Omega) \to \mathbb{R}$$

such that if $u \in C^\infty(X;\Omega)$ has support in a coordinate patch, $u = u(x)\,|dx|$, $|dx|$ being Lebesgue measure,

$$\int_X u = \int_{\mathbb{R}^n} u(x)\,dx.$$

<u>Exer</u> Check these are unique up to a p.o.i.

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Exercise Show that the definition (*) of the density bundle can be modified to yield α-densities for any $\alpha \in \mathbb{R}$:

$$v \in \Omega^\alpha_p X \iff u : \Lambda^n_p(T_pX) \setminus 0 \longrightarrow \mathbb{R},$$

$$u(s\gamma) = |s|^\alpha \gamma \quad \forall\, s \in \mathbb{R}\setminus\{0\},$$

$$\gamma \in \Lambda^n_p(T_pX)\setminus 0 .$$

Show that these always give trivial line bundles over X. Show that there is a canonical isomorphism

$$\Omega^0 X \cong X \times \mathbb{R}$$

and that for any α, β there is a canonical isomorphism

$$\Omega^\alpha X \otimes \Omega^\beta X \cong \Omega^{\alpha+\beta} X.$$

Use this to show that there is a canonical pre-Hilbert space structure on $C^\infty(X; \Omega^{1/2})$

$$\langle u, v \rangle = \int_X u \bar{v}.$$

Using the integral and the observation that if $\overset{u}{\underset{j}{}}\!\!\in C^\infty(X;\Omega)$ and $f\in C^\infty(X)$ then $fu\in C^\infty(X;\Omega)$ we define a bilinear map

$$C^\infty(X)\times C^\infty(X;\Omega)\longrightarrow \mathbb{C}\quad(\text{or }\mathbb{R})$$

(P)
$$(f,u)\longmapsto \int_X fu.$$

<u>Exercise</u> Think for a minute (no longer) about the topology on $C^\infty(X;\Omega)$ and $C^\infty(X)$ of 'uniform convergence of all derivatives on compact subsets of coordinate patches'.

Thus if $f\in C^\infty(X)$ then

$$M_f : C^\infty(X;\Omega)\longrightarrow \mathbb{R}$$

$$u\longmapsto \int_X fu$$

is a <u>continuous</u> linear map.

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<u>Exercise</u> Show that $M_f = 0$ (i.e. $M_f(u) = 0 \ \forall u \in C^\infty(X;\Omega)$) implies $f = 0$ in $C^\infty(X)$.

Distributions on the compact manifold without boundary are defined to be <u>exactly</u> and continuous linear maps

$$C^{-\infty}(X) = \{ u : C^\infty(X;\Omega) \to \mathbb{C} \text{ linear } \& \text{ continuous} \}.$$

The exercise above shows that

$$C^\infty(X) \ni f \longmapsto M_f \in C^{-\infty}(X)$$

is an injection. We regard this as an identification and write

$$C^\infty(X) \subset C^{-\infty}(X).$$

The reason this is useful to do, to regard $C^\infty(X)$ as a subset of $C^{-\infty}(X)$, is that many (but quite all) operations on $C^\infty(X)$ extend naturally to $C^{-\infty}(X)$. The most important examples of these are

1. Multiplication by $C^\infty(X)$: If $v \in C^\infty(X)$ and $f \in C^\infty(X)$ then

$$M_{vf}(u) = \int_X vfu = \int_X f(vu)$$

$$= M_f(vu) \quad \forall u \in C^\infty(X; \Omega)$$

For $w \in C^{-\infty}(X)$ we <u>define</u> $vw \in C^{-\infty}(X)$, $v \in C^\infty(X)$ being given, by

$$vw(u) = w(vu) \quad \forall u \in C^\infty(X; \Omega).$$

This defines a bilinear map

$$C^\infty(X) \times C^{-\infty}(X) \longrightarrow C^{-\infty}(X).$$

2. Action of differential operators. If $P \in \mathrm{Diff}^m(X)$, so $P : C^\infty(X) \to C^\infty(X)$, then we can extend P to an operator

$$P : C^{-\infty}(X) \to C^{-\infty}(X).$$

First we define $P^t \in \mathrm{Diff}^m(X; \Omega)$ by the formula

$$(t) \qquad \int_X f(P^t u) = \int_X Pf \cdot u \quad \forall f \in C^\infty(X).$$

<u>Exercise</u> Show that (t) does define a differential operator $P^t \in \mathrm{Diff}^m(X; \Omega)$. First suppose that f has support in a coordinate patch and use integration by parts to show that if $u = u(x)\,|dx|$ and

$$P = \sum_{|\alpha| \le m} p_\alpha(x) D_x^\alpha$$

then

$$ {}^tPu = \sum_{|\alpha| \le m} (-D_x)^\alpha p_\alpha(x) u .$$

Use a partition of unity to show that P^t exists, satisfying (†) as then use an easy exercise to show that it must be unique.

We define $P : C^{-\infty}(X) \to C^{-\infty}(X)$

by

$$Pv(u) = v(P^t u) \quad \forall u \in C_c^\infty(X; \Omega),$$

Exercise Show that P restricts to the original meaning on $C^\infty(X) \subset C^{-\infty}(X)$.

III/11

I will not go into the theory of general distributions here, it is mainly convenient to have a space that contains 'everything' we are interested in. Recall however, the basis

<u>Theorem</u> (Schwartz representation). If $u \in C^{-\infty}(X)$ then $\exists\, m \in \mathbb{N}_0$ and $v \in L^2(X)$ and $P \in \mathrm{Diff}^m(X)$ s.t.

$$u = Pv. \tag{m}$$

We write

$$H^{-m}(X) = \{u \in C^{-\infty}(X) \text{ of the form (m)}\}.$$

$m \in \mathbb{N}_0$.

Thus $H^0(X) = L^2(X)$. We also set

$$H^m(X) = \{u \in C^{-\infty}(X);\ Pu \in L^2(X)\ \forall\, P \in \mathrm{Diff}^m(X)\}.$$

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The structure theorem then shows (for a compact manifold without boundary) that

$$C^{-\infty}(X) = \bigcup_{m \in \mathbb{Z}} H^m(X).$$

A (somewhat simple) regularity theorem says

$$C^{\infty}(X) = \bigcap_{m \in \mathbb{Z}} H^m(X).$$

I will define and describe classes of distributions with geometric singularities, these are called conormal distributions, closely related to pseudodifferential operators, which I will describe first (but not straight away). / Pseudodifferential operators;

The

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acting from sections of one bundle to another, $\Psi^m(X;E,F)$, on a space of operators

$$P: C^\infty(X;E) \to C^\infty(X;F)$$

including, but considerably generalizing, the differential operators. I will start with a fundamental property which indicates their usefulness.

<u>Theorem</u> Suppose X is a compact manifold without boundary and $P \in \mathrm{Diff}^m(X;E,F)$ is <u>elliptic</u> then $P: C^\infty(X;E) \to C^\infty(X;F)$ is <u>Fredholm</u> in the sense that

① $N(P) = \{u \in C^\infty(X; E) ; Pu = 0\}$ is finite dimensional

② $R(P) = P \cdot C^\infty(X; E) \subset C^\infty(X; F)$ is closed

③ $R(P)$ has a finite dimensional complement, i.e. $\exists\ g_1, \dots ; g_N \in C^\infty(X; F)$

s.t.

(C) $C^\infty(X, F) = P \cdot C^\infty(X; E) + \mathrm{span}\{g_1, \dots, g_N\}$

For the moment I will not discuss the proof of this. In (C) we can always assume that the g_i are independent modulo $P \cdot C^\infty(X; E)$. It follows directly from ① - ③ that P has a generalized inverse

$Q : C^\infty(X; F) \to C^\infty(X; E)$

which is a continuous linear map

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Such that

$$Q \cdot Pu = u + u', \quad u' \in N(P)$$
$$\forall u \in C^\infty(X;E)$$

and

$$PQf = f + \sum_{l=1}^{N} c_l g_l \quad \forall f \in C^\infty(X;F).$$

Exercise Construct such a Q by choosing a complement to $N(P)$, $D \subset C^\infty(X;E)$ s.t.

$$C^\infty(X;E) = D \dot{+} N(P)$$

(i.e. $D \cap N(P) = \{0\}$), show that

$P_D : D \to R(P)$ is an isomorphism and then $Q = P_D^{-1}$ on $R(P)$, $Q \geq 0$ (?) on $Q g_j = 0$ $\forall j = 1, \ldots, N$ defines such a generalized inverse.

Such a generalized inverse Q is a 'typical' (well, except for the fact it is elliptic) element of $\Psi^{-k}(X; F, E)$.

In fact it is much better to construct Q first, using properties of the Ψ^k's and from this deduce the Theorem.

Let me apply this Theorem to get a basic result in differential topology/analysis. Recall that we have defined

$$d : C^\infty(X; \Lambda^k) \longrightarrow C^\infty(X; \Lambda^{k+1})$$

$\forall k$ by starting for $k=0$ and extending to the general case using

(∧) $$d(u \wedge v) = du \wedge v + (-1)^p u \wedge dv$$
$$u \in C^\infty(X; \Lambda^p),\ v \in C^\infty(X; \Lambda^l).$$

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It follows directly from this definition (indeed this is one of the important reasons for that definition) that $d^2=0$. This

$$(d)\quad C^\infty(X) \xrightarrow{d} C^\infty(X;\Lambda^1) \xrightarrow{d} C^\infty(X;\Lambda^2) \to \cdots$$

is a complex. We can 'roll up' this complex to get

$$d: C^\infty(X;\Lambda^*) \to C^\infty(X;\Lambda^*)$$

$$\Lambda^*X = \sum_{j=0}^{n} \bigoplus \Lambda^j X .$$

Let me compute the symbol of d. Recall that to do so we look at $d(e^{i\lambda f}u)$, $f\in C^\infty(X)$, $u\in C^\infty(X;\Lambda^*)$. Using (1) we find

$$d(e^{i\lambda f}u) = e^{i\lambda f}(i\lambda df\wedge u + du).$$

The symbol is the coefficient of λ, so at

$\xi = df(p) \in T^*_pX$

(σ) $$\sigma_\xi(d)\, u = i\xi \wedge u$$

as a homomorphism on Λ^*_pX. Clearly this is not elliptic, since for instance $\xi \wedge \xi = 0$.

To get an elliptic operator we can follow Hodge and define

$$\delta \in \mathrm{Diff}^1(X, \Lambda^*), \quad \delta = d^*$$

using the Riemannian inner products $\langle\ ,\ \rangle_p$ on Λ^k_pX and Riemannian density dg, $(u,v)_p = \langle u, \bar{v}\rangle_p$,

(δ) $$\int_X (v, du)\, dg = \int_X (\delta v, u)\, dg$$

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$\forall u \in C^\infty(X; \Lambda^k),\ v \in C^\infty(X; \Lambda^{k+1}).$

<u>Exercise</u> Generalize the early exercise (†) to show that δ is well-defined by this formula, $\delta \in \mathrm{Diff}^1(X; \Lambda^{k+1}, \Lambda^k)$ $\forall k$.

Show that $\delta^2 = 0$ and that

$$(\sigma^*) \qquad \sigma_\xi(\delta) = (\sigma_\xi(d))^*.$$

Thus we can compute the form of $\sigma_\xi(\delta)$ directly from (σ). If $\xi \in T_p^*X$ let $v \in T_pX$ be its image under the metric coupling $G: T_p^*X \to T_pX$

$$\langle G\xi, w\rangle = \xi(w) \quad \forall w \in T_pX.$$

then define

$$r_\xi : \Lambda_p^{k+1} X \longrightarrow \Lambda_p^{k} X$$

by $r_\xi \cdot u(v_1, \dots, v_k)$

$$= u(G\xi, v_1, \dots, v_k).$$

It follows from (σ^*) that

$$\sigma_\xi(\delta) = -i\, r_\xi \quad \text{of } \xi \in T_p^* X.$$

Let me write this out even more explicitly, assuming that $\xi \neq 0$. Set $\hat{\xi} = \xi/|\xi|$ and decompose $T_p^* X = \mathbb{R}\cdot\hat{\xi} + \xi^\perp$. Then

$$\Lambda_p^k X = \Lambda_p^{k-1} \xi^\perp + \Lambda^k \xi^\perp .$$

That is,

$$u = \hat{\xi} \wedge u_1 + u_2$$

with $u_1 \in \Lambda_p^{k-1} V$, $u_2 \in \Lambda^k V$, $r_\xi u_i = 0$.

In terms of these decomposit

$$\sigma_\xi(d)\cdot\begin{pmatrix}u_1\\u_2\end{pmatrix} = i|\xi|\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

$$\sigma_\xi(\delta)\cdot\begin{pmatrix}u_1\\u_2\end{pmatrix} = -i|\xi|\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

$$\Rightarrow\quad \sigma_\xi(d+\delta)\begin{pmatrix}u_1\\u_2\end{pmatrix} = i|\xi|\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}.$$

where $d+\delta \in \mathrm{Diff}^1(X;\Lambda^*)$.

<u>Proposition</u> $d+\delta$ is a (generalized) Dirac operator.

<u>Proof</u> $\sigma_\xi((d+\delta)^2) = (\sigma_\xi(d+\delta))^2$

$$= -|\xi|^2\begin{pmatrix}0&1\\-1&0\end{pmatrix}^2 = |\xi|^2\cdot \mathrm{Id}.$$

In particular, $d+\delta$ is elliptic. Thus, the elliptic Theorem above applies. In fact $d+\delta$ is (formally) self-adjoint i.e.

$$\text{(Sa)} \quad \int_X (u, (d+\delta)v) = \int_X ((d+\delta)u, v) \quad \forall u, v \in C^\infty(X; \Lambda^*)$$

as follows from (8). This means that ③ of the theorem becomes

$$\text{(H1)} \quad C^\infty(X; \Lambda^*) = (d+\delta)\, C^\infty(X, \Lambda^*) \dotplus N(P).$$

~~Certainly we can choose the g_i's in ③ so~~ that Indeed $u \in N(P)$ implies, using (Sa) that

$$(\perp) \quad \int_X (u, (d+\delta)v) = 0 \quad \forall v \in C^\infty(X; \Lambda^*)$$

so $u \in N(P) \cap (d+\delta)\, C^\infty(X; \Lambda^*) \Rightarrow u = 0$.

Conversely $\perp$ implies $(d+\delta)u = 0 \Rightarrow u \in N(P)$.

III/22 If we replace the α's by a maximal set $N(P)$ we get (H1) means $\exists\ g \in C^\infty(X;\Lambda^*)$, $g \perp N(d+\delta)$, $g \notin (d+\delta)C^\infty(X;\Lambda^*)$ [Not quite right.]

Now (H1) is a form of the Hodge decomposition

(H2) $$C^\infty(X;\Lambda^*) = dC^\infty(X;\Lambda^*) + \delta C^\infty(X;\Lambda^*) + H^*_{Ho}(X)$$

where $H^*_{Ho}(X) = N(d+\delta)$ by definition.

<u>Theorem</u> (Hodge) There is a natural isomorphism

$$H^k_{dR}(X) = \{u \in C^\infty(X;\Lambda^k);\ du=0\}/dC^\infty(X;\Lambda^{k-1})$$

$$\longleftrightarrow H^k_{Ho}(X) = H^*_{Ho}(X) \cap C^\infty(X;\Lambda^k)$$

given by (H2).

<u>Proof</u>. First observe that $u \in H^*_{Ho}(X) = N(d+\delta)$

implies that $du = 0$ & $\delta u = 0$. Firstly

$$0 = \int_X (du, (d+\delta)u) = \int_X (du, du) + \int_X (d^2u, u)$$

$$= \|du\|^2_{L^2} \Rightarrow du = 0.$$

From this it follows that

$$H^*_{\text{dR}}(X) = \bigoplus_{j=0}^{n} H^k_{\text{dR}}(X)$$

is graded, ~~positive~~

Exercise Check this.

If we apply (H2) to $u \in C^\infty(X, \Lambda^*)$

$$u = dv_1 + \delta v_2 + h, \quad h \in H^*_{\text{dR}}(X)$$

and then suppose $du = 0$ we would

$$0 = d^2v_1 + d\delta v_2 + dh, = d\delta v_2$$

$$\Rightarrow 0 = \int_X (v_2, d\delta v_2) = \int_X (\delta v_2, dv_2) = \|dv\|^2_{L^2}$$

$$\Rightarrow \delta v_2 = 0 \Rightarrow u = dv_1 + h,$$

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Furthermore, this decomposition is unique, since $h \perp dv$, so defines the map we want

$$H^k_{dR}(X) \longrightarrow H^k_{H}(X), \quad u \longmapsto h.$$

It is also clearly an isomorphism.

Replace Elliptic theory by (3)

$$C^\infty(X;F) = P\,C^\infty(X;E) \dotplus N(P^*)$$

and inner products. etc.

IV/1 Pseudodifferential operators.

As I said on Tuesday, pseudodifferential operators are a special class of operators between sections of vector bundles that allow one to prove the ellipticity theorem I wrote down, and used to prove the Hodge Theorem. Today I want to

1) Talk briefly about the Schwartz kernel theorem, which describes general operators

2) Talk about smoothing operators

3) Define the properties of pseudodifferential operators and, without proving, show how these can be used to prove the ellipticity theorem.

Last time I describe distributions,

$$C^{-\infty}(X) = \{u : C^\infty(X;\Omega) \underset{cts}{\longrightarrow} \mathbb{C}\}$$

without giving many examples, or even defining

IV/2 the topology on $C^\infty(X;\Omega)$. At least I want to remedy the first problem, if not the second. However, first let me generalize by defining the 'distributional sections' of a vector bundle.

If E is a vector bundle and E^* is its dual bundle then we have a pairing

$$E_p \times E_p^* \longrightarrow \mathbb{C} \qquad \forall p \in X$$

and hence

$$C^\infty(X;E) \times C^\infty(X;E^*) \ni (u,v)$$

$$\longmapsto f \in C^\infty(X)$$

$$f(p) = u(p)\cdot v(p) \in \mathbb{C}.$$

More generally if F is another bundle we get

$$C^\infty(X;E) \times C^\infty(X;E^*\otimes F) \to C^\infty(X;F)$$

In particular if we apply this to $F = \Omega X$

$$C^\infty(X;E) \times C^\infty(X;E^*\otimes\Omega) \longrightarrow C^\infty(X;\Omega) \xrightarrow{\int} \mathbb{C}$$

gives a non-degenerate pairing ($\mathbb{C}$ linear, not sesquilinear) i.e.

$$u \in C^\infty(X;E), \quad \int u\cdot v = 0 \ \forall v \in C^\infty(X;E^*\otimes\Omega)$$

$$\Leftrightarrow u \equiv 0$$

This generalizes the case $E = \mathbb{C}$ discussed last time. In particular we <u>define</u>

$$C^{-\infty}(X;E) = \{u : C^\infty(X;E^*\otimes\Omega) \to \mathbb{C}$$

$$\text{cts linear maps}\}$$

and as before we get an injection

$$C^\infty(X;E) \hookrightarrow C^{-\infty}(X;E)$$

that we regard as an identification.

IV/4

Now, suppose we consider two manifolds (for the moment compact without boundary) X, Y with vector bundles E, F over them. On the product $X \times Y$ we define the bundle $\mathrm{Hom}(F, E)$ by

$$\mathrm{Hom}_{(p,q)}(F, E) = \mathrm{hom}(F_p, E_q)$$

$$\cong E_q \otimes F_p^* \quad \forall\, (p,q) \in X \times Y.$$

Exercise Check that there is ~~also~~ a C^∞ vector bundle on $X \times Y$.

Now suppose $K \in C^{-\infty}(X \times Y; \mathrm{Hom}(E,F) \otimes \Omega Y)$

where ΩY is the density bundle on Y

$$\Omega_{(p,q)} Y = \Omega_q Y.$$

but as a bundle on $X\times Y$.

<u>Proposition</u> ~~Thm~~ If $u\in C^\infty(Y;F)$ and $K\in C^{-\infty}(X\times Y;\mathrm{Hom}(E,F)\otimes\Omega Y)$ then

$$K\cdot u\in C^{-\infty}(X\times Y;E\otimes\Omega Y)$$

is well-defined and so the Y-integral

$$\int_Y K\cdot u\in C^{-\infty}(X;E).$$

The map so defined

$$C^\infty(Y;F)\longrightarrow C^{-\infty}(X;E)$$

is continuous and linear <u>and</u> every such continuous linear map transports to a <u>unique</u> $K\in C^{-\infty}(X\times Y;\mathrm{Hom}(E,F)\otimes\Omega Y)$ in this way.

IV/6

As usual I do not plan to prove this, because I am not really going to use it. The identification of operators with distributional sections of Hom ⊗ Ω is the Schwartz kernel theorem. The main difficulty (not really so hard) is the construction of the kernel K from the operator. The uniqueness is not so hard, nor is the fact that the kernel defines an operator.

Despite the fact that it is a trifle imprecise we write the operator, corresponding to $K \in C^{-\infty}(X \times Y; \mathrm{Hom}(F,E) \otimes \Omega Y)$ as

$$Ku = \int_Y K(x,y) \cdot u(y)$$

even generally using the same notation for kernel and operator

<u>Example</u> The identity operator

$$C^\infty(X) \ni u \longmapsto u \in C^\infty(X)$$

has kernel $K \in C^{-\infty}(X^2; \Omega_R X)$ (where Ω_R is on the right factor) which in <u>any</u> local coordinates is

$$(\mathrm{Id}) \qquad K = \mathrm{Id} = \delta(x-x')\,|dx'|$$

<u>Exercise</u> Show that under coordinate change (Id) takes the same form. (Note, the coords are the <u>same</u> in both factors)

It is very important to observe that

$$\mathrm{Id} = 0 \text{ restricted to } U \subset X^2 \text{ open}$$

$$U \cap \mathrm{Diag} = \phi$$

i.e. $\operatorname{supp}(\mathrm{Id}) \subset \mathrm{Diag}$.

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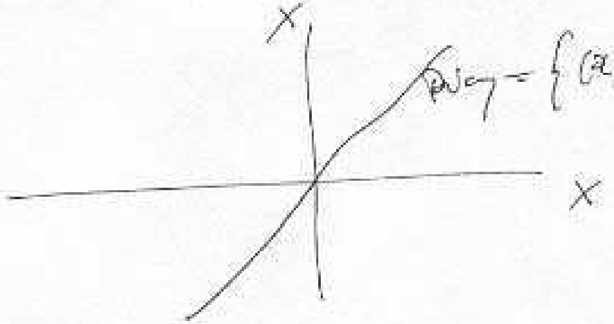


Now, it follows easily from the various definitions that if $P \in \mathrm{Diff}^m(X;E,F)$ then if $K : C^\infty_c(Y;G) \to C^{-\infty}(X;E)$ then $P \circ G : C^\infty_c(Y;G) \to C^{-\infty}(X;F)$ has kernel

$$P \cdot K \in C^{-\infty}(X \times Y; \mathrm{Hom}(F;G) \otimes \Omega Y).$$

Here ~~P∈~~ P may be reinterpreted as an element

$$P \in \mathrm{Diff}^m(X \times Y; \mathrm{Hom}(E;G) \otimes \Omega Y; \mathrm{Hom}(F;G) \otimes \Omega Y)$$

since it only acts 'in X' for E to F.

<u>Exercise</u> Try to write this down in some convincing way.

We apply this in the case $K = \mathrm{Id}$. Then we find that the kernel of P itself is just, in local coordinates

$$P(x,D)\cdot\delta(x-x')$$

– which of course supports in the diagonal.

Now let us consider another extreme case of Schwartz kernel theorem. Namely the elements $K\in C^\infty(X\times Y; \mathrm{Hom}(E,F)\otimes\Omega Y)$ must define operators

$$K: C^\infty(Y;E)\to C^{-\infty}(X;F)$$

in fact they define much better operators

$$K: C^{-\infty}(Y;E)\to C^\infty(X;F)$$

which is called 'smoothing operators'.

$\overrightarrow{\mathrm{IV}}/10$ thus we start off with a definition

$$\Psi^{-\infty}(X, E, F) \longleftrightarrow C^{\infty}(X^2; \mathrm{Hom}(E, F) \otimes \Omega)$$

smoothing operators.

- If $A \in \Psi^{-\infty}(X; E, F)$ & $B \in \Psi^{-\infty}(X; F, G)$ then

  $BA \in \Psi^{-\infty}(X; E, G)$

- If $K \in C^{-\infty}(X^2; \mathrm{Hom}(E, F) \otimes \Omega)$ is any cont. lin operator then $A \in \Psi^{-\infty}(X; \tilde{E}, E)$, $B \in \Psi^{-\infty}(X; F, \tilde{F}) \Rightarrow$

  $BKA \in \Psi^{-\infty}(X; \tilde{E}, \tilde{F})$

- If $P \in \mathrm{Diff}^m(X; E, F)$, $A \in \Psi^{-\infty}(X; \tilde{E}, E)$ &

$B \in \Psi^{-\infty}(X; F, \tilde{F})$ then $PA \in \Psi^{-\infty}(X; \tilde{E}, F)$ &

$BP \in \Psi^{-\infty}(X; E, \tilde{F})$.

- If $A \in \Psi^{-\infty}(X; E)$ then

- $N(\mathrm{Id} - A) \subset C^{-\infty}(X; E)$ is f.d. in $C^{\infty}(X; E)$
- $R(\mathrm{Id} - A) \in C^{\infty}(X, E)$ is closed & has fin. complement $N(\mathrm{Id} - A^{*})$.

We are going to insert that

- $$\bigcap_m \Psi^m(X;E,F) = \Psi^{-\infty}(X;E,F)$$

- For $A \in \Psi^m(X;E,F)$, A (its kernel) is singular only at $\mathrm{Diag}(X) \subset X^2$.

Let's look at the kernel of $P \in \mathrm{Diff}^m(X)$ in local coordinates. This is

$$P(x,D) = \sum_{|\alpha| \le m} p_\alpha(x) D_x^\alpha \delta(x-x').$$

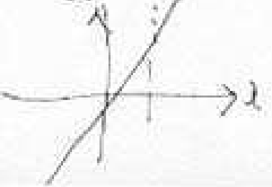
If we take the (minus) Fourier transform in x' in the sense we get

$$\sum p_\alpha(x) \xi^\alpha e^{ix\cdot\xi}$$

$$\cancel{P(x,D) e^{ix\cdot\xi} u(x)^{|\alpha|}}$$

$$u = (2\pi)^{-n} \int e^{ix\cdot\xi} \hat{u}(\xi)$$

$$P(x,D)u = (2\pi)^{-n} \int e^{ix\cdot\xi} \underbrace{p(x,\xi)}_{\text{iFT of } P(x,D) \text{ in } x'!} \hat{u}(\xi) d\xi$$



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One way to think of pseudodifferential operators is that we replace 'polynomial' functions $p(x,\xi)$ on T^*X by more general "symbolic" functions. Fortunately it is easy to introduce them! Namely, a polynomial in ξ, which has C^∞ coefficients in x of degree m satisfies the estimates (on compact x-sets)

$$(S) \qquad \left| D_x^\alpha D_\xi^\beta p(x,\xi) \right| \le C_{\alpha\beta} (1+|\xi|)^{m-|\beta|}.$$

In fact of course it vanishes identically if $|\beta| > m$. Hence for $m \in \mathbb{R}$ we take (S) as the definition

$$S^m_\infty(\mathbb{R}^n;\mathbb{R}^p) = \{ p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^p) \text{ satisfying } (S)\ \forall\ \alpha,\beta \}.$$

Exercise: If $L(x,\xi\cdot) : \mathbb{R}^p \to \mathbb{R}^p$ is an invertible linear map which is C^∞ in $x \in \mathbb{R}^n$ with all derivatives bounded then

$$\phi \in S^m_{cl}(\mathbb{R}^n;\mathbb{R}^p)$$

$$\iff \phi(x, L(x,\eta)) \in S^m_{cl}(\mathbb{R}^n;\mathbb{R}^p)$$

as smooth for global diffeo on $\mathbb{R}^n$,

$x_i = X_i(x)$ s.t. $|D^\alpha X_i| \le C_\alpha \quad \forall\ |\alpha| \ge 1$

with inverse satisfying the same condition.

This means that we can define

$$S^m(\bar{E}) \subset C^\infty(\bar{E})$$

for any real vector bundle E.

So now let me list some properties of

$\Psi^m(X;E,F)$, $m \in \mathbb{R}$. As already noted

these are operators

$$C^\infty(X;E) \to C^\infty(X;F).$$

① $\Psi^m(X;E,F) \subset \Psi^{m'}(X;E,F) \quad m' \ge m$

② $\Psi^m(X;F,G) \circ \Psi^{m'}(X;E,F)$

$$\subset \Psi^{m+m'}(X;E,G)$$

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(3) $\mathrm{Diff}^m(X,E,F) \subset \Psi^m(X,E,F)$ $m \in \mathbb{N}$.

(4) There is a short exact sequence

$$\Psi^{m-1}(X;E,F) \hookrightarrow \Psi^m(X;E,F) \xrightarrow{\sigma_m} \frac{S^m(T^*X;\hom(E,F))}{S^{m-1}(T^*X;\hom(E,F))}.$$

(5) $$\sigma_{m+m'}(AB) = \sigma_m(A) \circ \sigma_{m'}(B)$$

if $A \in \Psi^m(X;F,G)$, $B \in \Psi^{m'}(X;E,F)$

(6) $\sigma_m(P)$ is consistent for $P \in \mathrm{Diff}^m(X;E,F)$.

(7) If $A_j \in \Psi^{m-j}(X;E,F)$ is any sequence then $\exists\, A \in \Psi^m(X,E,F)$ s.t.

$$A - \sum_{j=0}^{N} A_j \in \Psi^{m-N-1}(X;E,F) \quad \forall N$$

(8) $A \in \Psi^m(X;E,F) \Rightarrow A^* \in \Psi^m(X;F,E)$, $\sigma(A^*) = \sigma(A)^*$

Let's try to use this to prove the existence of a generalized inverse to $P \in \mathrm{Diff}^m(X;E,F)$

where is elliptic.

<u>Step 1.</u> $\sigma_m(P) \in P^m(T^*X; \hom(E,F))$ is invertible for $\xi \in T^*X \setminus 0$. Then

$$a_0 = \sigma_m(P)^{-1} \cdot (1 - \rho(x,\xi)) \in S^{-m}(T^*X; \hom(F,E))$$

$\rho \in C^\infty(T^*X)$, $\rho \equiv 1$ near 0.

<u>Step 2</u> Choose $A_0 \in \Psi^{-m}(X; F, E)$, $\sigma_{-m}(A_0) = a_0$. Then $A_0 P \in \Psi^0(X;E)$,

$\sigma_0(A_0 P) = a_0 \cdot \sigma_m(P) = Id + \rho = Id$ in S^0/S^{-1}. So $A_0 P - Id = B_1 \in \Psi^{-1}(X;E)$.

<u>Step 3</u> Choose A_1 with $\sigma_{-m-1}(A_1) = -\sigma(B_1)\, a_0$

$\in \Psi^{-m-1}(X; F, E)$

$\Rightarrow (A_0 + A_1) P = Id + B_1 + A_1 P = Id + B_2$,

$B_2 \in \Psi^{-2}(X;E)$.

<u>Step 4</u> Do this $\forall j$, $A_j \in \Psi^{-m-j}(X;E)$ s.t.

$(A_0 + \dots + A_j) \cdot P = Id + B_{j+1}$, $B_{j+1} \in \Psi^{-j-1}(X;E)$

by induction (same as step 3).

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Step 5 Now choose $A \sim \sum_{j=0}^{\infty} A_j$; it follows that

$$AP - Id \in \bigcap_j \Psi^{-j}(X;E) = \Psi^{-\infty}(X;E).$$

$$\Rightarrow AP = Id + B, \quad B \in \Psi^{-\infty}(X;E).$$

Step 6 We could do exactly the same thing on the left to construct $\tilde{A}$ s.t. $P\tilde{A} = Id + \tilde{B}$, $\tilde{B} \in \Psi^{-\infty}(X;F)$.

Hence then

$$A = A(Id + \tilde{B}) - A\tilde{B}$$
$$= AP\tilde{A} - A\tilde{B}$$
$$= (Id + B)\tilde{A} - A\tilde{B}$$
$$= \tilde{A} + B\tilde{A} - A\tilde{B}$$

So $\tilde{A} - A \in \Psi^{-\infty}(X;F,E) \Rightarrow A$ also satisfies

$$PA = Id + B', \quad B' \in \Psi^{-\infty}(X;F)$$

Step 7 · $N(P) \subset N(Id + B)$ is f.d.

· $R(P) \supset R(Id + B')$ has closed and f.d complement.

$\Rightarrow$ Elliptic thm. Step 8 Gen'l case over $\Psi^m(X;F,E)$.

Lecture 5: Conormal distributions.

What's in the black box?

Next I want to talk about the radial compactification of a vector space. Why? Wait and see! For $\mathbb{R}^n$ we do this by a sort of 'stereographic projection' but not the usual one that gives the one point compactification. Instead map

$$(1) \quad \mathbb{R}^n \ni x \longmapsto (1,x) \in \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1}.$$

Then project onto the unit sphere

$$\mathbb{R}^{n+1} \ni Z \longmapsto \frac{Z}{|Z|} \in \mathbb{S}^n = \{Z \in \mathbb{R}^{n+1};\ |Z|=1\}.$$

Since $Z_0 > 0$ on the range in (1) the composite map

$$RC: \mathbb{R}^n \ni x \longmapsto \frac{(1,x)}{|(1,x)|} \in \mathbb{S}^n$$

has range in $\mathbb{S}^{n,1} = \{Z \in \mathbb{S}^n;\ Z_0 \geq 0\}$.

In fact RC is a diffeomorphism of $\mathbb{R}^n$ onto $\{Z\in\mathbb{S}^n;\ Z_0>0\}$, the inverse being

$$x=\frac{Z'}{Z_0},\quad Z=(Z_0,Z').$$

Thus RC embeds $\mathbb{R}^n$ as the interior of the compact manifold with boundary $\mathbb{S}^{n,1}$.

On a compact manifold with boundary we considered earlier the Lie algebra of vector fields $\mathcal{V}_b(X)$ which are tangent to the boundary. Consider

(3) $$\mathcal{A}^0(X)=\{u\in C^\infty(\mathring{X})\cap L^\infty(X);\ \mathcal{V}_b^k(X)\cdot u\subset L^\infty(X)\ \forall k\}.$$

Thus, for any vector fields $V_1\ldots V_k\in\mathcal{V}_b(X)$ and for any k, $V_1\cdots V_k u\in L^\infty(X)$.

This is the space of conormal functions (with respect to the boundary) & with respect to L^∞.

<u>Proposition</u>. If $RC : \mathbb{R}^n \hookrightarrow \mathbb{S}^{n,1}$ is radial compactification then

(I) $(RC)^* A^0(\mathbb{S}^{n,1}) = S^0(\mathbb{R}^n)$

is precisely the space of symbols of order 0.

<u>Proof</u> Let me do this reasonably carefully. First consider the right side, the space of symbols. If $\xi = (\xi_1, \ldots, \xi_n)$ are the usual coordinates then, by definition,

$$u \in S^0(\mathbb{R}^n) \iff |D_\xi^\alpha u| \le C_\alpha (1+|\xi|)^{-|\alpha|} \quad \forall \alpha.$$

The estimate here can be written

(E) $\quad |\xi^\beta D_\xi^\alpha u| \le C_{\alpha,\beta} \quad \forall\ |\beta| \le |\alpha|.$

Let us suppose separately that $u \in C^\infty(\mathbb{R}^n)$

V/4 (which follows from this). Then the estimates (E) get weaker if $|\beta|$ is decreasing, i.e. it suffices to suppose that (E) holds for $|\beta| = |\alpha|$:

$$(E') \quad u \in \mathcal{S}(\mathbb{R}^n) \Longleftrightarrow u \in C^\infty(\mathbb{R}^n) \ \& \ \sup_\xi |\xi^\beta D^\alpha u| < \infty \ \forall\ |\beta| = |\alpha|.$$

Now let $V_{ij} = \xi_i D_{\xi_j}$ be the obvious basis of <u>linear</u> vector fields. Another way of writing (E') is

$$(E'') \quad u \in \mathcal{S}(\mathbb{R}^n) \Longleftrightarrow u \in C^\infty(\mathbb{R}^n) \ \& \ \left(\prod_{l=1}^{N} V_{i_l j_l}\right) u \in L^\infty(\mathbb{R}^n) \quad \forall\ i_l, j_l \in \{1, \dots, n\}^N.$$

<u>Exercise</u> Prove (E'') by showing that the space of differential operators

$$P = \sum_{|\beta| = |\alpha| \le M} c_{\alpha\beta}\, \xi^\beta D_\xi^\alpha$$

is equal to the enveloping algebra of the $\bar{V}$'s linear vector fields, i.e.

$$P = \sum c_i \prod_{\ell=1}^{\bar{M}} V_{i_\ell j_\ell}$$

[But, need some induction argument!].

In the form (E'') we can easily transfer the arguments to v where $u = RC^* v$. Namely, let $\tilde{V}_{ij}$ be the ~~vector~~ vector fields on $S^{n,1}$ (really just the interior) which are the images of the V_{ij}. Then

(E''') $u \in \mathcal{S}^0(\mathbb{R}^n) \iff u = RC^* v,$

$$v \in \dot{C}^\infty(S^{n,1}), \quad \prod_{\ell=1}^{N} (\tilde{V}_{i_\ell j_\ell}) v \in L^\infty(S^{n,1})$$

$\forall\, i_\ell j_\ell, N.$

So, we just have to see what the $\tilde{V}_{ij}$ are.
Since we already know that $v \in C^\infty$ in the interior, we are only interested in the

V/6 form of the $\tilde{V}_{ij}$ <u>near</u> the boundary. That is in the region where $Z_0 > 0$ is small.

The map

$$(D)\quad \mathbb{S}^{n,1} \ni (Z_0, Z') \longmapsto \left(\frac{1}{|x|}, \frac{x}{|x|}\right) \in [0,\delta] \times \mathbb{S}^{n-1}$$

is a diffeomorphism. Indeed,

$$Z_0 = \frac{1}{(1+|x|^2)^{1/2}} = \frac{1/r}{(1+1/r^2)^{1/2}}, \quad r = |x|$$

$$= s/(1+s^2)^{1/2}$$

is C^∞ and invertible, from which (D) follows, since $Z' = \frac{x}{(1+|x|^2)^{1/2}}$ etc.

Then we just check that the $\tilde{V}_{ij}$ span $\mathcal{V}_b(\mathbb{S}^{n,1})$, near the boundary, are C^∞.

This follows from homogeneity, since

$$x\partial_x = \sum_{j=1}^{n} \xi_j \partial_{\xi_j}$$

$$\tilde{V}_{ij} = a_{ij}(\omega)\, x\partial_x + W_{ij}$$

W_{ij} C^∞ on $\mathbb{S}^{n-1}$ & spanning.

This (E''') does what just require that 9/17

$$\mathcal{V}_b^k(\mathbb{B}^{n,1})v \subset L^\infty \quad \forall k.$$

The argument is reversible so we have proved the proposition.

This is an idea worth remarking upon, that we can identify certain spaces of functions 'geometrically' using a compactification.

The same argument shows that if we define

$$A^m(X) = \{u \in C^\infty(\mathring{X});$$
$$\mathcal{V}_b^k u \in x^m L^\infty(X) \ \forall k\}$$

then

$$RC^*: A^m(\mathbb{B}^{n,1}) \longleftrightarrow S^{-m}(\mathbb{R}^n) \ \forall m \in \mathbb{R}.$$

Exercise: Check this!

V/8 So, in a sense at least, symbols and conormal functions are the same thing!

Notice that

$$C^\infty(X) \subset \mathcal{A}^\infty(X),$$

with the containment not equality. The image of $x^{-m}C^\infty(\overline{\mathbb{R}^n_+})$ in $S^m(\mathbb{R}^n)$ is often called the space of 'classical' symbols.

Why are conormal functions important for us? There are several reasons, not least the identification with symbols. However another important reason is the following

<u>Theorem</u> (Conic regularity) If δ is the adjoint of d wrt a <u>conic</u> metric

is a compact manifold with boundary then $(d+\delta)u = 0 \Rightarrow u$ is constant.

Of course, to give this real meaning we need to define conormal sections of vector bundles, etc. One cheap way to do this is to observe that

$$\mathcal{A}^m(X) \text{ is a } C^\infty(X)\text{-module}$$

and then to set

$$\mathcal{A}^m(X;E) = \mathcal{A}^m(X) \otimes_{C^\infty(X)} C^\infty(X;E)$$

for any vector bundle. What all this means that $u \in \mathcal{A}^m(X;E)$ is a section with $\mathcal{A}^m(X)$ coefficients.

Recall $d+\delta \in x^{-1}\,\mathrm{Diff}_b^1(X;{}^c\Lambda^*)$ is b-elliptic.

We want to replace everything by 'b-'!

V/10 We will try to do this directly, hoping even to shed some light on the 'usual' case. Trying to define

$\Psi_b^m(X;E,F)$ etc so that if $P \in x^{-l}\mathrm{Diff}_b^m(X;E,F)$

to define the $\exists$ $Q \in x^l \Psi_b^{-m}(X;E,F)$ s.t.

$$QP = \mathrm{Id} + E_L, \quad PQ = \mathrm{Id} + E_R,$$

E_L : Everything $\to A^*(X;\cdot)$!

Let's start with the remainder terms

$\Psi_b^{-\infty}(X;E,F)$. What should they be?

Since $d+\delta \in x^{-1}\mathrm{Diff}^1_b(X; {}^c\Lambda^*)$ in the case of a conic metric on a compact manifold with boundary, it is reasonable to expect that an element of the null space of $d+\delta$ — so a harmonic form — must be 'fully b-regular' i.e. $u \in x^H \mathcal{A}^*(X;\Lambda^*) = \mathcal{A}^H(X;{}^c\Lambda)$ for some H. So, we would also expect to prove this by finding a parametrix

$$Q \in x\Psi_b^{-1}(X;\Lambda^*), \quad Q(d+\delta) = \mathrm{Id} + E,$$

where $E : \{\text{Everything}\} \to \mathcal{A}^*(X; {}^c\Lambda^*)$. What should they be?

$$\{\text{Everything}\} = C^{-\infty}(X, {}^c\Lambda^*)$$

$$= \dot{C}^\infty(X; {}^c\Lambda^* \otimes \Omega)^*.$$

So, on a compact manifold with boundary consider

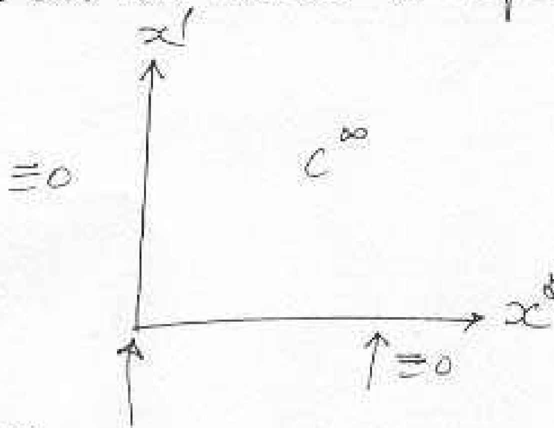
$$\dot{C}^\infty(X) = \{u \in C^\infty(X);\ u \equiv 0 \text{ at } \partial X\}$$

a submodule are $C^\infty(X)$.

VII/2 The obvious very smoothing operators are

$$\dot{\Psi}^{-\infty}(X;\Omega_b) = \dot{C}^{\infty}(X^2;\Omega_R).$$

These map $C^{-\infty}(X)$ to $\dot{C}^{\infty}(X)$. This of course is a lot too much to hope for!



Behaviour at the corner is still open to question.

Claim One such class of 'b-smoothing' operators correspond to the kernels as above which are smooth when the corner is blown up.

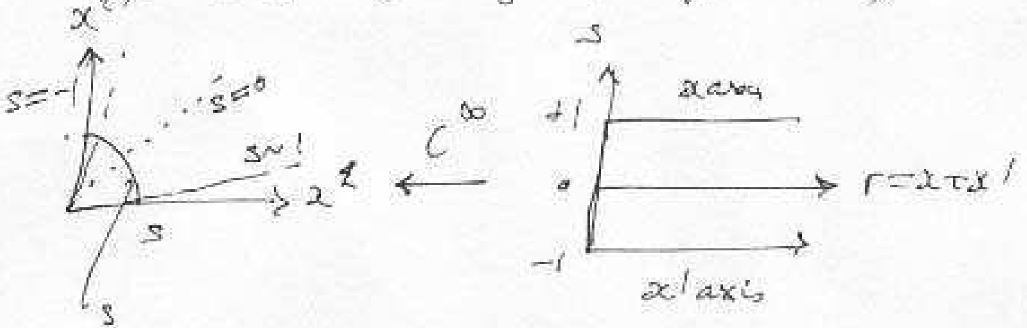
Near ∂X, $X \sim [0,\varepsilon) \times \partial X$

X^2 near corner $(\partial X)^2$, $X^2 \sim [0,\varepsilon)^2 \times (\partial X)^2$

Introduce singular coords near $(\partial X)^2$ — simplest type are projective / polar coords.

$$r = x + x' \quad (\text{"distance from } (\partial X)^2\text{"})$$

$$s = \frac{x - x'}{x + x'} \quad (\text{"angle of approach"})$$



$$x = \tfrac{1}{2} r (1+s) \qquad C^\infty, \text{ invertible when}$$
$$x' = \tfrac{1}{2} r (1-s) \qquad r > 0, \ (x, x' \geq 0).$$

Claim: The manifold $X_b^2 = [X^2, (\partial X)^2]$, X^2 with the corner blown up, is a well-defined C^∞ compact manifold with corners; we identify

$$\Psi_b^{-\infty}(X) = \{ A \in C^\infty(X_b^2; {}^b\tilde{\Omega}_R), \ A \equiv 0 \text{ at } \underline{\text{all}} \text{ boundaries} \}$$

where ${}^b\tilde{\Omega}_R$ = Right b-density bundle.

Last time I described, somewhat informally, the space of b-smoothing operators, $\Psi_b^{-\infty}(X)$. I want to go on and discuss, in some detail the spaces $\Psi_b^m(X)$ of finite order operators. I will do so, but first I will go ahead and examine some of the consequences of elliptic regularity in this sense.

Let me recall the setting. We work with a compact manifold with boundary, X, on which we consider the 'cone' structure epitomized by

$$(7.1) \qquad \mathcal{V}_C = \{u \in C^\infty(X);\ u|_{\partial X} = \text{const.}\}.$$

More precisely we consider a cone metric, g. This is a metric in the interior of X which near the boundary takes the form

$$(7.2) \qquad \begin{aligned} g_c &= dx^2 + x^2 h(x, y, dy, dx), \\ h_1 &= h(0, y, dy, 0) >> 0 . \end{aligned}$$

Thus, h_0 is a metric on ∂X. The question I want to make precise, and answer, is:–

What is the Hodge theory of g_c?

Elliptic regularity is supposed to tell us the following things:

- If $u \in L^2_{\phi c}(X; {}^c\Lambda^*)$ and $(d+\delta)u = 0$ then

(7.3) $$u \in x^{-\frac{1}{2}} H^\infty_b(X; {}^c\Lambda^*).$$

- If $u \in L^2_c(X; {}^c\Lambda^*) = x^{-\frac{n}{2}} L^2_b(X; {}^c\Lambda^*)$ and

$$(d+\delta)u \in L^2_c(X; {}^c\Lambda^*)$$

then $$u \in x^{-\frac{n}{2}} H^{\text{small}\,1}_b(X; {}^c\Lambda^*).$$

We want to use this to get a Hodge decomposition

$$L^2_c = H^*_? \oplus d? \oplus \delta?$$

First I want to use the Mellin transform to see what we can say about $u \in L^2_c(X; {}^c\Lambda^*)$

which satisfy $du = \delta u = 0$, using (7.3)

Let me remind you about the 1-dimensional Fourier transform, normalized by

$$\mathcal{F}u = \hat{u}(\tau) = \int_{\mathbb{R}} e^{-it\tau} u(t)\, dt .$$

(7.4) $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is an isomorphism

with inverse $u(t) = (2\pi)^{-1} \int e^{it\tau} \hat{u}(\tau)\, d\tau.$

(7.5) $\mathcal{F}$ extends to an isomorphism of $L^2(\mathbb{R})$.

Also recall the Paley–Wiener Theorem.

$\mathcal{F} \cdot \{u \in L^2(\mathbb{R});\ u = 0 \text{ in } t < 0\}$

(7.6) $\longleftrightarrow \{\hat{u} \in L^2(\mathbb{R})$ s.t. $\hat{u}$ is holomorphic in

$\operatorname{Im} \tau < 0$ &

$$\sup_{\sigma \in \mathbb{R}(0,\infty)} \int_{\mathbb{R}} |\hat{u}(\tau - i\sigma)|^2\, d\tau < \infty \}.$$

By continuity, or duality, $\mathcal{F}: \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is also an isomorphism and

$$\begin{cases} \mathcal{F}(D_t u) = \tau \mathcal{F}u, & D_t = \frac{1}{i}\frac{d}{dt}. \\ \mathcal{F}(tu) = -D_\tau \mathcal{F}u. \end{cases}$$

—

It is convenient to 'translate' these results to the Mellin transform, for functions on $[0,\infty)$.

$$v_M(\tau) = \int_0^\infty v(x)\, x^{i\tau} \frac{dx}{x}.$$

Since $(0,\infty) \ni x \longmapsto -\log x = t \in \mathbb{R}$ is a diffeomorphism, with $\frac{dx}{x} = -dt$ this is in fact the Fourier transform:

$$v_M(\tau) = \int_{-\infty}^{\infty} v(e^{-t})\, e^{-i\tau t}\, dt.$$

So $v \longmapsto v_M$ is an isomorphism of

$$L^2_b(0,\infty) = \left\{u \in \text{measurable on } (0,\infty), \int |u|^2 \frac{dx}{x} < \infty\right\} \to L^2(\mathbb{R})$$

Scratch

$\{u \in L^2_b(0,1);\ u = 0 \text{ in } x > 1\} \ni v \longmapsto v_M$ has range

as in (7.6).

Now, $u \in L^2_c(X, {}^c\Lambda^*)$ means that u has (locally) square-integrable coefficients w.r.t. an orthonormal basis, for g_c, of ${}^c\Lambda^*$. Thus 1, dx, $x\,dy_j$ are close to orthonormal. However the Riemannian volume form is $dg \simeq x^{n-1}\,dx\,dy \simeq x^n \frac{dx}{x}\,dy$, so

$u \in L^2_c(X, {}^c\Lambda^k) \iff u = x^k u_t + x^{k-1}\,dx \wedge u_n$, near ∂X

$u_t, u_n \in x^{-\frac{n}{2}} L^2_b$

+ integrability in the interior, since

$$\int_0^1\int_{\partial X} |x^k u_t|_c^2\, x^n \frac{dx}{x}\,dy < \infty$$

$$\Rightarrow \int_0^1\int_{\partial X} |x^{\frac{n}{2}} u_t|_h^2 \frac{dx}{x}\,dy < \infty.$$

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So this is how we will write forms near the boundary:

$$u = x^k u_t + x^{k-1} dx \wedge u_n$$

u_t & u_n tangential $k, k-1$ forms (depending on x).

I computed d in terms of this decomposition before:

$$du = x^k d_t u_t + x^{k-1} dx \wedge \left(-d_t u_n + x\partial_x u_t + k u_t\right).$$

$$= x^{k+1}\left(\tfrac{1}{x} d_t u_t\right) + x^k\left(-\tfrac{1}{x} d_t u_n + \left(\partial_x + \tfrac{k}{x}\right) u_t\right)$$

or as a matrix:

$$d\begin{pmatrix} u_n \\ u_t \end{pmatrix} = \frac{1}{x}\begin{pmatrix} -d_t & x\partial_x + k \\ 0 & d_t \end{pmatrix}\begin{pmatrix} u_n \\ u_t \end{pmatrix}.$$

So, $du = 0$ near ∂X becomes

$$\begin{cases} d_t u_t = 0 \\ -d_t u_n + (x\partial_x + k) u_t = 0 \end{cases}$$

Let us compute the form of δ following the idea of Hodge. It suffices to work locally and use the defining identity

$$\text{(IBP)} \quad \int_U \langle d\varphi, u\rangle\, dg = \int_U \langle \varphi, \delta u\rangle\, dg$$

where we can assume U is open and oriented.

If $e_1, \dots, e_n$ is an orthonormal basis of forms, Hodge defines

$$* e_{i_1}\wedge \dots \wedge e_{i_k} = \pm\, e_{j_1}\wedge \dots e_{j_{n-k}}$$

where $\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$ and the sign is given by the associated permutation. That is,

$$*(e_{i_1}\wedge \dots \wedge e_{i_k}) \wedge (e_{i_1}\wedge \dots \wedge e_{i_k})$$

~~sign~~

$$= e_1 \wedge \dots \wedge e_n = dg$$

This just means that

$$*u \wedge v = \langle u, v\rangle\, dg \quad \forall\ k \text{ forms } \forall h.$$

$$= \langle v, u\rangle\, dg = *v \wedge u.$$

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The ident (IBP) can be written

$$\int_U *d\varphi \wedge u = \int *\varphi \wedge \delta u = \int *du \wedge \varphi$$

$\forall$ $k-1$ forms $\varphi \in C^\infty_c$, u k-form.

$$= \int_U *u \wedge d\varphi$$

$$= \int_U (-1)^{n-k} d(*u \wedge \varphi) - (-1)^{n-k} (d*u) \wedge \varphi$$

$$= \int_U \varphi \wedge (d*) u$$

$$\iff \quad *\delta u = (-1)^{n-k+1} d*u.$$

since $*^2 = \cancel{k(n-k)} (-1)^{k(n-k)}$ on k-form.

$$\delta u = (-1)^{n-k+1+(k-1)(n-k+1)} *d*u.$$

$$\boxed{\delta u = (-1)^{k(n-k+1)} *d*u.}$$

For our conic metric, or the model conic metric   11/9

$$dx^2 + x^2 h_0(y, dy)$$

dx, $x e_i$ is orthonormal if e_i is o.n. for h_0.

$$\Rightarrow *u = *(x^k u_t + x^{k-1} dx \wedge u_n)$$

$$= x^{n-k-1} *(dx \wedge *_t u_t) + x^{n-k} (-1)^{n-k} *_t u_n .$$

so $d*u$

$$= x^{n-k} (-1)^{n-k} d_t *_t u_n + x^{n-k-1} dx \wedge$$

$$\left(-d_t *_t u_t + (-1)^{n-k}\left(\partial_x *_t u_n + \frac{n-k}{x} *_t u_n\right)\right).$$

$*d*u$

$$= (-1)^{k-1} x^{k-2} \left(-*d_t * u_t + (-1)^{n-k}\left(\partial_x *_t^2 u_n + \frac{n-k}{x} *_t^2 u_t\right)\right)$$

$$+ x^{k-1} (-1)^{n-k} dx \wedge *d_t * u_n .$$

Anyway, $\delta u = 0 \Leftrightarrow d*u = 0 \Leftrightarrow$

$$\delta_t u_n = 0 \quad \& \quad \delta_t u_t \pm (x\partial_x + n - k) u_n = 0 .$$

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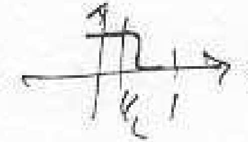
To understand the behavior of $u \in L^2_c$ satisfying $du = d^*u = 0$, we cut it off near $x = 1$ and apply the Mellin transform:

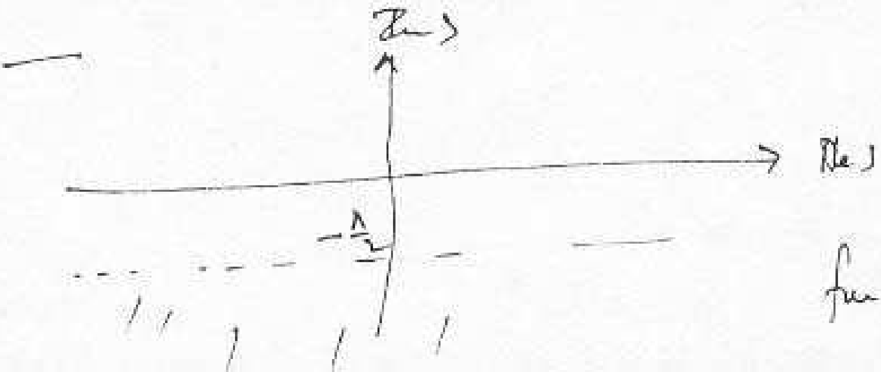
$$u_M(s,y) = \int_0^\infty \varphi u(x,y)\, x^{is} \frac{dx}{x}.$$



$$(x\partial_x u)_M = -is\, u_M + v_M$$

Here we know that $v \in C^\infty$ and supp in $\frac{1}{2} < x < 1$.

<u>Rp^h</u>: $u \in L^2_c(X; {}^c\Lambda^*)$, $du = d^*u = 0$ $\Rightarrow$ the Mellin transforms $(u_t)_M$, $(u_n)_M$ are meromorphic in the <u>entire</u> complex plane, holomorphic in $\operatorname{Im} s < -\frac{n}{2}$ and v does in C^∞ and rapidly decreasing as $|\operatorname{Re} s| \to \infty$ with $|\operatorname{Im} s|$ bdd.



from Paley-<u>Wiener</u>

v_Γ take an entire.

$$d(u_t)_\Gamma = 0$$

$$(-is+k)(u_t)_\Gamma - d_t(u_n)_\Gamma \text{ entire.}$$

$$\delta(u_n)_\Gamma = 0$$

$$(-is+(n-k))(u_n)_\Gamma \pm \delta_t(u_t)_\Gamma \text{ entire.}$$

$$(-is+k)(-is+(n-k))(u_t)_\Gamma \pm d_t\delta_t(u_t)_\Gamma$$

is entire.

$$d_t(u_t)_\Gamma = 0.$$

$$\Rightarrow (u_t)_\Gamma = (\pm\Delta_t + (-is+k)(-is+(n-k))^{-1} \text{ ent.}$$

is meromorphic.

Where are the poles exactly? Most interesting poles are in the strip $-\frac{n}{2} < \operatorname{Im} s \leq -\frac{n}{2}+1$. There cannot be any on $\operatorname{Im} s = -\frac{n}{2}$ by L^2 condition.

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— The domain of $d+\delta$.

We consider two distinct domains for $d+\delta$, the absolute & relative domains:–

$$\mathrm{Dom}_A(d+\delta) = \{u \in L^2_c(X;\Lambda^*)\,;\ du \in L^2_c \ \& \ \exists\ \varphi_j \in \dot{C}^\infty(X;{}^c\Lambda^*),\ \varphi_j \to u \text{ in } L^2_c(X;\Lambda^*),\ \delta\varphi_j \to \delta u \text{ in } L^2_c(X;\Lambda^*)\}.$$

$$\mathrm{Dom}_R(d+\delta) = \{u \in L^2_c(X;\Lambda^*)\,;\ \delta u \in L^2_c \ \& \ \exists\ \varphi_j \in \dot{C}^\infty(X;{}^c\Lambda^*),\ \varphi_j \to u \text{ in } L^2_c,\ d\varphi_j \to du \text{ in } L^2_c\}.$$

__Theorem__ (Cheeger-Goresky-MacPherson) The null space of $d+\delta : \mathrm{Dom}_R \longrightarrow L^2_c(X;\Lambda^*)$ is isomorphic to the L^2 cohomology and dual to the lower-middle intersection homology of $X/\partial X$;

$$H^k_{L^2} = \{u \in L^2_c(X;{}^c\Lambda^k)\,;\ du = 0\} \Big/ d\{u \in L^2_c(X;\Lambda^{k-1}),\ du \in L^2_c(X;\Lambda^k)\}$$

Start to put some of these computations together!
We decompose forms near the boundary on a conic manifold as

(1) $u = x^k u_t + x^{k-1} dx \wedge u_n$ $\quad$ k-form

where u_t & u_n are tangential (but x-dependent) forms of degree $k, k-1$. In terms of this decomposition

$$(2) \quad d = \begin{pmatrix} -\frac{1}{x} d_t & \frac{d}{dx} + \frac{k}{x} \\ 0 & \frac{1}{x} d_t \end{pmatrix} \quad \begin{pmatrix} u_n \\ u_t \end{pmatrix}$$

The x-factors in (1) are designed so that for the model conic metric

$$(3) \quad g_0 = dx^2 + x^2 h_0(y, dy)$$

$$\langle u, v \rangle_k = \langle u_t, v_t \rangle_{t,k} + \langle u_n, v_n \rangle_{t,k-1}$$

in terms of the tangential metric inner product.
The definition of δ

$$\int \langle d\varphi, u \rangle_k \, dg_0 = \int \langle \varphi, \delta u \rangle_{k-1} \, dg_0$$

VIII/2 becomes

$$\int_{\partial X}\int_0^1 \Big\{ \langle -\tfrac{1}{x} d_t \varphi_n + (\tfrac{d}{dx} + \tfrac{k-1}{x}) \varphi_t , u_t \rangle_{t,k-1} + \langle \tfrac{1}{x} d_t \varphi_t , u_t \rangle_{t,\text{both}} \Big\} x^n \frac{dx}{x}\, dh_0$$

$$= \int_{\partial X}\int_0^1 \Big\{ \langle \varphi_n , -\tfrac{1}{x} \delta_t u_n \rangle_{t,k-2} \quad x^n \frac{dx}{x} dh_0$$

$$+ \langle \varphi_t , \tfrac{1}{x} \delta_t u_t + (-\tfrac{d}{dx} - \tfrac{(n-k)}{x}) u_n \rangle_{t,k-1} \Big\}$$

$$= \int_{\partial X}\int_0^1 \big(\langle \varphi_n , (\delta u)_n \rangle_{t,k-2} + \langle \varphi_t , (\delta u)_t \rangle_{k-1} \big) x^n \frac{dx}{x}\, dh_0 .$$

Thus

$$\delta = \begin{pmatrix} -\frac{1}{x}\delta_t & 0 \\ -\frac{d}{dx} - \frac{n-k}{x} & \frac{1}{x}\delta_t \end{pmatrix} \quad \text{on } k-1, k \text{ forms}$$

Combining this

$$d+\delta = \begin{pmatrix} -\frac{1}{x}(d_t+\delta_t) & \frac{d}{dx} + \frac{k}{x} \\ -\frac{d}{dx} - \frac{n-k}{x} & \frac{1}{x}(d_t+\delta_t) \end{pmatrix} \quad \text{on } k-1, k \text{ forms.}$$

— Now, solve for the model metric $(d+\delta)u = 0$

with $(d+\delta)(\varphi u) = v \in \dot{C}^\infty(X; \Lambda^*)$

where $\varphi \in C^\infty(x)$, $\varphi(x) = 1$ in $x < \frac{1}{4}$, $\varphi(x) = 0$ in $x \geq \frac{1}{2}$, $\sqrt{\text{III}}/3$

so
$$u_M = \int x^{is} (\varphi u) \frac{dx}{x}$$

satisfies

$$\begin{pmatrix} -(d_t + \delta_t) & -is + k \\ is - (n-k) & (d_t + \delta_t) \end{pmatrix} \begin{pmatrix} u_{n,M} \\ u_{t,M} \end{pmatrix}$$ as entire values in $C^\infty(\partial X, \Lambda^*)$

- We can discuss the full invertibility of these operators but for the moment let me just make the overall claim that it is invertible except for a discrete set of poles of finite multiplicity and rank

So $\begin{pmatrix} u_{n,M} \\ u_{t,M} \end{pmatrix}$ is meromorphic as desired (modulo conditions)

Where are the poles, especially in the strip

$$-\frac{n}{2} < \operatorname{Im} s \leq -\frac{n}{2} + 1 \; ?$$

And for which of $du = 0$, $\delta u = 0$?

Here is a slightly more explicit form of what we are after.

Theorem For a conic metric on a compact manifold with boundary, $d+\delta : D_B \to L^2$ is Fredholm and self-adjoint for $B = A, R$ &

$$(\dagger) \qquad H^k_{A\text{-}L^2}(X) \underset{\text{natural}}{\cong} H^k_{A\text{-}Ho}(X)$$

$$\| $$

$$\{u \in D_A ;\ (d+\delta)u = 0\}.$$

Also

① If n is even $\quad D_A = D_R$

② If n is odd $\quad D_A = D_R \iff H^{\frac{n-1}{2}}(\partial X) = \{0\}.$

Exercise (straightforward) Show that

$$* D_A = D_R \quad \text{always.}$$

Thus, if n is even we have Poincaré duality,

$$* \; H^k_{A\text{-}L^2} = H^{n-k}_{A\text{-}L^2}$$

if n is odd this may not hold.

Proof of (†) given the rest of the theorem.

As already remarked we do this using the Hodge decomposition:

(§) $$L^2_c(X;\Lambda^*) = H^k_{B\text{-Ho}} \oplus d D^{k-1}_B \oplus \delta D^{k+1}_B \quad \forall k.$$

Let's prove this first, for $B = A$ for a start.

$$u, v \in D_A \Rightarrow (du, \delta v)_{L^2} = \lim_{j\to\infty} (du, \delta v_j)$$

$$= \lim_{j\to\infty} (d^2 u, v_j) = 0$$

since $v \in D_A$ means $\exists v_j \in C^\infty(X;\Lambda^*)$, $v_j \to v$ in L^2, $\delta v_j \to \delta v$ in L^2. Thus the second two terms on the right in (§) are orthogonal, so

$$(d+\delta) D_A = d D_A \oplus \delta D_A$$

is closed, hence both are closed. By essential selfadjointness $((d+\delta)D_A)^\perp = H^k_{A\text{-Ho}}$, giving (§).

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The identification (†) now follows directly from (8). Namely we define

(P) $\{u \in L^2 ; du = 0\} \longrightarrow H^{*}_{A\text{-}Ho}(X)$

by projecting u into its harmonic part in (8).

Since we know that $u \in D_A \Rightarrow du \perp \delta u$, $u \in H_{A\text{-}Ho}$ does satisfy $du = \delta u = 0$; it follows that (P) is surjective. If $u \in L^2_g(X; \Lambda^k)$ and $du = 0$ then the Hodge decomposition

$$u = u_H + dv + \delta w$$

necessarily has $\delta w = 0$. Indeed, $du = 0$ implies $d\delta w = 0$ and

$$0 = \langle w, d\delta w \rangle = \lim_{j\to\infty} \langle w_j, d\delta w \rangle$$
$$= \lim_{j\to\infty} \langle \delta w_j, \delta w \rangle = \|\delta w\|$$

where $w_j \in C^\infty(X; \Lambda^{k+1})$, $w_j \to w$ in L^2 & $\delta w_j \to \delta w$ in L^2. Thus,

(C) $\quad u = u_H + dv, \quad v \in D_A,$

so the null space of (P) consists of those $u \in L^2$

with $u = dv$, $v \in D_A$. Conversely, if $u = dw$ with $w \in L^2$ it follows that

$$\langle u_H, du\rangle = \lim_{j\to\infty} \langle w_j, du\rangle$$
$$= \lim_{j\to\infty} \langle \delta w_j, u\rangle = \langle \delta u_H, u\rangle = 0$$

where as usual $w_j \to u_H$ in L^2 with $\delta w_j \to \delta u_H = 0$.

Thus, $u = dv$ so the null space of (P) is precisely $\{u \in L^2_g(X;\Lambda^k);\ u = dv,\ v \in L^2_g(X;\Lambda^{k-1})\}$ and we see that (T) is correct.

<u>Exercise</u> Go through the analogous argument which shows that

$$\frac{\{u \in L^2_g(X;\Lambda^k);\ \exists\, u_j \to u \in L^2,\ u_j \in C^\infty(X;\Lambda^k),\ du_j \to du = 0 \text{ in } L^2\}}{d\,\{v \in L^2_g(X;\Lambda^{k-1});\ \exists\, v_j \in C^\infty(X;\Lambda^{k-1}),\ dv_j \to dv \text{ in } L^2\}}$$

$$\simeq H^k_{R-Ho}(X) = \{u \in D_R;\ (d+\delta)u = 0\}.$$

So, it remains to establish the first part of Theorem F, showing that $d+\delta$ is Fredholm and self-adjoint on the domains D_A & D_R.

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Now, we return to the question of the regularity of $u \in \mathfrak{D}$ satisfying $du=0$, $\delta u = 0$ for the model metric? (L^2) First, consider the projection of u_t & u_n onto the harmonic forms. It follows directly that

$$(-is+k)\, u^H_{t,H} \quad \text{and} \quad (is-(n-k))\, u^H_{n,H} \quad \text{are}$$

entire (with values in C^∞ and rapid decay as $|\mathrm{Re}\, s| \to \infty$).

So we conclude that

(A_t) $u^H_{t,H}$ can have at most a simple pole

$$\text{at} \quad -is = -k$$

(A_n) $u^H_{n,H}$ can have at most a simple pole

$$\text{at} \quad -is = -(n-k).$$

Note that these poles are in the 'critical strip'

$-\frac{n}{2} < \mathrm{Im}\, s \leq -\frac{n}{2}+1$ iff

(A) if n is <u>odd</u> $\quad -is = -k = -\frac{n}{2}+\frac{1}{2}$ $\quad \{ k = \frac{n}{2}-\frac{1}{2}$

$\quad$ & $-is = -(n-k) = -\frac{n}{2}+\frac{1}{2}$ $\quad \{ k = \frac{n}{2}+\frac{1}{2}$

if n is <u>even</u> $\quad -is = -k = -\frac{n}{2}+1$ $\quad \{ k = \frac{n}{2}-1$

$\quad -is = -(n-k) = -\frac{n}{2}+1$ $\quad \{ k = \frac{n}{2}+1.$

Next observe that

$$(B)\quad \begin{cases} d_t u_t = 0 \Rightarrow u^{\delta}_{t,M} = 0 \\ \delta_t u_n = 0 \Rightarrow u^{d}_{n,M} = 0 \end{cases}$$

where we write $u = u^H \oplus du^d \oplus \delta u^\delta$

for the Hodge decomposition for h_0 on the boundary.

Thus we can write the remaining conditions

as

~~kls+k~~

$$\left.\begin{aligned} &-d_t u^{\delta}_{n,M} + (-is+k)\, u^{d}_{t,M} \\ &(is-(n-k))u^{\delta}_{n,M} + \delta_t u^{d}_{t,M} \end{aligned}\right\} \text{ are entire}$$

Applying δ_t to the 1st & using the second

we find that

$$-\delta_t d_t u^{\delta}_{n,M} + (-is+k)\, \delta_t u^{d}_{t,M} \text{ is entire}$$

$$\text{or}\quad -\delta_t d_t u^{\delta}_{n,M} - (-is+k)(is-(n-k))\, u^{\delta}_{n,M} \text{ is entire.}$$

Acting on $u^{\delta}_{n,M}$, $\delta_t d_t = \Delta$ so

$$\left(\Delta - (-is+k)(-is+n-k)\right) u^{\delta}_{n,M} \text{ is entire}$$

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The inverse, $\left(\Delta - (-is+\frac{n}{2})^2 + (\frac{n}{2}-k)^2\right)^{-1}$

is therefore meromorphic, with only (simple) poles at the points where

$$(-is+\tfrac{n}{2})^2 = (\tfrac{n}{2}-k)^2 + \lambda_j^2$$

where λ_j^2 is one of the discrete eigenvalues of Δ (so it is strictly positive):-

$$-is = -\tfrac{n}{2} \pm \sqrt{(\tfrac{n}{2}-k)^2 + \lambda_j^2}\,.$$

So, these are points on the pure imaginary axis strictly above ~~below~~ $-\frac{n}{2} + |\frac{n}{2}-k|$ or strictly below $-\frac{n}{2} - |\frac{n}{2}-k|$. The latter are in the region where we know $u^{\delta}_{n,\mu}$ to be holomorphic so the only possible poles are at

$$(C_0) \qquad -is = -\tfrac{n}{2} + |\tfrac{n}{2}-k| + e_j$$

$$e_j = \sqrt{(\tfrac{n}{2}-k)^2+\lambda_j^2} - |\tfrac{n}{2}-k| > 0.$$

These are only in the critical strip if

$$n \text{ odd}, \quad k = \tfrac{n}{2} \pm \tfrac{1}{2}, \quad 0 < e_j \le \tfrac{1}{2}$$

8. Lecture VIII, 2 October, 2003

Handwritten notes: Pages 1-10

$$n \text{ even } k = \frac{n}{2},\ 0 < e_j \leq 1. \tag{1}$$

The poles of $u^d_{t,M}$ can only be in the same places.

9. Lecture IX, October 7, 2003

Let me start today by doing a little piece of analysis using the Mellin transform.

Lemma 1. *If $u \in x^{-t}L^2_{\mathrm{b}}([0,\infty))$ for some $t > 0$ has support in $x < 1$ and is such that $x\frac{d}{dx}u \in L^2_{\mathrm{b}}([0,\infty))$ then there exists $u_j \in \dot{\mathcal{C}}^\infty([0,\infty))$ such that $u_j \to u$ in $x^{-t}L^2_{\mathrm{b}}([0,\infty)$ and $x\frac{d}{dx}u_j \to x\frac{d}{dx}u_j$ in $L^2_{\mathrm{b}}([0,\infty)$.*

Exercise 1. If you are so inclined, find a proof which does not use the Mellin transform!

Proof. First note that if $u \in L_{\mathrm{b}}(X)$ then $x^\epsilon u \to u$ in $L^2_{\mathrm{b}}(X)$ as $\epsilon \downarrow 0$ for any compact manifold with boundary.

Exercise 2. Write out a careful proof of this.

Since we know that $x\frac{d}{dx} : H^1_{\mathrm{b}}([0,\infty) \longrightarrow L^2_{\mathrm{b}}([0,\infty))$ is continuous and that $\dot{\mathcal{C}}^\infty([0,\infty))$ is dense in $H^1_{\mathrm{b}}([0,\infty))$ it suffices to show that the sequence can be chosen in this space. So, the obvious way to get such a sequence is to take $u_j = x^{\epsilon_j}u$, with $\epsilon_j \downarrow 0$. From the Paley-Wiener theorem, the Mellin transform of u is holomorphic in $\operatorname{Im} s < -t$ and is square integrable on real lines in this half space with uniformly bounded L^2 norm. On the other hand,

$$(x\frac{d}{dx}u)_M = -isu_M \tag{1}$$

must be similarly holomorphic and L^2 in $\operatorname{Im} s < 0$. Thus certainly, $u_j \to u$ in $x^{-\delta}L^2_{\mathrm{b}}([0,\infty)$ for any fixed $\delta > 0$ (and in particular u lies in this space.) Now

$$x\frac{d}{dx}(x^\epsilon u) = x^\epsilon(x\frac{d}{dx}u) + \epsilon x^\epsilon u$$

and as already noted, the first term converges in L^2_{b} to $x\frac{d}{dx}u$ as $\epsilon \downarrow 0$ so it suffices to show that the second term converges to 0 in this space. The square of the L^2 norm of its Mellin transform may be estimated as follows:

$$\epsilon^2 \int_{\mathbb{R}} |u(s-i\epsilon)|^2 ds$$
$$\leq \epsilon \int_{|s|\geq\epsilon^{\frac12}} |(s-i\epsilon)u(s-i\epsilon)|^2 ds + \int_{|s|\leq\epsilon^{\frac12}} |(s-i\epsilon)u(s-i\epsilon)|^2 ds$$

where in the first term the estimate $|s-i\epsilon|^2 \geq \epsilon$ is used and in the second $|s-i\epsilon| \geq \epsilon^2$. By the assumed square-integrability of $x\frac{d}{dx}u$ both terms tend to 0 with ϵ. □

Using this and some related analysis I next want to write down the domains of $d+\delta$ that we have been discussing. First, we always have

$$x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) \subset D_A \cap D_R. \tag{2}$$

This is a direct result of the fact (discussed further below) that

$$\dot{\mathcal{C}}^\infty(X;\Lambda^*) \subset x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) \tag{3}$$

is dense with respect to the natural Sobolev norm and

$$d,\delta : x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) \longrightarrow L^2_{\mathrm{c}}(X : {}^{\mathcal{C}}\Lambda^*) \tag{4}$$

are continuous. Thus for an element $u \in x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$ there is an approximating sequence $\phi_j \in \dot{\mathcal{C}}^\infty(X;\Lambda^*)$, with $\phi_j \to u$, $d\phi_j \to du$ and $\delta\phi_j \to \delta u$ all in $L^2_g(X;\Lambda^*)$.

From the behaviour of solutions to $du = 0$ and $\delta u = 0$ in the model case we can add a few more pieces to the domains. These are all determined by the eigenfunctions of the limiting metric, h_0, on the boundary. Choose $\chi \in \mathcal{C}^\infty(X)$, a cut-off function supported very near the boundary and identically equal to 1 in some neighbourhood of it. Then set, for n odd

$$\begin{aligned} E_A &= \chi \cdot H^{\frac{n}{2}-\frac{1}{2}}_{\mathrm{Ho}(h_0)}(\partial X), \\ E_R &= \chi \cdot dx \wedge H^{\frac{n}{2}-\frac{1}{2}}_{\mathrm{Ho}(h_0)}(\partial X) \end{aligned} \tag{5}$$

in terms of the Hodge cohomology, i.e. harmonic forms, on the boundary for the metric h_0.

Similarly we fix spaces associated to non-harmonic eigenforms of the tangential Laplacian. If λ is such an eigenvalue for exact k forms on the boundary, so there is a non-trivial

$$0 \neq e_\lambda \in \mathcal{C}^\infty(\partial X; \Lambda^k),\ e_\lambda = de'_\lambda,\ d\delta e_\lambda = \lambda e_\lambda \tag{6}$$

we consider

$$\begin{aligned} &f_\lambda = x^{-is_\lambda}(x^k(-is_\lambda + k)e_\lambda + x^{k-1}dx \wedge e'_\lambda),\ \text{where} \\ &\quad -is_\lambda = -\frac{n}{2} + |\frac{n}{2} - k| + e_\lambda,\ e_\lambda = \sqrt{(\frac{n}{2}-k)^2 + \lambda} - |\frac{n}{2} - k| \end{aligned} \tag{7}$$

and then let

$$\begin{aligned} G^{\frac{n}{2}-\frac{1}{2}} &= \sum_{0<e_\lambda<\frac{1}{2},\ k=\frac{n}{2}-\frac{1}{2}} \mathbb{C}\chi f_\lambda, \\ G^{\frac{n}{2}} &= \sum_{0<e_\lambda<1,\ k=\frac{n}{2}} \mathbb{C}\chi f_\lambda, \\ G^{\frac{n}{2}+\frac{1}{2}} &= \sum_{0<e_\lambda<\frac{1}{2},\ k=\frac{n}{2}+\frac{1}{2}} \mathbb{C}\chi f_\lambda \end{aligned} \tag{8}$$

be the corresponding finite dimensional subspace of k-forms on X. Here, each eigenvalue of the boundary Laplacian on exact k forms, with e_λ in the indicated range, is repeated with its (finite) multiplicity, as the e_λ run over a basis. Of course the first and third spaces only make sense when n is odd, and the second when n is even.

Observe that each of these spaces is contained in $L^2_g(X; \Lambda^*)$ but intersects the smaller space $xL^2_g(X; \Lambda)$ in 0. The point here is that

$$du,\ \delta_0 u \in \dot{\mathcal{C}}^\infty(X; \Lambda^*),\ u \in G^k \tag{9}$$

provided δ_0 corresponds to a product-type conic metric, equal to $dx^2 + x^2 h_0$ near the boundary.

Exercise 3. Check (9) carefully! It follows from the formulæ for d and δ and the fact that the 2-vector implicit in (7) is a null vector of the 2×2 matrix implicit in the computation of the joint (formal) null space of d and δ_0 above

$$\begin{pmatrix} -1 & -is_\lambda + k \\ is_\lambda - (n-k) & \lambda \end{pmatrix} \begin{pmatrix} -is_\lambda + k \\ 1 \end{pmatrix} = 0. \tag{10}$$

Here of course s_λ has been chosen so the matrix has rank 1.

This takes care, as we shall see, of all the possible poles we discovered within the 'critical strip' $-\frac{n}{2} < \operatorname{Im} s < -\frac{n}{2} + 1$ for the Mellin transform of a form annihilated by d and δ_0. We need also to consider the poles on the line $\operatorname{Im} s = -\frac{n}{2} + 1$. To handle these we consider an infinite-dimensional space of functions on the line

$$\mathcal{L} = \left\{ h \in x^{-\epsilon} L^2_{\mathrm{b}}([0,\infty)), \epsilon > 0; h = 0 \text{ in } x > 1,\ x\frac{d}{dx} h \in L^2_{\mathrm{b}}([0,\infty)) \right\}. \tag{11}$$

Notice that Lemma 1 applies to elements of this space and shows in particular that it is independent of the choice of ϵ. With these functions as coefficients we consider spaces related to those in (5) and determined by the harmonic $\frac{n}{2}-1$ forms on the boundary with respect to h_0 :

$$\begin{aligned} E^{\frac{n}{2}-1}_{\mathcal{L}} &= \mathcal{L}(x) \cdot H^{\frac{n}{2}-1}_{\mathrm{Ho}(h_0)}(\partial X), \\ E^{\frac{n}{2}+1}_{\mathcal{L}} &= x^2 \mathcal{L}(x) \cdot dx \wedge H^{\frac{n}{2}-1}_{\mathrm{Ho}(h_0)}(\partial X). \end{aligned} \tag{12}$$

Exercise 4. Again you should do the little computation to see that if n is even then

$$E^{\frac{n}{2}\pm 1}_{\mathcal{L}} \subset L^2_g(X;\Lambda^*) \text{ and } u \in E^{\frac{n}{2}\pm 1}_{\mathcal{L}} \Longrightarrow du,\ \delta_0 u \in L^2_g(X;\Lambda^*). \tag{13}$$

Similarly we consider spaces closely related to those in (8) involving the form (7) corresponding to an exact boundary k-form which is an eigenform for the boundary Laplacian:

$$\begin{aligned} G^{\frac{n}{2}-\frac{1}{2}}_{\mathcal{L}} &= \sum_{e_\lambda = \frac{1}{2},\ k=\frac{n}{2}-\frac{1}{2}} \mathcal{L} \cdot f_\lambda, \\ G^{\frac{n}{2}}_{\mathcal{L}} &= \sum_{e_\lambda = 1,\ k=\frac{n}{2}} \mathcal{L} \cdot f_\lambda, \\ G^{\frac{n}{2}+\frac{1}{2}}_{\mathcal{L}} &= \sum_{e_\lambda = \frac{1}{2},\ k=\frac{n}{2}+\frac{1}{2}} \mathcal{L} \cdot f_\lambda. \end{aligned} \tag{14}$$

Notice that the non-triviality of these spaces corresponds to an 'accident' in which there is a positive eigenvalue for which e_λ takes on a specific value.

Exercise 5. If you haven't thought about this already, given an example of a function which is in $\mathcal{L}$ but is not in $L^2_{\mathrm{b}}([0,\infty))$.

Finally we get to an explicit description of the domains.

Proposition 1. *For a conic metric on a compact manifold with boundary*

$$\begin{aligned} D &= \left\{ u \in L^2_g(X;\Lambda^*); du, \delta u \in L^2_g(X;\Lambda^*) \right\} \\ &= x^{-\frac{n}{2}+1} H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) + E^*_A + E^*_R + G^* + E^*_{\mathcal{L}} + G^*_{\mathcal{L}}; \end{aligned} \tag{15}$$

and D_A and D_R are the same without the summands G^*_R and G^*_A respectively.

Remark 1. a) Before proceeding to the proof of this, note that the difference between D_A and D_R amounts to the repalacement of a finite dimensional subspace of the domain by another, of the same dimension − because by Poincaré duality $H^{\frac{n}{2}\pm\frac{1}{2}}(\partial X)$ have the same dimension.

b) The 'complicated' (in particular infinite-dimensional) extra terms in (15), $E^*_{\mathcal{L}}$ and $G^*_{\mathcal{L}}$, are really rather insignificant. As follows from the discussion below, if we give D the obvious norm

$$\|u\|_D^2 = \|u\|^2_{L^2_g} + \|du\|^2_{L^2_g} + \|\delta u\|^2_{L^2_g} \tag{16}$$

then $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*) + E^*_{\mathcal{L}} + G^*_{\mathcal{L}}$ is the closure of $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*)$ (and hence also of $\dot{\mathcal{C}}^\infty(X;\Lambda^*)$) in D.

c) In particular this means that the quotient of D by D_0, the closure of $\dot{\mathcal{C}}^\infty(X;\Lambda^*)$ in D, is finite dimensional.

Exercise 6. Check the statement following (16); the discussion below shows that this set is contained in the closure; the converse amounts to the exclusion of the other sets E^*_A, E^*_R and G^*. For the first two this is done below and a similar argument also works for the third.

Exercise 7. Show that the bilinear form

$$W : D \times D \ni (u,v) \longmapsto \int_X ((du+\delta u, v) - (u, dv + \delta v))\, dg \tag{17}$$

is antisymmetric (if we are dealing with real forms and the real pairing) and vanishes on D_0. Let $D_m = D_A \cap D_R$ be the subspace of D consisting of the elements have approximating sequences $u_j \in \dot{\mathcal{C}}^\infty(X;\Lambda^*)$ such that $u_j \to u$ and $du_j \to du$ in L^2_g and also are approximable in L^2_g by a possibly different sequence v_j for which δv_j converges in L^2_g. Show that W vanishes on D_m and that that D/D_m is a symplectic vector space in which D_A/D_0 and D_R/D_0 are complementary Lagrangian subspaces.

Proof. From elliptic regularity (which I still have to prove) we 'know' that

$$u \in L^2_g(X;{}^{\mathcal{C}}\Lambda^*),\ du, \delta u \in L^2_g(X;{}^{\mathcal{C}}\Lambda^*) \Longrightarrow u \in x^{-\frac{n}{2}}H^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*). \tag{18}$$

Thus, we start off with one factor of x less than we need to get into the first term in the putative expansion of D.

Exercise 8. Check again that you know why all the terms in (15) are in $L^2_g(X;{}^{\mathcal{C}}\Lambda^*)$.

Now, we are dealing with a conic metric which is not necessarily of product type near the boundary. On the other hand, the result we are looking for only depends on the limiting metric h_0 and not the higher perturbations

$$g = dx^2 + x^2 h(x,y,dy,dx) = g_0 + xq(x,y,xdy,dx),\ g_0 = x^2 + x^2 h_0(y,dy). \tag{19}$$

To see this directly observe that the Hodge star operator has a similar property

$$\star_g = \star_{g_0} + xA \tag{20}$$

where A is a smooth homomorphism of ${}^{\mathcal{C}}\Lambda^*$.

Exercise 9. See if you can do this reasonably neatly!

This in turn implies that

$$\delta_g = \delta_{g_0} + B,\ B \in \mathrm{Diff}^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*). \tag{21}$$

Thus B has no $1/x$ factor. Now,

$$D \subset x^{\frac{n}{2}}H^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*) \xrightarrow{B} L^2_g(X;{}^{\mathcal{C}}\Lambda^*) \tag{22}$$

from which it follows that D, D_A and D_R for the metric g are the same as they are for a product metric g_0 with the same limiting metric h_0. Thus we are reduced to the case of a product-type metric for which we were able to do computations using the Mellin transform.

All the terms in the expansion of D, apart from the first, correspond to the poles we discovered in examining the condition $du = \delta_0 u = 0$. We are now working with weaker regularity, namely that du, $\delta_0 u \in L^2_g$. Thus, writing out $u \in D$ in terms of its normal and tangential parts as before tangential parts

$$\begin{gathered}\begin{pmatrix} -d_t & x\partial_x + k \\ 0 & d_t \end{pmatrix} \begin{pmatrix} u_n \\ u_t \end{pmatrix} \in x^{-\frac{n}{2}+1} L^2_{\mathrm{b}}([0,\infty); L^2(\partial X)) \\ \begin{pmatrix} -\delta_t & 0 \\ -x\partial_x - (n-k) & \delta_t \end{pmatrix} \begin{pmatrix} u_n \\ u_t \end{pmatrix} \in x^{-\frac{n}{2}+1} L^2_{\mathrm{b}}([0,\infty); L^2(\partial X))\end{gathered} \tag{23}$$

The analysis of the (truncated) Mellin transform proceeds very much as before except that the right side in (23) leads only to a holomorphic Mellin transform in $\operatorname{Im} s < -\frac{n}{2} + 1$ with L^2 integral on real lines in this set uniformly bounded. Moreover, the invertibility of the full matrix

$$\begin{pmatrix} -d_t - \delta_t & x\partial_x + k \\ -x\partial_x - (n-k) & d_t + \delta_t \end{pmatrix} \tag{24}$$

off the imaginary axis follows as before and it only has a finite number of poles in $-\frac{n}{2} < \operatorname{Im} s < -\frac{n}{2} + 1$ of finite multiplicity. Thus we conclude that u_M is meromorpic as a function in $\operatorname{Im} s < -\frac{n}{2} + 1$ with values in $H^1(\partial X; {}^{\mathcal{C}}\Lambda^*)$ and su_M is square-integrable, with values in L^2, on real lines except possibly near $\operatorname{Re} s = 0$.

Writing out the steps in the argument we find

(1) From the tangential part of the first condition and the normal part of the second, the coexact part of u_t and the exact part of u_n, in terms of the Hodge decomposition with respect to h_0, must be the Mellin transforms of functions in $x^{-\frac{n}{2}+1} H^1_{\mathrm{b}}(\partial X; {}^{\mathcal{C}}\Lambda^*)$.
(2) The harmonic parts must be such that $(-is + k)u^H_{n,M}$ and $(is - n + k)u^H_{t,M}$ are the Mellin transforms of functions in $x^{-\frac{n}{2}+1}([0,\infty))$ with values in this vector space.
(3) For the exact part of $u_{n,M}$ and the coexact part of $u_{t,M}$ the projection onto the span of the eigenforms with eigenvalues larger than some R are necessarily in $x^{-\frac{n}{2}+1} H^1_{\mathrm{b}}$. Each of the components corresponding to an eigenvalue λ satisfy the same equation as before with an error in $x^{\frac{n}{2}+1} H^1([0,\infty))$.

So the poles in $\operatorname{Im} s < -\frac{n}{2} + 1$ of the Mellin transform of $u \in D$ are therefore precisely the same as thos of the solutions of $du = \delta_0 u = 0$ as analysed before. The terms in the spaces G^*_A, $G^*_R$and E^* have exactly these poles, with arbitrary coefficients of the appropriate type. Thus, subtracting them we may arrange that u_M has no poles below $\operatorname{Im} s = -\frac{n}{2} + 1$. However the result may still not be the Mellin transform of a function in $x^{-\frac{n}{2}+1} H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$. However, a similar argument for the poles lying on $\operatorname{Im} s = -\frac{n}{2} + 1$ gives rise to terms in $G^*_{\mathcal{L}}$ and $E^*_{\mathcal{L}}$. After subtracting these terms the result is a form in $x^{-\frac{n}{2}+1} H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$ which shows that D is indeed given by (15).

To shows that D_R is as indicated, we need to show that all terms apart from E_A are contained within it, and that $E_A \cap D_R = \{0\}$. The first requires the construction of approximations $u_j \in \dot{\mathcal{C}}^\infty(X; \Lambda^*)$ such that $u_j \to u$and $du_j \to du$ in L^2_g. For terms in E^*_R a simple cut-off suffices. For terms in $E^*_{\mathcal{L}}$ and $G^*_{\mathcal{L}}$ approximability follows from Lemma 1. For the terms in G^*, which are of the form

$$\chi x^{-is_\lambda}(x^k(-is_\lambda + k)e_\lambda + x^{k-1}dx \wedge e'_\lambda)$$

we first approximate the normal term using a simple cut-off setting

$$u_{j,n} = x^{-is_\lambda}(1 - \chi(x/\epsilon))(-is_\lambda + k)dx \wedge e'_\lambda$$

and then fix the tangential part by solving

$$u_{j,t} = (-is_\lambda + k)\chi(x)x^{-k}\int_0^x t^{k-1-is_\lambda}(1 - \chi(t/\epsilon)dte_\lambda. \tag{25}$$

That $\square$

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Last time I described, for a conic metric, the space

$$D = \{u \in L^2_g(X;\Lambda^*);\ du,\ \delta u \in L^2_g(X;\Lambda^*)\}.$$

It consists of four pieces. The largest is simply the closure of $\dot{C}^\infty(X;\Lambda^*)$ (or the Hilbert space $x^{-\frac{n}{2}+1} H^1_b(X;{}^c\Lambda^*)$) with respect to the norm

(ND) $$\|u\|_D^2 = \|u\|^2_{L^2_g} + \|du\|^2_{L^2_g} + \|\delta u\|^2_{L^2}.$$

Apart from this there are three finite dimensional pieces, two associated to boundary cohomology

$$(*)\ \begin{cases} E_A^{\frac{n}{2}-\frac{1}{2}} = x \cdot H^{\frac{n}{2}-\frac{1}{2}}_{Ho(h_0)}(\partial X) \\ E_R^{\frac{n}{2}+\frac{1}{2}} = x \cdot dx \wedge H^{\frac{n}{2}+\frac{1}{2}}_{Ho(h_0)}(\partial X) \end{cases}$$

which only exist for n odd and

a 'non-cohomologic' part G^*. I show, rather briefly, that $G^* \subset D_A \cap D_R$, i.e. can be appropriately approximated.

We wish to show that $E_A^{\frac{n}{2}-\frac{1}{2}} \cap D_R = \{0\}$, which will complete the description of D_R and hence D_A. To do this we compute directly the quadratic form

$$Q(u,v) = \int_X \left(\langle du, v \rangle - \langle u, \delta v \rangle \right) dg, \quad u, v \in D.$$

Observe that this vanishes on D_R, since if $u_n, v_n \in \dot{C}^\infty(X; \Lambda^*)$ ~~then $Q(u_n, v) = 0$~~ $Q(u_n, v_n) = 0$.

<u>Lemma</u>. If $u = x\varphi \in E_A^{\frac{n}{2}-\frac{1}{2}}$ and $v = x\, dx \wedge \psi \in E_R^{\frac{n}{2}-\frac{1}{2}}$ then

$$Q(u,v) = \int_{\partial X} \langle \varphi, \psi \rangle \, dh_0.$$

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Proof For $v \in E_R^{\frac{n}{2}+\frac{1}{2}}$, $dv = 0$ and $u \in E_A^{\frac{n}{2}-\frac{1}{2}}$, $\delta u = 0$. Then

$$Q(u,v) = \int_X (\langle du, v\rangle - \langle u, \delta v\rangle)\, dg$$

$$= \int_0^1 \int_{\partial X} 2xx' \langle \varphi, \psi\rangle_{\partial X}\, dx\, dh_0$$

$$= \int_{\partial X} \langle \varphi, \psi\rangle_{h_0}\, dh_0 .$$

From this it follows that $E_A^{\frac{n}{2}-\frac{1}{2}} \cap \mathcal{D}_R = \{0\}$, since $E_R^{\frac{n}{2}+\frac{1}{2}} \subset \mathcal{D}_R$.

Now we are in a position to prove the main proposition leading to the Hodge decomposition, namely that $d+\delta$ is self-adjoint and Fredholm on $\mathcal{D}_A$ and $\mathcal{D}_R$.

For self-adjointness, recall that we define

$D_R^* = \{ u \in L^2_g ;\ D_R \ni \varphi \longmapsto \langle (d+\delta)\varphi, u \rangle$ extends by continuity to $L^2_g \}$.

Recall that D_R is graded by degree so we see that $u \in D_R^*$ implies that each form component $u_{(k)} \in D_R^*$ and

$$D_R \ni \varphi \longmapsto \langle d\varphi, u \rangle = \langle \varphi, v_d \rangle$$
$$D_R \ni \varphi \longmapsto \langle \delta\varphi, u \rangle = \langle \varphi, v_\delta \rangle$$

both extend by continuity to L^2_g — simply restrict to the appropriate form degree $k \pm 1$.

Thus, $u \in D$ since $du, \delta u \in L^2_g$. So

$$D_R^* \subset D.$$

It is also clear that $D_R \subset D_R^*$ since if $u \in D_R$,

$$\langle (d+\delta)\varphi, u \rangle = \langle d\varphi, u \rangle + \langle \delta\varphi, u \rangle$$
$$= \lim_{n \to \infty} (\langle d\varphi_n, u \rangle + \langle \delta\varphi, u_n \rangle)$$
$$= \lim_{n \to \infty} (\langle \varphi_n, \delta u \rangle + \langle \varphi, du_n \rangle) = \langle \varphi, (d+\delta) u \rangle$$

8/5 using the approximations both to φ and u.
Thus we only need show that $E_A \cap D_R^* = \{0\}$
and this is the same argument as before.
Namely, essentially by definition,

$$\Phi(\varphi, u) = ((d+\delta)\varphi, u) - (\varphi, (d+\delta)u)$$

$$= 0 \quad \text{on } D_R \times D_R^*.$$

Taking $\varphi \in E_R^*$ it follows that $E_A^* \cap D_R^* = \{0\}$.

Finally then, we need to check the Fredholm property for $d+\delta$ on D_R, say.
The main point here is that D (or D_R) with the norm (KD) inject <u>compactly</u> into L^2_g:

(I) $\quad I : D \hookrightarrow L^2_g$, $\overline{I(B)} \subset L^2_g$ is
compact if $B \subset D$ is bdd.

This is the L^2 Ascoli–Arzela theorem,

namely $D \subset x^{-\frac{n}{2}} H^1_b(X, \Lambda^*) \cap x^{-\frac{n}{2}+\epsilon} L^2_b(X, \Lambda^*)$

and the latter already rejects compactly into L^2_g.

Exercise Check this!

From this compactness we deduce immediately that

$$H^*_{Ho(R,g)}(X) = \{u \in D_R ; (d+\delta)u = 0\}$$

is finite dimensional, since it is closed in L^2_g (by the comp continuity of $d+\delta$ into distributions) and has compact unit ball.

Similarly, the range

$$(d+\delta) D_R \subset L^2_g \text{ is closed.}$$

Indeed, if $(d+\delta)u_k \to v$ in L^2_g then we can assume $u_k \perp H^*_{Ho(R,g)}(X)$. The orthogonality $du \perp \delta u$, $u \in D_R$, shows $du_k \to v_d$, $\delta u_k \to v_\delta$, $v = v_d + v_\delta$.

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If we go back to the derivation of the structure of D we can apply the same argument, first concluding that $u_n \to u$ in $x^{-\frac{n}{2}} H_b^1(X; {}^b\Lambda^*)$, using elliptic regularity. From this we deduce that $(d+\delta_0)u_n \to (d+\delta_0)u \in L_g^2$ and hence that $u_n \to u$ in D (and hence D_R), using the Mellin transform.

Finally then we have most of what we set out to get for the cones — Hodge decomposition and identification of Hodge as L^2 cohomology; subject however to elliptic regularity (which despite my delaying the proof, is not supposed to be hard!) We still need to check the identification with intersection cohomology, but I will get back to that.

So, back to more geometric analysis.

If you recall I had introduced a space X_b^2 by blowing up the corner in X^2, where X is a (compact) manifold with boundary. If we go back to the beginning where we thought a little about identifying X with, or from, $C^\infty(X)$. From this point of view we can <u>define</u>

$$C^\infty(X_b^2) = \{ u \in C^\infty(X^2 \setminus (\partial X)^2);\ u \text{ is } C^\infty \text{ in polar coords locally around } (\partial X)^2 = \{x = x' = 0\} \}.$$

Of course we either have to say <u>all</u> polar coordinates or show that this condition is independent of <u>which</u> polar coordinates (that is, which coordinates we take before we introduce polar coordinates). I already did this.

Namely, if we just take a compact

8/9 manifold with corners and focus on a particular border face F, of codimension k, we are reduced to the following

<u>Lemma</u> If $F: U, 0 \longrightarrow \mathbb{R}^{n,k}, 0$ is a diffeomorphism from a neighbourhood U of $0 \in \mathbb{R}^{n,k}$ onto a neighbourhood of 0 then u C^∞ in $x_1 + \cdots + x_k$, $\dfrac{x_i - x_{i+1}}{x_1 + \cdots + x_k}$, $i = 1, \ldots, k-1$

and $y_1, \ldots, y_{n-k}$ implies the same is true of $F^* u$.

<u>Proof</u> It suffices to show that the 'polar' coordinate functions $x_1 + \cdots + x_k = r$ and $t_i = (x_i - x_{i+1})/r$ pull back to C^∞ functions (of these coordinates). For the y_i's this is trivial. Reversing the coordinate change we see that

$$x_i - x_{i+1} = t_i r \quad i = 1, \ldots, k-1$$

$$\Rightarrow \quad x_i = (L_i t) r, \quad L_i t = l_i t + l_i'$$

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For vectors t_i, l_i which I will leave you to evaluate, however these do form a basis of $\mathbb{R}^k$, with $L_i t \geq 0$ of ~~all [illegible]~~.

By assumption, F preserves $\mathbb{R}^{n,k}$ locally; let's assume for simplicity that

$$F^* x_j = a_j x_j, \quad 0 < a_j \in C^\infty$$

since this must be true up to rearrangement.

Thus

$$F^* r = \sum_j (a_j L_j t) \cdot r = \alpha r,$$

$$0 < \alpha \in C^\infty.$$

Here I leave you to check the strict positivity of α. From this it follows that

$$F^* t_i = \frac{F^* x_i - F^* x_{i+1}}{\alpha r}$$

$$= \frac{1}{\alpha}(a_i L_i t - a_{i+1} L_{i+1} t_i)$$

is C^∞ as claimed.

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Then we know that $[X;F]$ is a well-defined compact manifold with corners, then

$$C^\infty([X;F]) = \{u \in C^\infty(X \setminus F);$$

$$u \text{ is } C^\infty \text{ in polar coords near each pt of } F\}$$

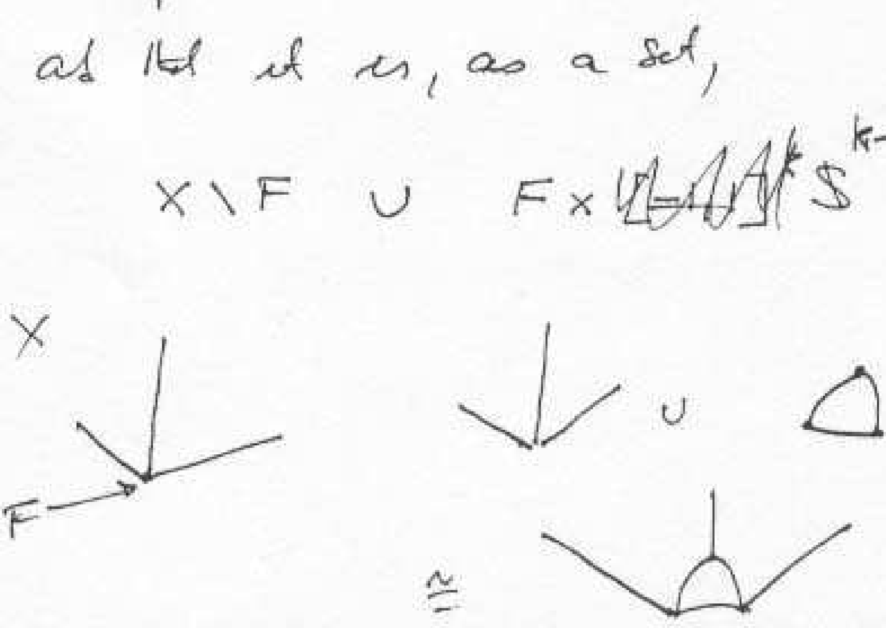
normal

It is unimportant to note that $[X;F]$ comes with a smooth blow down map

$$\beta : [X;F] \to X$$

and that it is, as a set,

$$X \setminus F \quad \cup \quad F \times S^{k-1,k-1}$$



10. Lecture X; in part

So, I had some difficulties in this lecture! Zhang Zhou pointed out subsequently that the proof of self-adjointness of $d+\delta$ with domain D_R is inadequate. This is a case of me trying to avoid work earlier, only to cause trouble later. The difficulty is that from the definition of D_R^*,

$$D_R^* = \big\{u \in L^2_g(X;\Lambda^*); D_R \ni \phi \longmapsto \langle (d+\delta)\phi, u\rangle \\ \text{extends by continuity to } L^2_g(X;\Lambda^*)\big\} \tag{1}$$

it does not follow directly that the domain is order-graded. That is, we certainly deduce that $(d+\delta)u \in L^2_g$ for the distributional action of the differential operator but we do not know that du, δu are separately in L^2_g. So, we are forced to go back to the earlier analysis and work harder to derive not just the structure of D but the structure of the, in general larger, space

$$D_{\max}(d+\delta) = \big\{u \in L^2_g(X;\Lambda^*); (d+\delta)u \in L^2_g(X;\Lambda^*)\big\}\,. \tag{2}$$

As I said, I should have done this directly but maybe it is better to postpone it to this point where we have the experience to do it relatively easily.

Proposition 2. *The maximal domain $D_{\max}(d+\delta) = D + U'$ where U' is the finite dimensional vector space which for even n is*

$$U' = \chi(x)\operatorname{sp}\{x^{-is_\sigma+k}((is-(n-k-2))\psi_k - dx\wedge d\omega_k) \tag{3}$$

where the linear span is over k and coexact k-forms ψ_k which are eigenforms of the boundary Laplacian, $\Delta\psi_k = \sigma\psi_k$ with

$$-is = -\frac{n}{2} + 1 - \sqrt{(\frac{n}{2}-1-k)^2+\sigma} > -\frac{n}{2}. \tag{4}$$

It follows immediately that such forms can occur only in for dimensions $k=\frac{n}{2}-1$ if n is even of $k = \frac{n}{2}-1\pm\frac{1}{2}$ if n is odd. The forms in (3) can never be degree-graded, unless zero of course.

Proof. Elliptic regularity applies as before to show that if $u \in D_{\max}(d+\delta)$ for a conic metric then $u \in x^{-\frac{n}{2}}H^1_{\mathrm{b}}(X;{}^{\mathcal{C}}\Lambda^*)$. Thus, exactly as with the discussion of D, the space $D_{\max}$ is the same for any two metrics with the same boundary metric h_0. We can therefore work with a product-type metric and analyse the conditions under which

$$v = \chi\sum_k (x^k u_{t,k}(x) + x^{k-1}dx\wedge u_{n,k-1}(x)) \tag{5}$$

is such that $(d+\delta)u \in L^2_g$, given that u itself is in L^2_g, which is just the condition

$$u_{n,*}, u_{t,*} \in x^{-\frac{n}{2}}L^2_{\mathrm{b}}(X;\Lambda^*(\partial X)). \tag{6}$$

The Hodge decomposition on the boundary allows these tangential and normal parts to be divided. Namely we set

$$L^2(\partial X;\Lambda^*) + L^2(\partial X;\Lambda^*) = H + G + U \tag{7}$$

where H is the harmonic part in all degrees, G is the part we discussed extensively before

$$u \in G \Longleftrightarrow u_n \in dH^1(\partial X;\Lambda^*),\ u_t \in \delta H^1(\partial X;\Lambda^*) \tag{8}$$

and U is the remaining part

$$u \in U \Longleftrightarrow u_n \in \delta H^1(\partial X; \Lambda^*),\ u_t \in dH^1(\partial X; \Lambda^*). \tag{9}$$

The harmonic part is finite dimensional, given by a smoothing operator applied to (u_t, u_n) whereas the components in G and U are given by the action of pseudodifferential projections of order 0. This means that, as necessary, we can track their regularity in Sobolev spaces.

The point of the decomposition (7) is that it is directly related to the action of $d+\delta$, in the product-conic case. Thus, $d_t + \delta_t$ maps exact to coexact forms and conversely so the components of (u_n, u_t) in H, G and U are mapped, respectively, into H, U and G so must separately take values in L^2_g. The H and G components were analysed earlier, so consider the component in U. From the form of d and δ (and taking care to get the value of k right for the action in a given form degree) we arrive at the condition

$$-\delta_0 u_{n,k+1} + (x\partial_x + k)u_{t,k} \in xL^2_g(-x\partial_x - (n-k-2))u_{n,k+1} + du_{t,k} \in xL^2_g \tag{10}$$

as conditions between the tangential component in degree k and the normal component in degree $k+1$ (as opposed to $k-1$ for G.)

As before we analyse the degree to which the condition (10) does *not* imply that $u \in x^{-\frac{n}{2}+1} H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$ by using the Mellin transform to find any possible poles in the strip $-\frac{n}{2} < \operatorname{Im} s \le -\frac{n}{2} + 1$. Such poles must satisfy

$$-\delta\phi_{k+1} + (-is + k)\psi_k = 0,\ (is - (n-k-2))\phi_{k+1} + d\psi_k = 0 \tag{11}$$

where ϕ_{k+1} is exact and ψ_k is coexact. Eliminating between the equations as before gives

$$\delta\delta\psi_k + (is - (n-k-2))(-is + k)\psi_k = 0 \tag{12}$$

which is to say that $(is - n + k + 2)(is - k) = \sigma$ must be a positive eigenvalue of Δ acting on coexact k-forms. Completing the square we find

$$-is = -\frac{n}{2} + 1 \pm \sqrt{(\frac{n}{2} - 1 - k)^2 + \sigma}. \tag{13}$$

This of course is pure imaginary and can lie in the 'critical strip' only when the sign is $-$ and then only when $k = \frac{n}{2} - \frac{3}{2}$, $k = \frac{n}{2} - 1$ or $k - \frac{n}{2} - \frac{1}{2}$ and only for correspondingly small eigenvalues, namely $\sigma_{\frac{n}{2}-\frac{3}{2}} < \frac{3}{4}$, $\sigma_{\frac{n}{2}-1} < 1$ and $\sigma_{\frac{n}{2}-\frac{1}{2}} < \frac{3}{4}$. In particular there are never such 'accidental poles' on the line $-is = -\frac{n}{2} + 1$. These poles can be removed by subtracting a term as in (3). □

Now, the defect form Q is defined on the whole of $D_{\max}$:

$$Q(u,v) = \int_X (\langle (d+\delta)u, v\rangle - \langle u, (d+\delta)v\rangle)\, dg. \tag{14}$$

Moreover by the approximability conditions already discussed it vanishes if either factor is in

$$\begin{aligned} D_{\min}(d+\delta) = \big\{ & u \in L^2_g(X;\Lambda^*); \\ & \exists u_n \to u \text{ in } L^2_g,\ u_n \in \dot{\mathcal{C}}^\infty(X;\lambda^*),\ (d+\delta)u_n \to (d+\delta)u \text{ in } L^2_g \big\}. \end{aligned} \tag{15}$$

For the moment we know at least that $D_{\min}$ contains all put the finite dimensional parts E^*_A, E^*_R, G^* in D and U^* in $D_{\max}$. There remains a little computation to do:

Lemma 2. *The defect (or boundary) pairing Q defines non-degenerate pairings between E^*_A and E^*_R and also between G^* and U^* and vanishes on all other pairings.*

Proof. Well, we already know the first part. The second part follows by a similar integration-by-parts argument and the eigendecomposition of boundary forms. □

Exercise 10. Carry through the argument here!

So, with this extra work we can see why

$$D^*_R = D_R. \tag{16}$$

Namely, $u \in D^*_R$ certainly implies that $u \in D_{\max}(d+\delta)$. Then the definition implies that Q must vanish on $D_R \times D^*_R$. The fact that $G^* \subset D_R$ and the lemma above then shows that $U^* \cap D^*_R = \{0\}$, which is to say $D^*_R \subset D$ where the previous argument, just the pairing argument for E^*_* takes over and shows that $D^*_R \subset D_R$, and hence they are equal.

Exercise 11. Use the same argument to decide on the exact identity of $D_{\min}(d+\delta)$. Show that an self-adjoint operator $\eth_B$ which is given by $d+\delta$ acting on some domain D_B with $D_{\min} \subset B \subset D_{\max}$ corresponds to a maximal subspace of $E^* + G^* + U^*$ on which Q vanishes.

Since my handwritten lecture notes for today are at best misleading on blow-up of a boundary face of a manifold with corners I have typed up something closer to what I actually said.

Rather than just define $X^2_{\mathrm{b}} = [X^2; (\partial X)^2]$, which we need for the definition of the algebra $\Psi^{-\infty}_{\mathrm{b}}(X)$, I will give the general definition of $[Z, F]$ where F is a boundary face of a compact manifold with corners, Z. By definition (and this is what makes this easier than the general case of an appropriately embedded submanifold) there are global defining functions for F. Namely, if H_i are the boundary hypersurfaces of Z which contain F and x_i are defining functions for the H_i, $i = 1, \dots, k$ then the simultaneous vanishing of the x_i defines F, at least locally near F (there may be other components of the intersection of these k hypersurfaces).

Now to define a manifold it suffices to give the space of smooth functions on it. We can set

$$\begin{aligned}(17)\quad &\mathcal{C}^\infty([Z; F]) = \\ &\{u \in \mathcal{C}^\infty(Z \setminus F); u \text{ is smooth in any normal polar coordinates at a point of } F\}.\end{aligned}$$

Here, by normal polar coordinates, I mean polar coordinates in the defining functions x_i. Thus the local coordinates are x_i, y_j, $j = 1, \dots, n-k$. By polar coordinates I will, for the moment, mean 'projective' polar coordinates. These are the k functions

$$r = x_1 + \cdots + x_k,\ t_i = \frac{x_i}{r},\ i = 1, \dots, k-1,\ y_j. \tag{18}$$

Notice that (as always locally near F) $r = 0$ only at F. It is only because it is a 'corner' of Z that we can do this. We can replace any one of the 'angular' variables t_i by $t_k = \frac{x_k}{r}$, or more generally take any $k-1$ of these k variables as coordinates. To see that the definition (17) really makes sense, we need to show that it does not actually depend on the choices of the x_i and y_j, although the latter is pretty obvious.

Lemma 3. *If $F : U, 0 \longrightarrow U', 0$ is a diffeomorphism of (relatively) open subsets of $\mathbb{R}^{n,k}$ with $F(0) = 0$ then the pull-back under F of any $\mathcal{C}^\infty$ function of the polar coordinates r, $t_1, \dots, t_{k-1}$, y_j is also a $\mathcal{C}^\infty$ function of these variables.*

Proof. It suffices to show that the pull-pack functions F^*r, F^*t_i and F^*y_j are $\mathcal{C}^\infty$ functions of r, t_i and y_j, since then the same is true for any smooth function of these variables. It is clear than any relabelling of the x_i has this property, so we can assume that F maps each of the boundary hypersurfaces x_i into itself (rather than permuting them). Thus $F^*x_i = a_i x_i$ with $0 < a_i$ a $\mathcal{C}^\infty$ functions near 0 on $\mathbb{R}^{n,k}$. Since $x_i = t_i r$, it follows that

$$F^*r = \sum_{i=1}^{k} F^*x_i = (\sum_{i=1}^{k} a_i t_i) r = \alpha r, \ 0 < \alpha. \tag{19}$$

Here we use the fact that the t_i, $1 = 1, \dots, k-1$ take values in the standard simplex in $\mathbb{R}^{k-1}$, i.e. $0 \le t_i \le 1$ and $t_1 + \cdots + t_{k-1} \le 1$. Since $t_k = 1 - t_1 - \cdots - t_{k-1}$ and the a_i in (19) are smooth and positive, it is indeed the case that the coefficient α is positive and a smooth function of the polar variables. From this the rest follows easily, since for instance

$$F^*t_i = \frac{F^*x_i}{F^*r} = \alpha^{-1} a_i t_i. \tag{20}$$

$\square$

Exercise 12. Check that these projective coordinates are equivalent to polar coordinates in the usual sense. That is, show that the functions $R = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$ and $\omega_i = x_i/R$ are smooth functions of r and the t_i and conversely. Notice that the ω_i are the coordinates of a vector in $\mathbb{S}^{k-1,k-1} = \mathbb{S}^{k-1} \cap \mathbb{R}^{k,k}$, and that any $k-1$ of the ω_i can be used as coordinates at a point on the sphere, except if there is one which takes the value 1 at the point, in which case only the others form a coordinate system.

Having shown that the definition (17) does actually make sense independent of coordinates, we need to check that the space of functions so defined is indeed the space of all $\mathcal{C}^\infty$ functions on a compact manifold with corners. To make this space concrete we can use the chosen defining functions x_i to identify it as

$$[Z, F] = (Z \setminus F) \cup \mathit{\Delta} \times F, \ \mathit{\Delta} = \{t \in \mathbb{R}^{k-1,k-1}; t_1 + \cdots + t_{k-1} \le 1\}. \tag{21}$$

Proposition 3. *Once the defining functions x_i are fixed, the space in* (17) *gives* $[Z; F]$ *in* (21)*, for F a boundary face of a compact manifold with corners Z, a natural structure as a compact manifold with corners.*

Proof. Already proved really. We can identify a neighbourhood of F in Z with the product $F \times U$ where U is a neighbourhood of 0 in $\mathbb{R}^{k,k}$ of the form $x_1 + \cdots + x_{<} \epsilon$, $\epsilon > 0$. Then the functions r and $t_i = x_i/r$ allows us to identify the part of the union in (21) consisting of $\backslash F$ and $\mathit{\Delta} \times F$ with $F \times \mathit{\Delta} \times [0, \epsilon)_r$. This is consistent with the definition of $\mathcal{C}^\infty([Z; F])$ and so gives the space a $\mathcal{C}^\infty$ structure. $\square$

Exercise 13. If you want to define $[Z; F]$ as a set, canonically and not as in (21) by reference to some particular collection of defining functions, it is not hard to do; so do it! The usual way is to introduce the normal bundle to F. This is the quotient of the tangent bundle to Z over F, $T_F Z$, by the tangent bundle to F. Thus $N_p F = T_p Z / T_p F$ for all $p \in F$. It is a bundle of rank k over F and has a positive 'quadrant' bundle, namely the image of the tangent vectors which satisfy $Vx_i \geq 0$ for all i. This condition is independent of the choice of defining functions x_i. If we let N^+F denote this 'quadrant bundle' we can pass to the corresponding 'fractional sphere bundle' – really a bundle of simplices – given by the quotient by the fibre $\mathbb{R}^+$ action, $SN^+F = (N^+F \setminus 0)/\mathbb{R}^+$. After all this we can set, as a set,

$$[Z; F] = (Z \setminus F) \cup SN^+F. \tag{22}$$

Check that the choice of defining functions x_i gives a natural identification $SN^+F = F \times \Delta$ and the $\mathcal{C}^\infty$ structure induced on $[X; F]$ in (22) by this choice is actually independent of the choice.

Now, the blown up space comes with a smooth map back to the original

$$\beta : [X; F] \longrightarrow Z,\ \beta(r, t, y) = (rt, y), \tag{23}$$

which is independent of any choices, since it is just the canonical identification on the first part of (21), or (22), and the projection onto F on the second part. Under this map, the 'new' boundary face $r = 0$ is identified with F; sometimes I call $r = 0$ the 'front face' of the blow up, or if a sudden algebraic wave overcomes me, the (exceptional) divisor. The lifts (proper transforms) of the old boundary faces $x_i = 0$ are the $t_i = 0$, which are mapped smoothly onto them; I will often use the notation $\beta^*(H)$ for the lift of a boundary hypersurface and $\mathrm{ff}(\beta)$ for the front face. Notice however that

$$\beta^{-1}(\{x_i = 0\}) = \{r = 0\} \cup \{t_i = 0\} = \mathrm{ff}(\beta) \cup \beta^*\{x_i = 0\}, \tag{24}$$

so the preimage of a boundary hypersurface containing F is the union of its lift (proper transform) and the front face (divisor).

Exercise 14. Make sure you see that in this real setting, blowing up a boundary hypersurface does absolutely nothing.

Now, back to the matter at hand. We want do identify the space of 'order $-\infty$ b-pseudodifferential operators' on X with a space of smooth kernels on $X^2_{\mathrm{b}} = [X^2, (\partial X)^2]$. To do so, we should be careful and include the obligator right density factors. Since we are in this 'b-category' it is natural (and wise) to take the density to be a b-density.

To do so, let me introduct another little bit of notation. Since we will need to talk about the projections of X^2 onto the factors, set

$$\pi_R : X^2 \longrightarrow X,\ \pi_R(x, x') = x',\ \pi_L : X^2 \longrightarrow X,\ \pi_L(x, x') = x. \tag{25}$$

Then we need the corresponding 'stretched' maps from X^2_{b} :

$$\pi_{\mathrm{b},R} : X^2_{\mathrm{b}} \longrightarrow X,\ \pi_{\mathrm{b},R} = \pi_R \circ \beta,\ \pi_{\mathrm{b},L} : X^2_{\mathrm{b}} \longrightarrow X,\ \pi_{\mathrm{b},L} = \pi_L \circ \beta. \tag{26}$$

Definition 1. On any compact manifold with boundary we set

$$\Psi^{-\infty}_{\mathrm{b}}(X) = \{A \in \mathcal{C}^\infty(X^2_{\mathrm{b}}; \pi^*_{\mathrm{b},R}\Omega_{\mathrm{b}}); A \equiv 0 \text{ at } \beta^*(\partial X \times X) \cup \beta^*(X \times \partial X)\}. \tag{27}$$

recall that $u \equiv 0$ at H for a smooth function u indicates that it vanishes with its Taylor series at each point of H, so all derivatives vanish there.

Now, as we shall see, these are operators and form an algebra.

Let me show first that the act on the smallest reasonable space, $\dot{\mathcal{C}}^\infty(X)$. It is easy to see that the Schwartz kernel theorem applies here and shows that $A \in \Psi^{-\infty}_{\mathrm{b}}(X)$ does define an operator from $\dot{\mathcal{C}}^\infty(X)$ to $\mathcal{C}^{-\infty}(X)$, the space of extendible distributions (the dual of $\dot{\mathcal{C}}^\infty(X;\Omega)$. This is pretty unimpressive, and would not allow us to compose the operators. To do so, observe how the action should go. We want to 'define'

$$Af = \int_X A(z,z')f(z') \tag{28}$$

where $f \in \dot{\mathcal{C}}^\infty(X)$ and I have formally written z, z' for the two 'variables' in X. Notice that the kernel A is supposed to carry within it the density factor needed to carry out the integral.

Trying to interpret (28) rigourously, we have to think of A as a smooth function on X^2_{b}. The functin f is on X but in (28) this is clearly supposed to be interpreted as the right factor of X^2. So we can consider the product

$$A \cdot \pi^*_{\mathrm{b},R} f \in \dot{\mathcal{C}}^\infty(X^2_{\mathrm{b}}; \pi^*_{\mathrm{b},R}\Omega_{\mathrm{b}}). \tag{29}$$

Here we are using the obvious fact that $\pi^*_{\mathrm{b},R} f \in \mathcal{C}^\infty(X^2_{\mathrm{b}})$ which is just the smoothness of the map, but also the fact that it vanishes to infinite order at $\mathrm{ff} \cap \beta^*(X \times \partial X) = \pi^{-1}_{\mathrm{b},R}(\partial X)$, simply because f, by assumption, vanishes to all orders at the boundary. Now, A already vanishes to infinite order at the old boundaries so the product in (29) does, as claimed, vanish to infinite order at all boundaries.

Now, one elementary property of the blow-up procedure is that it induces an isomorphism on functions that are smooth and vanish to infinite order at all boundaries

$$\beta^* : \dot{\mathcal{C}}^\infty(Z) \longleftrightarrow \dot{\mathcal{C}}^\infty([Z;F]) \tag{30}$$

for the blow-up of any boundary face. This allows us to interpret the product in (29) as a section of the b-density bundle

$$Af \in \dot{\mathcal{C}}^\infty(X^2; \pi^*_R\Omega_{\mathrm{b}}) = \dot{\mathcal{C}}^\infty(X^2; \pi_R\Omega) \tag{31}$$

where we use the fact that the sections of Ω_{b} which vanish to infinite order at the boundary are the same, naturally, as the sections of Ω, the ordinary density bundle. Finally then we see that

$$\Psi^{-\infty}_{\mathrm{b}}(X) \times \dot{\mathcal{C}}^\infty(X) \ni (A,f) \longmapsto \int_X Af \in \dot{\mathcal{C}}^\infty(X) \tag{32}$$

is actually a continuous bilinear map. In particular we get the desired operator interpretation

$$A : \dot{\mathcal{C}}^\infty(X) \longrightarrow \dot{\mathcal{C}}^\infty(X),\ A \in \Psi^{-\infty}_{\mathrm{b}}(X). \tag{33}$$

Exercise 15. Check that this action is faithful, i.e. if A vanishes as an operator (33) then it vanishes as an element of the space $\Psi^{-\infty}_{\mathrm{b}}(X)$.

XI/1 Continuing from TeX notes.

Recall the composition of smoothing operators which works just as well on compact manifolds with boundary (or corners for that matter):

$$A\cdot B(z,z') = \int_X A(z,z'')B(z'',z')$$

Here, $A,B \in C^\infty(X^2;\pi_R^*\Omega)$ so there is a density in the central factor of X to integrate. In terms of the three projections

$$\pi_F, \pi_S, \pi_C : X^3 \longrightarrow X^2$$

$$\pi_F(z,z',z'') = (z',z'')$$
$$\pi_S(z,z',z'') = (z,z')$$
$$\pi_C(z,z',z'') = (z,z'')$$

This can be written

$$A\cdot B = (\pi_C)_*(\pi_S^* A\cdot \pi_F^* B)$$

and product as

$$\begin{tikzcd}
 & X^2 & \\
 & X^3 \arrow[u, "\pi_C"'] \arrow[dl, "\pi_S"'] \arrow[dr, "\pi_F"] & \\
X^2 & & X^2
\end{tikzcd}$$

What we need is a similar space related to X^2_b.

<u>Proposition</u> For any compact manifold with boundary the 'triple b-space' $\cancel{X^3_{b}}$

$$X^3_b = [X^3; (\partial X)^3; X\times(\partial X)^2; \partial X\times X\times\partial X; (\partial X)^2\times X]$$

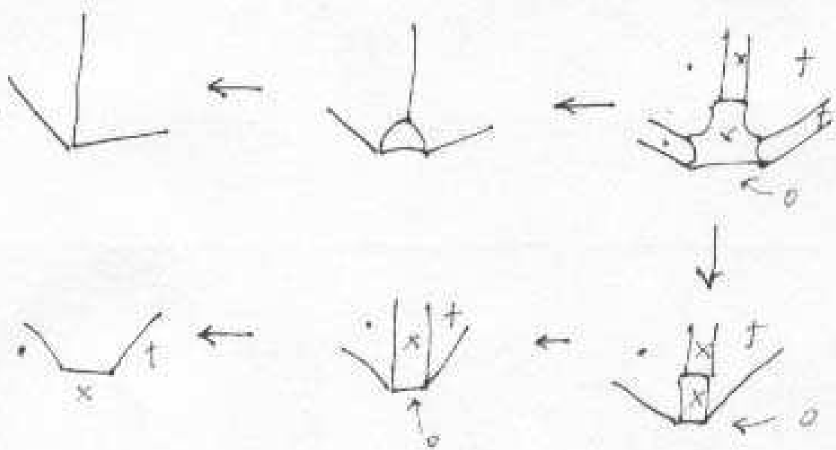
is such that there are 3 simple b-fibrations

$$\pi_{O,b} : X^3_b \longrightarrow X^2_b \qquad O = F, S, C$$

giving a commutative diagram

$$\begin{tikzcd}
 & X^2_b \arrow[r] & X^2 & \\
 & X^3_b \arrow[u] \arrow[r] \arrow[dl] \arrow[d] & X^3 \arrow[u] \arrow[d] \arrow[dr] & \\
X^2_b \arrow[r] & X^2 & X^2 & X^2
\end{tikzcd}$$

XI/3 Assuming, as I suddenly realize I am doing implicitly, that ∂X is connected, X_6^S has 7 boundary hypersurfaces — 3 original ones and four from the blow-ups — see picture



<u>Exercise</u>: try to check for yourself that the push-forward theorem applies to show that

$$(\pi_{6b})_* \left\{ (\pi_{S,6})^* \psi_6^{-\infty}(X) \cdot (\pi_{F,6})^* \Psi^{-\infty}(X) \right\}$$

$$\subset \Psi_6^{-\infty}(X).$$

[or at least think about what you would need to check].

11. Lecture XI: October 14

Last time I defined (again) the space $\Psi_{\mathrm{b}}^{-\infty}(X)$ and showed that its elements are operators on $\dot{\mathcal{C}}^\infty(X)$. Today I want to prove that they form an algebra and discuss some of its properties, relating them in general to geometric properties of X_{b}^2 ('the b-double space'). As a warm-up exercise, that turns out to be close to the proof of the composition theorem, let me discuss

Proposition 4. *The elements of* $\Psi_{\mathrm{b}}^{-\infty}(X)$ *act on* $\mathcal{C}^\infty(X)$.

Of course this is also important in its own right, as a further justification that the elements of $\Psi_{\mathrm{b}}^{-\infty}(X)$ act on 'almost everything'.

Proof. We are supposed to get this action in the same way as the action on $\dot{\mathcal{C}}^\infty(X)$. Notice that $\pi_{\mathrm{b},R}^* : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X_{\mathrm{b}}^2)$ so the big difference is that we do not have vanishing at the preimage of the boundary, hence not at $\mathrm{ff}(X_{\mathrm{b}}^2)$. We can summarize the operations in the little diagram

$$\begin{tikzcd}[column sep=small] & X_{\mathrm{b}}^2 \arrow[dl, "\pi_{L,\mathrm{b}}"'] \arrow[dr, "\pi_{R,\mathrm{b}}"] & \\ X & & X \end{tikzcd} \tag{1}$$

In fact it is clear from the definition that

$$\Psi_{\mathrm{b}}^{-\infty}(X) \text{ is a } \mathcal{C}^\infty(X_{\mathrm{b}}^2)\text{-module.} \tag{2}$$

Thus, in trying to show that when we integrate $A\pi_{\mathrm{b},R}^* f$ over the right factor of X we get an element of $\mathcal{C}^\infty(X)$, we might as well forget about f and just integrate a general A. Thus we are trying to show that the push-forward map to the left factor gives

$$(\pi_{\mathrm{b},L})_* : \Psi_{\mathrm{b}}^{-\infty}(X) \longrightarrow \mathcal{C}^\infty(X). \tag{3}$$

Note that if the map in question was a fibration, as the left projection from X^2 is, then this is a version of Fubini's theorem. However $\pi_{\mathrm{b},L}$ is *NOT* (quite) a fibration. If it were a fibration then (3) would be true without the vanishing conditions on the 'old boundary faces' which are inherent in the definition of $\Psi_{\mathrm{b}}^{-\infty}(X)$ and $\mathcal{C}^\infty(X_{\mathrm{b}}^2; \pi_{\mathrm{b},R}^*\Omega_{\mathrm{b}})$ itself would push-forward into $\mathcal{C}^\infty(X)$; it does not, so we have to make use of this vanishing.

So, to business. To prove (3) we can work locally on X_{b}^2. Indeed using a partition of unity we can cut the kernel, A, up into small pieces and assume that it has support in the preimage of the product of coordinate neighbourhoods in the two factors of X. If we are away from the front face of X_{b}^2, so away from the corner 'downstairs' in X^2, then (3) is obvious – the map is locally a fibration and in any case we are back to the previous result and the image is actually in $\dot{\mathcal{C}}^\infty(X)$.

Thus, we can assume that A has its support in a 'polar coordinate' neighbourhood $[0,\epsilon)_r \times [-1,1]_t \times U \times U'$ where U, U' are open neighbourhoods of $0 \in \mathbb{R}^{n-1}$ with coordinates y, y' and $r = x + x'$, $t = (x - x')/r$ are projective polar coordinates. Then

$$A = a\frac{dx'}{x'}dy',\ a \in (1-t)^k(1+t)^{k'}\mathcal{C}_c^\infty([0,\epsilon) \times [-1,1] \times U \times U')\ \forall\ k, k'. \tag{4}$$

Here the factors of $1-t$ and $1+t$ reflect the assumed rapid vanishing at the old boundaries, which are $t=\pm 1$. In fact, because we want to discuss the map back to x,y variables, it is convenient to introduce the *singular* projective coordinates

$$s=x'/x, x, y, y' \text{ so } t=\frac{1-s}{1+s},\ r=(s+1)x. \tag{5}$$

These are valid coordinates in some region $0\le r\le \epsilon$, $t\in(-1,1]$ with $t=1,-1$ corresponding to $s=0,\infty$. My claim is that, despite the singularity of these coordinates, we can translate the conditions on a to imply

$$a'(s,x,y,y')=a(r,t,y,y')\Longrightarrow a'\in\mathcal{S}([0,\infty);\mathcal{C}^\infty_c([0,\delta)\times U\times U')). \tag{6}$$

By this I just mean that a' is $\mathcal{C}^\infty$ has support contained in $[0,\infty)\times K$ for some compact $K\subset[0,\delta)\times U\times U'$ and all derivatives (meaning in s, x, y and y' of all orders) vanish rapidly as $s\to\infty$.

This is clear in $0\le s\le S$ where the coordinates are legitimate. In $s\ge S>0$ for any fixed S, we can introduce $s'=1/s$ $(=x/x')$ taking values in $(0,1/S)$. We still do not quite get legitimate coordinates since $t=\frac{s'-1}{s'+1}$ is fine, but $r=(1+s')x/s'$ is not smooth. Since $x=x's'$, s',x' *are* legitimate coordinates in this region, with $r=(1+s')x'$ so we do get a smooth function, b, of s',x',y,y' which vanishes to infinite order at $s'=0$ and has bounded support in x'. Notice that such a function can indeed be written as a smooth function of s',x,y,y' :

$$b'(s',x,y,y')=b(s',\frac{x}{s'},y,y') \tag{7}$$

because the singularities in the second variable, as $s'\to 0$ are swamped by the rapid decay in s'. For instance we can write

$$b=(s')^N b_N(s',\frac{x}{s'},y,y'),\ b_N\ \mathcal{C}^\infty,$$

from which it follows that the first $N-1$ derivatives in x are continuous down to $s=0$. Now, for a function to be smooth and vanish to infinite order at $s'=0$ is equivalent to its being 'Schwartz' in the variable $s=1/s'$ near $s=\infty$. Thus we do really have (6).

Exercise 16. Prove the converse to (6) that this (with the correct support constraints) does actually characterize the kernels of elements in $\Psi^{-\infty}_{\mathrm{b}}(X)$.

Finally then we can write our push-forward integral as

$$\int_0^\infty\int_{\mathbb{R}^{n-1}} a(\frac{x'}{x},x,y,y')\frac{dx'}{x'}dy' \tag{8}$$

where the supports in x' and y' are actually bounded. Changing variable from x' to $s=x'/x$ this becomes

$$\int_0^\infty\int_{\mathbb{R}^{n-1}} a(s,x,y,y')\frac{ds}{s}dy' \tag{9}$$

Note that the measure has 'miraculously' become regular except at $s=0,\infty$ where we have corresponding rapid vanishing (or decay) in the integrand. Thus the integral (9) converges absolutely and uniformly to a smooth function of x and y. This is what we need to prove. □

This proof is a bit hands-on for my taste! For later purposes I will generalize this result and make it more geometric. The results I will formulate next will first (as usual you might say) be used to prove something worthwhile, in this case the composition theorem, and then later it will be proved. The proof can be based on computations in singular coordinates just like that above, but there are other approaches too.

First think about the properties of smooth maps between compact manifolds with corners. We know what smoothness means already, but we need to add some conditions as to how boundaries are mapped. Recall that the boundary hypersurfaces each have defining functions (if you like these are simply generators of the $\mathcal{C}^\infty$-module of functions which vanish on the boundary hypersurface in question), ρ_H for each $H \in M_1(X)$.

Definition 2. A smooth map $F : X \longrightarrow X'$ is a *b-map* if each boundary defining function $\rho'_{H'}$, $H' \in M_1(X')$ pulls back to a product of boundary defining functions for X :

$$f^*\rho'_{H'} = a_{H'} \prod_{H \in M_1(X)} \rho_H^{e(H,H')},\ a < a_{H'} \in \mathcal{C}^\infty(X). \tag{10}$$

It is an interior b-map if it maps the interior of X into the interior of X'. It is a *simple* b-map if it is a b-map and in addition the exponents $e(H, H')$ take only the values $0, 1$. It is a *b-normal* map if it is a b-map and in addition for each $H \in M_1(X)$ there is at most one $H' \in M_1(X')$ such that $e(H, H') \neq 0$.

A simple b-normal map is one which is simple and b-normal, etc, duh.

Exercise 17. Translate these definitions into statements about the behaviour of the ideals corresponding to boundary faces.

Now recall that the b-cotangent bundle ${}^{\mathrm{b}}T^*X$ is the ordinary cotangent bundle in the interior, but near a boundary face has as local basis the 'logarithmic differentials' dx_i/x_i and dy_j in terms of our usual adapted coordiantes. The b-tangent bundle, its (pre-)dual, has corresponding basis $x_i\partial_{x_i}$, ∂_{y_j}.

Proposition 5. *Any interior b-map the usual differential on the interior extends by continuity to a 'b-differential' and its dual*

$$f^{*\,\mathrm{b}} : {}^{\mathrm{b}}T^*_{f(p)}X' \longrightarrow {}^{\mathrm{b}}T^*_pX,\ f_{*\,\mathrm{b}} : {}^{\mathrm{b}}T_pX' \longrightarrow {}^{\mathrm{b}}T_{f(p)}X, forall p \in X. \tag{11}$$

Note that despite some danger of confusion, I will generally denote this 'new' differential by f^* or f_*, just like the usual one.

Exercise 18. See if you can carry the proof through.

Definition 3. An interior b-map $f : X \longmapsto X'$ is said to be a *b-submersion* if it is surjective and $f_{*\,\mathrm{b}} = f_* : T_pX \longrightarrow T_{f(p)}X'$ is surjective for each $p \in X$. A b-submersion which is also b-normal is said to be a *b-fibration*

Exercise 19. Check that these definitions are not at all vacuous!

(1) Show that the blow-down map $\beta : [X, F] \longrightarrow X$ for F a boundary face of a manifold with corners is always a b-submersion but not a submersion in the usual sense unless F is a boundary hypersurface (in which case it is the identity map).

(2) Show that this blow-down map is never b-normal, and hence is not a b-fibration, unless H is a boundary hypersurface.
(3) Show that the 'stretched projection' $\pi_{L,\mathrm{b}} : X^2_{\mathrm{b}} \longrightarrow X$ is a b-fibration but is not a fibration in the usual sense.

I will discuss the general structure of b-fibrations, and so on, later. For the moment I will just quote a push-forward result

Theorem 1. *For a simple b-fibration f, suppose for each $H' \in M_1(X')$ for which $e(H,H') \neq 0$ for some $H \in M_1(X)$ a particular such $H = p_f(H')$ is chosen, then push-forward (fibre-integration) gives a map*

$$f_* : \big\{u \in \mathcal{C}^\infty(X;\Omega_{\mathrm{b}}); u \equiv 0 \text{ at } H \in M_1(X) \\ \text{unless } H = p_f(H') \text{ for some } H' \in M_1(X')\big\} \longrightarrow \mathcal{C}^\infty(X';\Omega_{\mathrm{b}}). \tag{12}$$

Of course you are very welcome to try to prove this, but it is easier when we have a little more machinery at our disposal. For the moment I suggest

Exercise 20. Show that this theorem does imply Proposition 4 in the form (3). Hint: Since the theorem deals with b-densities and (3) is about 'partial' b-densities, something has to be done! First show that there is a natural isomorphism

$$(\pi_{L,\mathrm{b}})^*\Omega_{\mathrm{b}}(X) \otimes (\pi_{R,\mathrm{b}})^*\Omega_{\mathrm{b}}(X) \equiv \Omega_{\mathrm{b}}(X^2_{\mathrm{b}}) \tag{13}$$

(Hint-within-a-hint, the corresponding statement on X^2 is true). Now to get (3), choose a positive b-density $0 < \nu_{\mathrm{b}} \in \mathcal{C}^\infty(X;\Omega_{\mathrm{b}})$ and show that Theorem 1 can be applied to $\Psi^{-\infty}_{\mathrm{b}}(X) \cdot \pi^*_{L,\mathrm{b}}\nu_{\mathrm{b}}$. Check that the result is independent of the choice of ν_{b}.

Despite appearances there is something going on here to do with b-densities as opposed to ordinary densities.

Let me start today with a geometric (almost algebraic-geometric) result that I mentioned at the end of last lecture. We already know how to blow up a boundary face F of a compact manifold with corners X. Namely, we can give X a product structure near F. If $x_1, \dots, x_k$ are defining functions for F (or if you prefer, for the boundary hypersurfaces containing F) then there is a neighbourhood U of F of the form

$$U = \{ x_1 + \dots + x_k < \varepsilon \}$$

$$(\quad) \qquad U \cong F \times \{ (x_1, \dots, x_k) \in \mathbb{R}^{k,k}_{} ; \; x_1 + \dots + x_k < \varepsilon \} .$$

This is a 'collar neighbourhood'. To blow up F we just blow up $0 \in \mathbb{R}^{k,k}$, setting

$$[X; F] = (X \setminus U) \cup (F \times [0, \varepsilon)_r \times \Delta)$$

where $\Delta = \{(x_1, \dots, x_k) \in \mathbb{R}^{k,k};$
$$x_1 + x_2 + \dots + x_k = 1\}.$$
The identification of the two pieces
$$[X;F] = ((X \setminus F) \cup (F \times [0,2)_r \times \Delta))/i$$

$$i(r, (t_1, \dots, t_k), p)$$
$$(\beta) \qquad = (rt_1, \dots, rt_k, p)$$
gives $[X;F]$ its natural C^∞ structure (i.e. independent of choices).

For any boundary face $G \in \mathcal{M}_*(X)$ we define its lift (proper transform) under the blow down map (also given by (β)) by
$$(\beta^*) \qquad \beta^*(G) = \begin{cases} \beta^{-1}(G) & \text{if } G \subset F \\ \overline{\beta^{-1}(G \setminus F)} & \text{if } G \not\subset F. \end{cases}$$
Note that if G is not contained in F then $G \setminus F$ is dense in G in X; in (β^*) we take the closure in $[X;F]$.

XII/3 Clearly, $\beta^*G \in \mathcal{M}_k([X;F])$ is always a boundary face. We can then ask the following question:

When does the natural identification of interiors extend to a diffeomorphism

(I) $$[[X;F];\beta_F^*G] \longleftrightarrow [[X;G];\beta_G^*F] \, ?$$

Since there is not much ambiguity here, we write $[X;F,G] = [[X;F];\beta^*G]$ etc. meaning first blow up F, then the lift of G.

Theorem The natural identification of the interiors extends to a ~~[illegible]~~ diffeomorphism (I), for two boundary faces $F, G \in \mathcal{M}_k(X)$ if and only if $F \subset G$ or $G \subset F$ or

(II) $$G \pitchfork F .$$

XI/4

Note that the transversality condn ($\pitchfork$) means that F & G either don't intersect at all or, if they do, then there is no boundary hypersurface containing them both – i.e. ρ_H with $\rho_H = 0$ on F & $\rho_H = 0$ on G.

Exercise check that this last condition does reduce to the usual notion of transversality, that $\forall p \in F \cap G$

$$T_pF + T_pG = T_pX$$

(natural tangent spaces).

In fact these two (& three) cases are the extremes, and in between are opposite extremes, and is as far from $F \pitchfork G$ as one can get.

XI/5

<u>Proof</u> (I will not do the necessity). Let's do the hard part first. We can suppose that $F \subset G$. Then $F = \{\rho_1 = \cdots = \rho_k = 0\}$ and $G = \{\rho_1 = \cdots = \rho_p = 0\}$ for some $p < k$ (the case $F = G$ is of course trivial.)
Since the desired diffeomorphism is given to us, we can work locally and show that it extends; in fact we only need consider points in $[X; F; G]$ and $[X; G; F]$ "above" F, since over $G \setminus F$ these are obviously the same (from the definition of lift).

So we can look at the local situation i.e. $X = \mathbb{R}^{n,k'}$, $k' \geq k$, $F = \{x_1 = \cdots = x_k = 0\}$, $G = \{x_1 = \cdots = x_p = 0\}$ for $0 < p < k$. The case $p = 1$ is also trivial,

Since then G is a hypersurface. In any XVI/6
case we get a local product decomposition

$$\mathbb{R}^{n,k'} = \mathbb{R}^{k,h} \times \mathbb{R}^{n-k,k'-h}$$

$$T = \{0\} \times \mathbb{R}^{n-k,k'-h}$$

$$G = \{x_1 = \cdots = x_p = 0\} \times \mathbb{R}^{n-k,k'-h} .$$

The last factor remains that throughout,
so we can ignore it too and suppose $n=k=k'$.

Finally then we are in a simple setting

where

$$F = \{0\} \subset G = \{x_1 = \cdots = x_p = 0\} \subset [0,\infty)^k = X$$

First consider

$$[X;F] = [0,\infty)_r \times \{t \in \mathbb{R}^{k,k} ; t_1 + \cdots + t_k = 1\}$$

$$[X;G] = [0,\alpha)_R \times \{z \in \mathbb{R}^{p,p} ; z_1 + \cdots + z_p = 1\} \times [0,\infty)^{k-p}$$

where $r = x_1 + \cdots + x_k$, $R = x_1 + \cdots + x_p$,
$x = rt$ in the 1st case $x = (Rz, x')$ in

XII/7 the second. For the lifts we have

$$\beta_F^* G = [0,\infty)_r \times \{t_1 = \dots = t_p = 0\}$$

$$\beta_G^* F = \{0\}_R \times \{(\mathbb{R}^{p,p} \cap \{z_1 + \dots + z_p = 1\}) \times \{0\}.$$

So it is mainly a matter of having enough letters! In $[[X;F];\beta_F^* G] = [X;F;G]$ the new 'radial' variable and 'tangential' variables are

$$\tilde{r} = t_1 + \dots + t_p \,,\quad \tilde{t}_j = \frac{t_j}{\tilde{r}},\ 1 \le j \le p$$

together with $z_{p+1}, \dots, z_p$. On the other hand in $[X;G;F]$ the new radial and tangential variables are

$$\tilde{R} = R + x_{p+1} + \dots + x_p \,,\ Z = \frac{R}{\tilde{R}},$$

$$\gamma_j = \frac{x_j}{\tilde{R}},\ j > p \text{ and } \mathfrak{z}_1, \dots, \mathfrak{z}_p.$$

Notice straight away that

$$\tilde R = R + x_{p+1} + \cdots + x_k$$
$$= x_1 + \cdots + x_p + x_{p+1} + \cdots + x_k = r.$$

Similarly,

$$\tilde r = t_1 + \cdots + t_p = \frac{x_1 + \cdots + x_p}{r}$$
$$= \frac{R}{r} = \frac{R}{\tilde R} = Z,$$

and so on, $\gamma_j = \dfrac{x_j}{\tilde R} = \dfrac{x_j}{r} = \tau_j$, $j > p$,

$$\tilde t_j = \frac{t_j}{\tilde r} = \frac{x_j}{r \tilde r} = \frac{x_j}{R} = z_j, \quad 1 \le j \le p.$$

Thus in fact the global coordinates are identified in the two spaces in such a way that the interactions between maps are the same:

$$x = (\tilde R Z z, \tilde R \gamma) = (r \tilde r t, r \tau).$$

So, it remains to consider the case when F and G are transversal. Again we only

XI/9 consider the region in the two blown-up spaces above $F \cap G$ and work locally. In this case the blow-up is in 'different variables' so that, locally, near $p \in F \pitchfork G$

$$X = \mathbb{R}^{k,h} \times \mathbb{R}^{n-k,k'}$$

$$F = \{x_1 = \dots = x_p = 0\} \times \mathbb{R}^{n-k,k'}$$

$$G = \{x_{p+1} = \dots = x_h = 0\} \times \mathbb{R}^{n-k,k'}$$

Regard X as the product $\mathbb{R}^{p,p} \times \mathbb{R}^{k-p,h-p} \times \mathbb{R}^{n-k,k'}$ we see that the blow-ups are completely 'independent' and each of the results follows.

So, I want to apply this to understanding the 'triple b-space' of a compact manifold with boundary:

$$X^3_b = [X^3; (\partial X)^3, (\partial X)^2 \times X, \partial X \times X \times \partial X, X \times \partial X)^2]$$

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where the notation now makes sense.

The first claim is that there is a b-fibration, $\pi_{F,b} : X^3_b \to X^2_b$ giving a commutative diagram

$$\begin{tikzcd}
X^3_b \arrow[r, "\pi_{F,b}"] \arrow[d, "\beta"'] & X^2_b \arrow[d, "\beta"] \\
X^3 \arrow[r, "\pi_F"] & X^2
\end{tikzcd}$$

We get $\pi_{F,b}$ as a composite of maps as follows:

$$\begin{aligned}
X^3_b &\longrightarrow [X^3; (\partial X)^3; X\times(\partial X)^2] \\
&\longrightarrow [X^3; X\times(\partial X)^2; (\partial X)^3] \\
&\longrightarrow [X^3; X\times(\partial X)^2] = X\times[X^2; (\partial X)^2] \\
&\longrightarrow [X^2; (\partial X)^2] = X^2_b.
\end{aligned}$$

In the first step we use the fact that the last three (lifted) boundary faces are disjoint, i.e.

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the second that $(\partial X)^3 \subset X\times(\partial X)^2$ and the last maps to the projection. Now all these maps are b-submersions, hence so is the product, $\pi_{F,b}$ (in particular it is a b-map). To check the b-normality it is convenient to follow all the boundary faces and see what the hyperfaces are where they map.

$$X^3_b \;-\; ff^3,\ ff^2_S,\ ff^2_C,\ ff^2_F,\ L,\ M,\ R$$

$$[X^3; X\times(\partial X)^2; (\partial X)^3] - ff^3,\ (\partial X)^2\times X,\ \partial X\times X\times\partial X,\ ff^2_F,\ L,\ M,\ R$$

$$[X^3; X\times(\partial X)^2]\ \partial X\times ff^2_H,\ (\partial X)^2\times X,\ \partial X\times X\times\partial X,\ ff_F,\ L,\ M,\ R$$

$$X^2_b \;-\; ff^2_F,\ \partial X\times X,\ X\times\partial X,\ ff^2_F,\ X^2_b,\ \partial X,\ X\times\partial X$$

Here, L, M, R are the 'old boundary' being, $\partial X\times X^2$, $X\times\partial X\times X$ and $X\times X\times\partial X$.

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The multiplicities are clearly 1 and and no boundary hypersurface is mapped into a face of codimension 2 or greater, so $\pi_{F,b}$ is indeed a b-fibration, being also a b-submersion.

Of course, by symmetry, we have 2 other such maps, just as I claimed last time.

If A and B are elements of $\Psi_b^{-\infty}(X)$ then the product

$$(P) \qquad \pi_{S,b}^{*}A \cdot \pi_{F,b}^{*}B$$

is a smooth section of some density bundle, with the coefficient vanishing at any boundary hypersurface which maps to either of the old boundaries under either stretched projection. Checking off the list, this includes $L, M, R, ff_S, ff_C^2, ff_F^2$ leaving only ff^3, which maps to ff_C^2

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under $\pi_{C,b}$ just as we want.

Finally then we only have to think about the densities. From $A \times B$ we have Ω_b factors from the middle and right factors. Let's just add (I mean multiply by) $\alpha\nu \in C^\infty(X;\Omega_b)$ on the left factor. This gives

$$\Omega_b(X)\otimes\Omega_b(X)\otimes\Omega_b(X) \equiv \Omega_b(X^3).$$

Now under each blow-up of a boundary face, Ω_b lifts canonically to itself. Thus multiplying (P) by ν_b on the left factor we get a smooth section of $\Omega_b(X^3_b)$, to which our push-forward theorem applies. Thus

$$(\pi_{C,b})_*(\nu_b\cdot A\cdot B) = \nu_b\cdot C,\ C\in\Psi_b^{-\infty}.$$

The proves the composition Theorem. — XI/14

$$\Psi_b^{-\infty} \cdot \Psi_b^{-\infty} \subset \Psi_b^{-\infty}.$$

Exercise Try cutting the kernels up into little pieces and introducing suitable singular coordinates to check this!

Finally, then I want to introduce

— $\Psi_b^k(X)$ as a space of operators on $C^\infty(X)$ to get our parametrix for diff.

Lemma The lifted diagonal $\beta^*\Delta = \Delta_b \subset X_b^2$ is a 'product-submanifold = p-submanifold' so the $\mathcal{V}_b$ lifts from left or right is transversal.

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Conormal distributions

The aim today is to properly introduce the space $\Psi_b^{r,k}(X)$ for $k \in \mathbb{R}$ and describe as many of the properties as I can get to. A little bit of standard geometry to start with.

A closed subset $Y \subset X$ is an embedded submanifold in a compact manifold (for the moment without boundary) if $C(X)|_Y = C^\infty(Y)$ is a C^∞ structure on Y. This is equivalent to the existence at each point of Y of local coordinates

$$(E) \qquad y_1, \ldots, y_p, z_1, \ldots, z_{n-p} \text{ in } U \subset X$$

$$\text{s.t. } Y \cap U = \{z_i = 0\}.$$

<u>Theorem</u> [Collar neighbourhood] For any embedded submanifold in a compact manifold without boundary there is an open neighbourhood $Y \subset U \subset X$ and $O_Y \subset U' \subset NY$, of the zero section of the normal bundle, and a diffeomorphism

$$F : U \longrightarrow U'$$

with

1. $F(y) = y \ \forall y \in Y$, i.e. $F: Y \to O_Y$ is the natural map

2. $F_*: NY \to NO_Y$ is the natural map

and any two such maps are homotopic through a smooth family with these properties.

Note that $N_yY = T_yX/T_yY$ and for $y \in O_Y \cong Y$ $T_y(N_yY) = N_yY \cong N_y O_Y$ since the fibre is complementary to the zero section.

<u>Proof</u> I am not going to do this in detail since it is very standard. The cleanest proof I know of uses a Riemann metric on X and fixes F through the exponential map. The Riemann metric allows N_yY to be identified with the orthocomplement $(T_yY)^\perp \subset T_yX$ and then one can check readily that

$$F(y,v) = \exp_y(v)$$

has the desired properties. This $F(y,v)$ is the point at parameter distance one along the geodesic with initial point y and initial tangent vector v, or if you prefer at distance $|v|$ for the geodesic with initial vector $v/|v|$.

The 'uniqueness' part of the result can be based on the existence of such a 'normal fibration' for the diagonal in $X \times X$.

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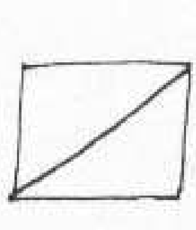
In the case of a submanifold $Y \subset X$ of a compact manifold with corners we assume that Y is a p-submanifold (the p- is for product). This is the condition that near each $y \in Y$ there are local coordinates, of the usual adapted sort, $x_1, \dots, x_k, y_1, \dots, y_{n-k}$ in terms of which Y is locally

(P) $\quad x_1 = \dots = x_j = 0, \quad y_1 = \dots = y_{p-j} = 0.$

Usually we demand $j = 0$, in which case Y is an interior p-submanifold (contained not in an interior p-submanifold in some boundary face).

<u>Exercise</u> Note that if we don't assume (P) explicitly but just require that $C^\infty(X)|_Y = C^\infty(Y)$ be the C^∞ structure we allow things like $\{x = x'\}$ in $[0,1]^2$:



Show how to recover (P) locally by assuming just that $C^\infty(X)|_Y = C^\infty(Y)$ and some condition of I_x for the embedding $I : Y \hookrightarrow X$.

<u>Theorem</u> The collar neighbourhood theorem goes through unchanged for a p-submanifold of/manifold with corners.
(closed) compact

Forget the boundary case for the moment, this is not the important part.

We introduce spaces of conormal distributions associated to an embedded submanifold $Y \subset X$. These are distributions on X, singular only at Y. Put

$$\mathcal{V}(X;Y) = \{V \in C^\infty \text{ vector field on } X, \text{ tangent to } Y\}$$

<u>Exercise</u> Show that in local coordinates (B), $\mathcal{V}(X;Y)$ is spanned by $\partial/\partial y_j$, $z_k\, \partial/\partial z_i$. Deduce that, whilst a $C^\infty(X)$-module it is <u>not</u> the space of sections of a vector bundle unless it has codimension 1 (or zero I suppose).

Now, given $m \in \mathbb{R}$ we define

$$I_\mathcal{V} H^m(X) = \{u \in H^m(X);\ \mathcal{V}^j u \subset H^m(X)\ \forall j,\ \mathcal{V} = \mathcal{V}(X;Y)\}$$

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These are the spaces we want, but the filtration is wrong! Not far wrong mind you. This is what we need to sort out.

The Fourier transform can be applied on the fibres of a real vector bundle to give an isomorphism

$$(FT)\qquad \mathcal{F}_{fib} : \begin{cases} \mathcal{S}'(V) \longrightarrow \mathcal{S}'(V', \Omega_{fib}). \\ \mathcal{S}(V) \longrightarrow \mathcal{S}(V', \Omega_{fib}) \end{cases}$$

Here Ω_{fib} is the space of fibre densities on V' and

$$\mathcal{F}_{fib} f = \hat{f}(x,w) = \int e^{-i w \cdot v} f(x,v)\,|dv|$$

in terms of a local trivialization. In fact everything here is well-defined except for $|dv|$.

<u>Exercise</u> Check this claim, that if $\hat{f}\,|du|$ is taken as a fibre density on V' then it is completely well defined, given (FT).

<u>Proposition</u> For any embedded submanifold

$$C^\infty(X;Y) \subset I_V H^u(X;Y) \subset C^\infty(X \setminus Y)$$

and if $\chi \in C^\infty(X)$ has support in a collar neighbourhood of Y then $\exists\, M \in \mathbb{R}$ s.t. $\mathcal{F}_{fu}(\chi u) \in S^M(V') \otimes \Omega_{fib}$

and conversely $\exists\, M' \in \mathbb{R}$ s.t.

$$\chi\left(\mathcal{F}^{-1}_{\xi\to} S^{M'}(V')\otimes\Omega_{fib}\right) \subset I_V H^m(X;Y);$$

in fact one can take ~~all~~ any $M' < -m - \frac{d}{2}$, $d = \dim Y$, and any $M \geq -m - d/2$.

Recall that $S^M(V')$ is the space of symbols on V'; in local coordinates and trivialization (x,η)

$$a \in S^M(V') \iff \left| D_x^\alpha D_\eta^\beta a(x,\eta)\right| \le C_{\alpha\beta}(1+|\eta|)^{M-|\beta|} \quad \forall \alpha,\beta.$$

<u>Exercise</u>: Try to check the proof of these – I do not actually use it below. Just see that in local coordinates $u \in I_V H^m(E)$ satisfies

$$\partial_{xy}^\alpha\, z^\gamma D_z^\beta u \in H^m \quad \forall\ \alpha,\beta,\gamma \text{ with } |\gamma| \ge |\beta|.$$

Under Fourier transform these turn into the estimates

$$\partial_y^\alpha D_\xi^\gamma\, \xi^\beta a \in (1+|\xi|)^M L^2 \quad \forall\ \alpha,\beta,\gamma,\ |\gamma| \ge |\beta|.$$

If you go back to an earlier lecture I think you will find I showed that this implies $a \in S^M$ and conversely $a \in S^{M'}$ implies these estimates.

Definition If $Y \subset X$ is embedded and α has support in the range of a normal fibration F we define

$$(\mathrm{I}) \qquad I^m(X;Y) = F^* \alpha\, \mathcal{F}^{Y-1}_{\mathrm{fib}}\, S^{m-\frac{n}{2}+\frac{c}{4}}(V'; \Omega_{\mathrm{fib}}) + C^\infty(X).$$

$n = \dim X$, $c = \operatorname{codim} Y = n-d$.

This definition is bogus until we show that the left side is independent of the choice of F (and α but that is actually easy). This amounts to the coordinate-invariance of $I^m(X;Y)$ which is not so obvious since the Fourier transform of an L^∞-based space is involved.

From the discussion above, $\forall\, \epsilon > 0$

$$(\mathrm{EF}) \quad \begin{cases} I^m(X;Y) \subset I_\nu H^{-m+\frac{n}{2}-\frac{c}{4}-\frac{d}{2}-\epsilon}(X;Y) \\ I_\nu H^{-m+\frac{n}{2}-\frac{c}{4}-\frac{d}{2}}(X;Y) \subset I^m(X;Y). \end{cases}$$

On the other hand the $I_\nu H^m$ spaces are manifestly coordinate-invariant so we just need to show

Lemma Given a 1-parameter family F_t, $t \in [0,1]$, of normal fibrations of Y and $u = F_0^* \alpha\, \mathcal{F}^{Y-1}_{\mathrm{fib}}\, a$, $a \in S^{m-\frac{n}{2}+\frac{c}{4}}$

there is a smooth family $a_t \in S^{m-\frac{n}{2}+\frac{c}{4}}$ with

(F) $\quad \frac{d}{dt} F_t^* \chi \mathcal{L}^{-1}_{fib}\, a_t \in C^\infty([0,1]; I_V H^N(X;Y))$

for any preassigned N and $a_t - a_0 \in S^{m-\frac{n}{2}+\frac{c}{4}-1}$ i.e.

Proof The derivative of such a family is given by the unique vector field V_t s.t. $\frac{d}{dt} F_t^* f = F_t^* V_t f$,

namely,

$$\frac{d}{dt} F_t^* \chi \mathcal{L}^{-1}_{fib}\, a_t = F_t^*\left(V_t \chi \mathcal{L}^{-1}_{fib}\, a_t + \chi \mathcal{L}^{-1}_{fib} \frac{da_t}{dt}\right)$$

Now, observe that the condition $F_t|_Y$ on a normal fibration implying that V_t is actually of the form

(V) $\quad \sum_{fin} g_j W_t^j \;,\quad g_j \in C^\infty(X),\ g_j = 0 \text{ on } Y,\quad W_t^j \in \mathcal{V}(X;Y)$

Exercise Check (V) by working in local coordinates (F).

Next from what we have seen before,

$$\mathcal{L}_{fib} V_t \chi \mathcal{L}^{-1}_{fib} : S^M \longrightarrow S^{M-1} \;,\quad \forall M$$

Thus, just taking a constant net t already gives

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$$\frac{d}{dt} F_t^* \chi \mathcal{L}_{fib}^{-1} a \in I^{m-1} \quad \text{at least formally.}$$

To do this properly, define a_t iteratively (to any finite order). Take $a_t^{(0)} =$ const,

$$a_t^{(1)} = \int_0^t \frac{d}{dt} \mathcal{L}_{fib}^{*} V_t \chi \mathcal{L}_{fib}^{-1} a \in C^\infty([0,1]; S^{M})$$

$$M = m - \tfrac{n}{2} + \tfrac{c}{4},$$

$$= B a^{(0)}$$

and then $a_t^{(2)} = B a_t^{(1)}$ etc. For N large we arrive at (F). $\subset$

This shows that (I) is independent of the choice of F. Even more, it gives us a completely natural short exact sequence

$$I^{m-1}(X;Y) \hookrightarrow I^{m}(X;Y) \xrightarrow{\sigma_m} \frac{S^{m-\frac{n}{2}+\frac{c}{4}}(N^*Y;\Omega_{fib})}{S^{m-\frac{n}{2}+\frac{c}{4}-1}}$$

<u>Exercise</u> Recall that the <u>classical</u> symbol

$$S^M_{cl} \subset S^M$$

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$$(\Psi_b)\quad \Psi_b^m(X;E,F) = \{A \in I^m(X_b^2; \Delta_b, \mathrm{Hom}(E\times F)\otimes \pi_{R,b}^*\Omega_b,$$

$$A \equiv 0 \text{ at old boundary}\}.$$

Of course this second definition requires some comment. I have set things up so that the definition of $I^m(X;Y)$ works just as well in case of Y an <u>interior</u> p-submanifold of a compact manifold with corners. We already know that Δ_b is such, as it does not meet the 'old boundaries' of X^2_b, so (Ψ_b) does make sense, and is consistent with $\Psi_b^{-\infty}$. Note that the weird orders have disappeared since Δ_b has codimension n when X^2_b has dimension $2n$. [Except I messed up the definition!]

Now we just have to check that things work as advertised.

In particular, $I^m(X;Y)$ is a $C^\infty(X)$-module, with $\sigma(fu) = f|_Y \cdot \sigma(u)$. This allows us to extend the definition to sections of vector bundles with no effect.

$$I^m(X;Y,E) = I^m(X;Y) \underset{C^\infty(X)}{\otimes} C^\infty(X;E)$$

and to get the same short exact sequence and 'local' 'action' formula:

$$I^{m-1}(X;E) \hookrightarrow I^m(X;E) \xrightarrow{\sigma_m} S^{m-\frac{n}{2}+\frac{c}{4}}(N^*Y;E)$$

$$P \in \mathrm{Diff}^l(X;E,F) \to P : I^m_Y(X;E) \to I^{m+l}_Y(X;F)$$

$$\sigma_{m+l}(Pu) = \sigma(P)|_{N^*Y} \cdot \sigma(u)$$

<u>Definition</u> For a compact manifold without boundary

$$\Psi^m(X;E,F) = \{A \in I^m(X^2;\Delta, \mathrm{Hom}(E,F) \otimes \pi_R^*\Omega)\}$$

and for a compact manifold <u>with</u> boundary are defined in terms of the radial compactification $\bar V$ of the vector bundle and ρ a defining function for the boundary by

$$S^M_{cl} = \rho^{-M} C^\infty(\bar V).$$

Show that $I^m_{cl} \subset I^m$, for any $Y \subset X$, given by the classical symbols is also well-defined.

<u>Proposition</u> If P is a differential operator on X, $P \in \mathrm{Diff}^k(X)$ then

$$P : I^m(X;Y) \longrightarrow I^{m+k}(X;Y) \quad \forall\, m \in \mathbb{R}$$

(PA)

$$\text{and} \quad \sigma_{m+k}(Pu) = \sigma_k(P)\big|_{N^*Y} \cdot \sigma_m(u).$$

<u>Proof</u>. It suffices to check this for vector fields and then iterate. From the coordinate-independence and the fact that everything is well-defined, it is enough to work locally. Then $Y = \{z_i = 0\}$ and

$$P = V = \sum_i a_i(y,z)\, \partial_{y_i} + \sum_j b_j(y,z)\, \partial_{z_j}$$

in terms of (F), the first part is in $\mathcal{V}(X;Y)$ and maps I^m to I^m. The second gives (PA) easily.

14. Lecture XIV, 23 October, 2003

What have I not done to complete the treatment of the 'conic case' at least as far as the identification of the L^2 cohomology, the relative Hodge cohomology and the appropriate intersection cohomology is concerned? I have not

(1) Treated the composition of finite-order b-pseudodifferential operators.
(2) This in turn is only really needed (for the moment) for the proof of the Sobolev continuity of such operators, that $A \in \Psi^k_{\mathrm{b}}(X)$ always defines a bounded linear operator from $x^s H^m_{\mathrm{b}}(X)$ to $x^s H^{m-k}_{\mathrm{b}}(X)$ for any m, s. This is what gives us elliptic regularity.
(3) I have not yet discussed intersection cohomology at all.

I will add to the notes, but probably not devote a lecture to, the first two of these. My reasoning here is that these reduce, given what we already know, to the same issues in the boundaryless case, so I do not feel the need to go through the discussion fully here.

Rather than go through the conic case again, I will now quickly describe the same sort of approach to another class of degenerate metrics which I will call 'cusps' but are often called 'horns'. I do not want to take the time to go through all the details, but I will attempt to write down everything to the point where it is 'straightforward' to check the claims that I make.

The metrics we consider again exist on any compact manifold with boundary, but with a somewhat different degeneration than for conic metrics. Thus, we suppose that in the interior, g is a metric and near the boundary there is a boundary defining function ρ such that

$$g = d\rho^2 + \rho^{2N} h \tag{1}$$

where h is as before, a smooth symmetric 2-cotensor which restricts to the boundary to a metric h_0 and $N \geq 2$; the conic case corresponds to $N = 1$. The extreme case, $N = 0$, is that of a regular boundary problem, which can also be handled the same way but leads to somewhat different analytic issues (and a different L^2 cohomology of course, namely the absolute cohomology).

Exercise 21. Let $Y^n \subset X^{n+1}$ be a singular submanifold of a compact manifold without boundary where Y has just an isolated singular point (or perhaps several) near which there are local coordinates z_j, in which it takes the form

$$z_0^{2N} = \sum_{j=1}^{n} z_j^2 + f(z_1, \ldots, z_n),\ \ z_0 \geq 0, \tag{2}$$

where f (real-valued) vanishes to order 3 at least at 0. Show that the introduction of the singular coordinates z_0, z_j/z_0^N resolves Y to a manifold with boundary to which a metric on X restricts to a 'horn' metric (note that z_0 might not quite be x.)

For extra credit (!) show that the same thing can be accomplished by repeatedly blowing up the singular point, namely it needs to be blown up N times.

Problem 1. Describe the L^2 and Hodge cohomology for a metric of this 'cusp' type. In fact we want to do 'everything' in a sense that should be getting clearer by now.

The approach I will use is, and of course this is one of the main points, essentially the same as in the conic case although some of the 'details' are necessarily different.

Namely, first look at the structure of $d+\delta$ in terms of a Lie algebra of vector fields such that we can give an elliptic regularity result for the associated enveloping algebra (and develop a full calculus of pseudodifferential operators to go along with this). This is used to analyse the relative and absolute domains, which have the same definitions as before, deduce self-adjointness and the Fredholm property and hence get the Hodge decomposition and identity of L^2 cohomology and relative Hodge cohomology. It turns out that in this case the Hodge cohomologies can be identified in terms of the usual relative/absolute cohomology and subsequently in terms of appropriate intersection cohomology. Rather surprisingly perhaps the spaces (say L^2 cohomology) on a fixed manifold for different metrics and different values of $N \geq 1$ turn out to be canonically isomorphic.

So, first we look for a Lie algebra of smooth vector fields with which to describe the Laplacian and $d+\delta$. If we look at vector fields of finite length they will generally be singular at the boundary, with the worst singularity being $O(\rho^{-N})$ (take $N=2$ if you want). So we can look at the vector fields V which are smooth and satisfy

$$|V|_g = O(\rho^N). \tag{3}$$

Since the part $\rho^{2N}h$ of the metric already gives such an order of vanishing for any smooth V, this is equivalent to

$$V\rho \in \rho^N \mathcal{C}^\infty(X) \Longleftrightarrow V \in \mathcal{V}_{\mathrm{Nc}}(X). \tag{4}$$

Locally, in adapted coordinates, in which $x=\rho$ must always be an *admissible* defining function, i.e. one for which (4) holds, this Lie algebra and $\mathcal{C}^\infty(X)$ module is spanned by

$$x^N\partial_x,\ \partial_{y_j}. \tag{5}$$

It follows that it is the space of all smooth sections of a vector bundle, ${}^{\mathrm{Nc}}TX$, for which (5) gives a local basis. In the case $N=2$ I introduced this Lie algebra long ago; it depends on the choice of ρ as a trivialization of the normal bundle to the boundary, but nothing more. For $N \geq 3$ it only depends on the choice of ρ modulo terms $O(\rho^N)$ as is clear from (4).

Exercise 22. Can you give a 'geometric' description of an N-cusp structure on a compact manifold with boundary, analogous to the trivialization of the normal bundle in case $N=2$?

Similarly we define the N-cusp cotangent, and form, bundles based on $\mathcal{C}^\infty$ combinations of the forms

$$dx,\ \rho^N dy_j \tag{6}$$

I will denote these bundles ${}^{\mathrm{Nc}}T^*X$ and ${}^{\mathrm{Nc}}\Lambda X$; note that they depend on more than N!

Since ${}^{\mathrm{Nc}}T^*X$ is, by definition, the dual of ${}^{\mathrm{Nc}}TX$, a smooth section of the latter, $V \in \mathcal{V}_{\mathrm{Nc}}(X)$, defines a smooth function on the former which is linear on the fibres; we normalize this by defining $\sigma(V)$ to be iV thought of as a linear function. A function $f \in \mathcal{C}^\infty(X)$ similarly defines a smooth function on ${}^{\mathrm{Nc}}T^*X$ which is constant on the fibres (and we do not put an i in the identification of $\sigma(f)$ with f in this sense). Let $\mathrm{Diff}^k_{\mathrm{Nc}}(X)$ be the space of N-cusp differential operators of order k (at most). Thus $P \in \mathrm{Diff}^k_{\mathrm{Nc}}(X)$ is an operator, for example on $\mathcal{C}^\infty(X)$, which can be written as a finite sum of up to k fold products of elements of $\mathcal{V}_{\mathrm{Nc}}(X)$; this one can think of as the enveloping algebra of $\mathcal{V}_{\mathrm{Nc}}(X)$ as a Lie algebra and $\mathcal{C}^\infty(X)$ module (in particular $k=0$ factors means the action of $f \in \mathcal{C}^\infty(X)$ by multiplication); the $\mathrm{Diff}^k_{\mathrm{Nc}}(X)$ clearly form a (n order-)filtered algebra. Moreover, from the fact that $\mathcal{V}_{\mathrm{Nc}}(X)$ is a Lie algebra

$$[\mathrm{Diff}^k_{\mathrm{Nc}}(X), \mathrm{Diff}^l_{\mathrm{Nc}}(X)] \subset \mathrm{Diff}^{k+l-1}_{\mathrm{Nc}}(X) \tag{7}$$

we see that the commutative product in $\mathcal{C}^\infty({}^{\mathrm{Nc}}T^*X)$ leads to a short exact sequence

$$\mathrm{Diff}^{k-1}_{\mathrm{Nc}}(X) \hookrightarrow \mathrm{Diff}^k_{\mathrm{Nc}}(X) \xrightarrow{\sigma_k} \mathcal{P}^k({}^{\mathrm{Nc}}T^*X). \tag{8}$$

Here the quotient space is the space of smooth functions on ${}^{\mathrm{Nc}}T^*X$ which are homogeneous (polynomials) of degree k on the fibres. The symbol map is *determined* by the fact that it is multiplicative and our earlier normalization on $\mathcal{C}^\infty(X) = \mathrm{Diff}^0_{\mathrm{Nc}}(X)$ and $\mathcal{V}_{\mathrm{Nc}}(X)$.

We can extend these definitions to sections of vector bundles without pain. Either localize everything, which is a bit painful, or interpret the tensor product in

$$\mathrm{Diff}^k_{\mathrm{Nc}}(X; E, F) = \mathrm{Diff}^k_{\mathrm{Nc}}(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X; \hom(E, F)). \tag{9}$$

Of course it is important that this defines a space of operators $\mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F)$.

Exercise 23. Check that there are no surprises and the symbol extends in the obvious way and gives rise to a short exact sequence as in (8) but with bundles inserted appropriately.

As usual, ellipticity means precisely that $\sigma_k(P)$ is invertible off the zero section of ${}^{\mathrm{Nc}}T^*X$.

Now we look at $d+\delta$ from this point of view.

Lemma 4. *For an N-cusp metric* (1)*, $d+\delta \in \rho^{-N}\,\mathrm{Diff}^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*)$ is elliptic in this sense.*

Proof. To check that $d \in \rho^{-N}\,\mathrm{Diff}^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*)$ just work out its action on the local basis of 1-forms (6):

$$\begin{aligned} d(adx + \sum_{k=1}^{n-1} b_k x^N dy_k) &= x^{-N}(\sum_{j=1}^{n-1} (\partial_{y_j} a) x^N dy_j \wedge dx \\ &+ \sum_{k=1}^{n-1} (x^N \partial_x b_k + N x^{N-1} b_k) dx \wedge x^N dy_k + \sum_{l,k=1}^{n-1} (\partial_{y_l} b_k) x^N dy_l \wedge x^N dy_k \end{aligned} \tag{10}$$

Remark 2. It is precisely at this point that we see a simplification arising in the cases $N \geq 2$ relative to the conic case, $N=1$. Namely in the middle, 'cross', term here the term of order 0, which arises from the x-differentiation of x^N in the basis of forms, vanishes at $x=0$ if $N>1$. This means that this term will not show up in the 'model' operator we later consider, as we considered the model cone earlier. For this reason the non-zero eigenvalue problems that appeared for the cone, and caused most of the computation work, do not show up at all for $N>1$. I did say the cone was the hardest case earlier! It also means that the 'model operator' for $d+\delta$, when we try to look at what happens to leading order at the boundary is not, or at least should not be thought of, as the operator for the model problem.

For the latter this x^{N-1} term would appear, but it is irrelevant to the analysis and is better dropped. Make of that what you will.

Then we can either check, exactly as before, that Hodge $\star$ is an isomorphism of ${}^{\mathrm{Nc}}\Lambda^* X$, which is essentially immediate from the definition, or else that the adjoints with respect to a measure such as

$$dg = \rho^{N(n-1)}\nu,\ 0 < \nu \in \mathcal{C}^\infty(X;\Omega), \tag{11}$$

and non-degenerate fibres inner products, of elements of $\mathrm{Diff}^k_{\mathrm{Nc}}(X;E,F)$ are in $\mathrm{Diff}^k_{\mathrm{Nc}}(X;F,E)$. Anyway, we easily conclude that $\delta \in \mathrm{Diff}^1_{\mathrm{Nc}}(X;{}^{\mathrm{Nc}}\Lambda^*)$.

The argument for ellipticity is the same as before. We can see from (10) that the symbol of d at (p,ξ), $\xi \in {}^{\mathrm{Nc}}T^*_pX$, is $i\xi\wedge$ acting on ${}^{\mathrm{Nc}}\Lambda^*_pX$. The symbol of the adjoint is the adjoint of the symbol (or use $\star$) so the symbol of δ is $-ic_\xi$ in terms of metric contraction. The result is a Clifford action of ${}^{\mathrm{Nc}}T^*X$ on ${}^{\mathrm{Nc}}\Lambda^*X$, and in any case is elliptic since its square is diagonal and given by multiplication by the metric (remember, this is a course on Dirac operators, except I have only talked about one so far!) □

We want uniform elliptic estimates (and more, we really want a way to write down the inverse of $d+\delta$). To get these we will work on a double space which resolves $\mathcal{V}_{\mathrm{Nc}}(X)$. This is supposed to be obtained through iterated blow-up, $\beta : X^2_{\mathrm{Nc}} \longrightarrow X^2$ and be such that we can lift $\mathcal{V}_{\mathrm{Nc}}$ smoothly from either factor of X and the resulting smooth vector fields are transversal to the lifted diagonal. We also want the stretched projections $\pi_{H,\mathrm{Nc}} = \pi_H \circ \beta$, $H = L, R$, to be b-fibrations which are transversal to the lifted diagonal (among other things this means that the lifted diagonal is diffeomorphic to X). Let's try to do it; maybe just sticking to $N = 2$ would be wise, but I will go on and outline the general case.

Locally our vector fields are $x^N\partial_x$ and ∂_{y_j}. Just as in the conic case we do not want, in fact cannot, do anything to the tangential ∂_{y_j} vector fields, since they are already non-degenerate. Basically this resolution problem is again 1-dimensional, or since we are in the double space, 2-dimensional, just involving x and x'. An obvious thing to look at is the lift of $x^N\partial_x$ to the space X^2_{b}, which resolves $\mathcal{V}_{\mathrm{b}}(X)$. In the coordinates $s = x/x'$ x', y_j and y'_j near a point on the lifted diagonal $s = 1$, $y = y'$ in X^2_{b} we know that $x\partial_x = s\partial_s$. (So that the lift of $\mathcal{V}_{\mathrm{b}}(X)$ is everwhere transversal to the diagonal). So of course, $x^N\partial_x$ lifts to $(x')^{N-1}s^N\partial_s$. Since $s = 1$ on the diagonal, this vanishes exactly at $x' = 0$ on the lifted diagonal, which is to say at its boundary. However, we cannot blow-up the boundary of the diagonal, since the ∂_{y_j} are not tangent to it! The smallest reasonable thing to blow up is

$$B_2 = \{x' = 0, s = 1\} \subset \mathrm{ff}(X^2_{\mathrm{b}}). \tag{12}$$

Exercise 24. Check that this is actually a well-defined submanifold of X^2_{b} which depends (only) on the choice of cusp structure, i.e. the defining function ρ. Note that it is a boundary p-submanifold, i.e. is an interior p-submanifold of the boundary hypersurface $\mathrm{ff}(X^2_{\mathrm{b}}$.

Even though the notation is not quite defined, we consider

$$X^2_{\mathrm{cu}} = [X^2_{\mathrm{b}}; B_2]. \tag{13}$$

I have not defined the blow up of a boundary p-submanifold such as B but it is a straightforward generalization of the blow up of a boundary face. We get a new boundary face ff and b-map as blow-down map. The main (big) difference is that this map is not a b-submersion, if fact it is a b-submersion exactly when B is a boundary face, which it is not here. This is replaced by the fact that

Lemma 5. *The Lie algebra $\mathcal{V}_{\mathrm{b}}(X;Y)$ of vector fields tangent to the boundary faces of X and to the p-submanifold Y lifts smoothly to $[X;Y]$ to span $\mathcal{V}_{\mathrm{b}}([X;Y])$ as a module over $\mathcal{C}^\infty([X;Y])$.*

Thus we do in fact know the range of β_*.

Exercise 25. Check this in the particular case of interest here, namely for $B_2 \subset X^2_{\mathrm{b}}$.

Proposition 6. *The cusp algebra $\mathcal{V}_{\mathrm{cu}}(X)$ defined by a choice of boundary defining function $\rho \in \mathcal{C}^\infty(X)$ on a compact manifold with boundary lifts, from the right or left factor, to a space of smooth vector fields on X^2_{cu} (defined of course by the same choice of ρ) to be transversal to the lifted diagonal, which is an interior p-submanifold $\mathrm{Diag}_{\mathrm{cu}} \subset X^2_{\mathrm{cu}}$. The left and right stretched projections, $\pi_{O,\mathrm{cu}} = \pi_O \circ \beta_{\mathrm{cu}}$, $O = L, R$, are b-fibrations which are transversal to $\mathrm{Diag}_{\mathrm{cu}}$.*

Proof. That the cusp algebra lifts, we know from Lemma 5. In any case it is a rather straightforward computation which I will do! We can ignore ∂_{y_j} throughout and we have only to deal with the one vector field, which starts off as $x^2\partial_x$. After we lift it to X^2_{b} it is $x's^2\partial_s$ in terms of the coordinates $x', s = x/x'$ which are valid near the boundary of the diagonal. We can drop the s^2 since it is non-vanishing and switch from s to $t = s-1$ which has the virtue of vanishing at $B_2 = \{x' = 0, t = 0\}$ and our vector field is a non-vanishing smooth multiple of $x'\partial_t$. The lifted diagonal is $y = y'$ $t = 0$ and near it we can use the singular coordinates x' and $t_2 = t/x'$ (together with y and y'). In terms of these our vector field has become ∂_{t_2}, so with ∂_{y_j} we do indeed get a set of smooth vector fields transversal to the interior p-submanifold $\mathrm{Diag}_{\mathrm{cu}}$.

So, X^2_{cu} does resolve $\mathcal{V}_{\mathrm{cu}}(X)$. Now we need to check that we haven't gone too far somehow. So, $\pi_{R,\mathrm{cu}}$ is a well-defined b-map. Why is it a b-submersion? Consider the vector field $x\partial_x + x'\partial_{x'}$. This is in $\mathcal{V}_{\mathrm{b}}(X^2)$ and so lifts to X^2_{b} to be smooth. Near B_2 it has become

$$(t-1)\partial_t + (x'\partial_{x'} - (t-1)\partial_t) = x'\partial_{x'}.$$

in terms of the coordinates $t = s-1 = (x-x')/x'$ and x'. Here the first term is the lift of $x\partial_x$ and the second is the lift of $x'\partial'_x = -s\partial_s + x'\partial_{x'}$ in the new coordinates. Thus it is certainly tangent to B_2 and so lifts to be smooth on X^2_{cu} by Lemma 5. But this means that the vector field to which it lifts pushes forward under $\pi_{L,\mathrm{cu}}$ (or $\pi_{R,\mathrm{cu}}$ for that matter) to $x\partial_x$ on X. So in fact $(\pi_{L,\mathrm{cu}})_* : {}^{\mathrm{b}}T_pX^2_{\mathrm{cu}} \longrightarrow {}^{\mathrm{b}}T_{p'}X$ must always be surjective! Since the image manifold is a manifold with boundary the additional condition of b-normality is void. Thus $\pi_{L,\mathrm{cu}}$ is a b-fibration. That this b-fibration is transversal to $\mathrm{Diag}_{\mathrm{cu}}$ is the statement that the null space of the (ordinary or b-) differential $(\pi_{R,\mathrm{cu}})_*$ contains a complement to the tangent space of $\mathrm{Diag}_{\mathrm{cu}}$ at each point. This we already know, since the lifts from the left factor of elements of $\mathcal{V}_{\mathrm{cu}}(X)$ must be killed by $\pi_{R,\mathrm{cu}}$, and this lift spans such a complement at each point. □

Now, having done this in the cusp case, $N = 2$, I may as well go on into the higher cusp cases. First try $N = 3$. Then we have $x^3\partial_x$ in place of $x^2\partial_x$. So, when we lift it up to X^2_{cu} from the left, we can see from the computation above that in the coordinates t_2, x' (and always y, y') that we get a smooth positive multiple of $x'\partial_{t_2}$. So, we have to blow up $B_3 = \{t_2 = 0, x' = 0\} \subset X^2_{\mathrm{cu}}$. Thus we conclude that

$$\mathcal{V}_{3\mathrm{c}}(X) \text{ is resolved on } X^2_{3\mathrm{c}} = [X^2_{\mathrm{cu}}; B_3]. \tag{14}$$

In fact the same argument clearly works for any N. Proceeding by induction we can claim that the functions

$$t_N = \frac{x - x'}{(x')^N}, x', y, y' \tag{15}$$

lift to X^2_{Nc} to give coordinates near the lifted diagonal in which it becomes $t_N = 0$, $y = y'$ and such that $x^{N+1}\partial_x$ lifts from the left factor to be a smooth positive multiple of $x'\partial_{t_N}$. Then we can define $B_{N+1} = \{x' = 0, t_N = 0\}$ and define the next space as

$$X^2_{(\mathrm{N+1})\mathrm{c}} = [X^2_{\mathrm{Nc}}; B_N]. \tag{16}$$

Proposition 7. *Proposition 6 carries over to the N-cusp algebra with X^2_{cu} replaced by X^2_{Nc}.*

With this behind us, we can define

$$\Psi^k_{\mathrm{Nc}}(X) = \{A \in I^k(X^2_{\mathrm{Nc}}, \mathrm{Diag}_{\mathrm{Nc}}; \pi^*_{R,\mathrm{Nc}}\Omega_{\mathrm{Nc}}); A \equiv 0 \text{ at } \partial X^2_{\mathrm{Nc}} \setminus \mathrm{ff}_{\mathrm{Nc}}\}, \tag{17}$$

where $\Omega_{\mathrm{Nc}} = \rho^{-Nn+1}\Omega_{\mathrm{b}} = \rho^{-Nn}\Omega$.

Exercise 26. I leave it to you to show how to define the operators on sections of vector bundles.

There are lots of things to say about these operators, and I will say at least some of them. The first thing is to see that $\mathrm{Diff}^k_{\mathrm{Nc}}(X) \subset \Psi^k_{\mathrm{Nc}}(X)$. The place to start here is the identity operator! In local coordinates it can be written

$$\mathrm{Id}\, u(x, y) = \int \delta(x - x')\delta(y - y')u(x', y')|dx'dy'|. \tag{18}$$

To lift the kernel up to X^2_{b} we need to introduce say $s = x/x'$ as variable in place of x. Since it is essentially a parameter we can use the fact that the delta 'function' is homogeneous of degree -1, so

$$\delta(x - x') = (x')^{-1}\delta(s - 1) = (x')^{-1}\delta(t). \tag{19}$$

But the factor of x' just turns dx' into dx'/x' and we get a coefficient b-density:

$$\delta(t)\delta(y - y')|\frac{dx'}{x'}dy'| \in \Psi^0_{\mathrm{b}}(X). \tag{20}$$

We can continue this way up to X^2_{Nc} to see that

$$\mathrm{Id} = \delta(t_n)\delta(y - y')|\frac{dx'}{(x')^N}dy'| \in \Psi^0_{\mathrm{Nc}}(X). \tag{21}$$

Clearly it is elliptic, since it has symbol 1.

Exercise 27. Now use the fact that $\mathcal{V}_{\mathrm{Nc}}(X)$ lifts from the left fact to be smooth vector fields inn $\mathcal{V}_{\mathrm{b}}(X^2_{\mathrm{Nc}})$ to show that $\mathrm{Diff}^k_{\mathrm{Nc}}(X) \subset \Psi^k_{\mathrm{Nc}}(X)$.

Proposition 8. *The elements of $\Psi^m_{\mathrm{Nc}}(X)$, for any $m \in \mathbb{R}$, define continuous linear operators on $\mathcal{C}^\infty(X)$.*

Proof. This is an application of the push-forward theorem using the fact that the stretched projections are b-fibrations; it is also necessary to sort out the behaviour of the density factors. □

Proposition 9. *The $\Psi^k_{\mathrm{Nc}}(X)$ form an order-filtered asymptotically complete *-algebra of operators on $\mathcal{C}^\infty(X)$ with multiplicative symbol map giving a short exact sequence*

$$\Psi^{m-1}_{\mathrm{Nc}}(X) \hookrightarrow \Psi^m_{\mathrm{Nc}}(X) \longrightarrow (S^m/S^{m-1})({}^{\mathrm{Nc}}T^*X) \tag{22}$$

and each $A \in \Psi^0_{\mathrm{Nc}}(X)$ is bounded on $L^2(X)$.

Proof. So, I have left a bit of a hole in the preparation for the product formula – in particular I don't quite have the machinery in place to prove even the composition formula in the boundaryless case, so I will have to talk about that too. So this whole proof will take a little while – maybe you should bypass it as I will do in the lecture!

First we consider the composition formula for operators of order $-\infty$, which of course is part of the claim. The idea here is exactly the same as before. We want to find a triple product with appropriate properties. This will involve a bit of an effort. Let's start with the triple b-product X^3_{b}. We already know that X^3_{b} maps back under $\pi O, \mathrm{b}$ to X^2_{b}. Now, inside X^2_{b} we have the submanifold B_2 that we need to blow up to turn X^2_{b} into X^2_{cu}. So, we consider the inverse image $\pi^{-1}_{O,\mathrm{b}}(B_2) = B_{2,O}$ for $O = F, S, C$. Two of the boundary faces of X^3_{b} are mapped into the front face of X^2_{b} under each of the stretched projections so we actually get two, intersecting, boundary p-submanifolds as the preimage of B_2 from each of the projections. Somewhere there are some pictures of what is going on!

□

In particular our elliptic construction from the boundaryless case carries over unchanged.

Proposition 10. *If $P \in \Psi^k_{\mathrm{Nc}}(X; E, F)$ is elliptic then there exists $Q \in \Psi^{-k}_{\mathrm{Nc}}(X; F, E)$ such that $P \circ Q - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{Nc}}(X; F)$ and $Q \circ P - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{Nc}}(X; E)$.*

One important point is that, as in the conic case, $(x/x')^s$ is a multiplier on the space $\Psi^k_{\mathrm{Nc}}(X; E, F)$, although as we shall see, much more is true for $N > 1$. Anyway, by conjugating and using ellipticity it follows that

$$u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*),\ \rho^N(d+\delta)u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*) \Longrightarrow u \in \rho^{-\frac{Nn}{2}} H^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*) \tag{23}$$

Here I have put in the weight that comes from the metric. The Sobolev space on the right is just that based on $\mathcal{V}_{\mathrm{Nc}}(X)$, so $u \in H^p_{\mathrm{Nc}}(X)$ just means that $\mathcal{V}_{\mathrm{Nc}}(X)^j u \subset L^2_{\mathrm{Nc}}(X)$ for all $j \le p$. This of course applies to the maximal, ungraded, domain

$$D_{\max} = \left\{u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*); (d+\delta)u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*)\right\} \subset \rho^{-\frac{Nn}{2}} H^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*) \tag{24}$$

So, we then want to work out more precisely what this domain, and the various smaller ones we have defined, D, D_A, D_R and $D_{\min}$ are. This turns out to be fairly straightforward when $N \ge 2$ using the finer conjugation property anticipated above. Namely, for any real τ,

$$\exp(i\tau(\frac{1}{x} - \frac{1}{x'}) \text{ is a multiplier on } \Psi^k_{\mathrm{cu}}(X). \tag{25}$$

More generally

$$\exp(i\tau(x^{-N} - x'^{-N}) \text{ is a multiplier on } \Psi^k_{\mathrm{Nc}}(X). \tag{26}$$

Exercise 28. Check this by seeing what happens to the function when lifted to X^2_{Nc}; i.e. show that it is smooth except at the part of the boundary where the kernels are assumed to vanish rapidly where it only has a singularity of finite order.

Now, the maximal graded domain

$$D = \left\{u \in L^2_g(X;\Lambda^*); du, \delta u \in L^2_g(X;\Lambda^*)\right\} \tag{27}$$

with norm $\|u\|^2_D = \|u\|^2_{L^2} + \|du\|_{L^2} + \|\delta u\|_{L^2}$ and the relative and absolute domains

$$D_R = \{u \in D; \exists\dot{\mathcal{C}}^\infty(X;\Lambda^*) \ni u_n \to u \text{ in } L^2_g(X;\Lambda^*) \text{ s.t. } du_n \to du \text{ in } L^2_g(X;\Lambda^*)\},\ D_A = *D_R. \tag{28}$$

Lemma 6. *If $k = \frac{n-1}{2}$ (so n is odd) and $U : H^k_{\mathrm{Ho},h_0}(\partial X;\Lambda^k) \longrightarrow D$, and $V : H^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X) \longrightarrow D$ such that*

$$U\phi = \phi + \rho\mathcal{C}^\infty(X;\Lambda^k),\ V\phi = d\rho \wedge \phi + \rho\mathcal{C}^\infty(X;\Lambda^{k+1}). \tag{29}$$

Proof. For $\phi \in H^k_{\mathrm{Ho},h_0}(X)$, let $\phi(x)$ be the representative harmonic with respect to the varying metric $h(x, y, dy, 0)$. The termss in dx can be suppressed since these are already $O(x^2)$ with respect to the metric. It then follows that $U\phi = \chi\phi(x)$, for an appropriate cut-off χ, is in D and then we can take $V\phi = *U\phi$. □

Basically there is nothing else!

Theorem 2. *For a cusp metric as in* (1) *the graded L^2 domain*

$$D = \overline{x^{-\frac{n}{2}+1}H^1_{\mathrm{cu}}(X;{}^{\mathrm{cu}}\Lambda^*)} + UH^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X) + VUH^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X), \tag{30}$$

where the closure is with respect to $\|\cdot\|_D$, and the relative domain

$$D_R = \overline{x^{-\frac{n}{2}+1}H^1_{\mathrm{cu}}(X;{}^{\mathrm{cu}}\Lambda^*)} + UH^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X) \tag{31}$$

and with this domain, $d + \delta$ is a self-adjoint Fredholm operator with consequent Hodge decomposition

$$L^2(X;{}^{\mathrm{cu}}\Lambda^*) = H^*_g(X) \oplus dD_R \oplus \delta D_R \tag{32}$$

and null space canonically isomorphic to the L^2 deRham cohomology.

Proof. This involves computations similar to, but easier than, those in the conic case. □

Lecture XV.

First today let me compute the cohomology groups I have been talking about. Consider a general conic-cusp metric

$$g = dx^2 + x^{2N} h.$$

If $u = u_t + dx \wedge u_n$ is a smooth form on X in the usual sense, $u \in C^\infty(X, \Lambda^k)$, then

$d = n-1$

$$k < \frac{d+1}{2} \Rightarrow u \in L^2_g \qquad N=1$$

or $$k < \frac{d}{2} + \frac{1}{N} \Rightarrow u \in L^2_g \quad \text{general } N.$$

To see this, check the normal and tangential part separately:

for $N=1$ $|x^k u_t|_{g\Lambda^k} < \infty$ so $u_t \in L^2$ if $\int_0^1 x^{-2k+d} dx\, dh < \infty$.

for general N,

$$|x^{kN} u_t|_{h\Lambda^k} < \infty \text{ so } \int_0^1 x^{-2kN+dN} dx\, dh < \infty$$
$$\Rightarrow u_t \in L^2.$$

Since dx has length around 1, the condition on u_n is $k-1 < \frac{d}{2} + \frac{1}{N}$ which of course is weaker.

Thus, we have a map given by the

Hodge decomposition, or projection onto harmonic part

$$(\text{I}) \qquad C^\infty(X;\Lambda^k) \longrightarrow H^k_{Ho,A}(X), \quad k < \frac{d}{2} + \frac{1}{N}.$$

By definition the absolute de Rham cohomology is

$$H^k_{dR,A}(X) = \{u \in C^\infty(X;\Lambda^k);\ du = 0\} / dC^\infty(X;\Lambda^{k-1}).$$

The relative de Rham cohomology is

$$H^k_{dR,R} = \{u \in \dot{C}^\infty(X;\Lambda^k);\ du = 0\} / d\dot{C}^\infty(X;\Lambda^{k-1}).$$

<u>Prop</u>^{n} For a conic-cusp metric $(*)$ (so $N \geq 1$),

$$(\text{E}) \qquad H^k_{Ho,A} \cong \begin{cases} H^k_{dR,A}(X) & k \leq \frac{n}{2} \\ \operatorname{Im}(i_R : H^k_{dR,R}(X) \longrightarrow H^k_{dR,A}(X)) & k = \frac{n}{2}, \frac{n+1}{2} \\ H^k_{dR,R}(X) & k \geq \frac{n}{2} + 1. \end{cases}$$

Here, i_R is the natural inclusion arising from

$$\dot{C}^\infty(X;\Lambda^k) \hookrightarrow C^\infty(X;\Lambda^k),$$

so $[u] \in \operatorname{Im}(i)$ is represented by $u \in C^\infty(X;\Lambda^k)$ with $du = 0$ modulo $dC^\infty(X;\Lambda^{k-1}) \cap \dot{C}^\infty(X;\Lambda^k)$.

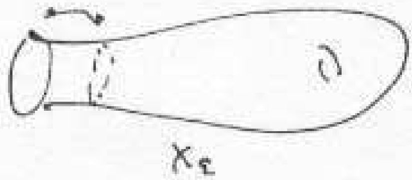
Proof From the inclusion $C^\infty(X;\Lambda^k) \hookrightarrow L^2_g(X;\Lambda^k)$, and $k < \frac{n}{2}$ we not only have an extension (I) but

$$H^k_{dR,A}(X) \hookrightarrow H^k_{Hg,A}(X) \qquad k \leq \frac{n}{2}.$$

Indeed, if u is mapped to zero then

$$\text{(A)} \qquad u = dv, \quad v \in D^{k-1}_A .$$

In fact we know considerably more than this, since $v \in C^\infty(X^\circ;\Lambda^*)$, by elliptic regularity. This however implies that $[u] = 0$ in $H^k_{dR,A}(X)$, although not quite trivially. To do so we use a retraction of X into its interior, $X_\varepsilon = \{p \in X;\ x(p) \geq \varepsilon\}$.



X_ε

Choose an inward-pointing vector field V, $Vx = 1$ near ∂X and consider the 1-parameter family $F_\varepsilon = \exp(\varepsilon V)$ of diffeos.

$$\frac{d}{dt} F_t^* f = F_t^* Vf, \quad \text{on functions}$$

$$\frac{d}{dt} F_t^* u = F_t^* \mathcal{L}_V u = F_t^* (d\iota_V + \iota_V d) u \quad \text{on forms.}$$

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Certainly $F_t^*: C^\infty(X;\Lambda^k) \to C^\infty(X;\Lambda^k)$ and $dF_t^* = F_t^* d$,

so F_t^* projects to a map on cohomology, in fact

$$F_t^*: H^k_{dR,A}(X) \to H^k_{dR,A}(X) \text{ is the identity.}$$

Indeed if u is closed, $\frac{d}{dt} F_t^* u = F_t^* d\, \iota_V u = d v_t$

so $[u] = [u_t]$ in cohomology.

Now, from this it follows that (A) implies $[u] = 0$,

since $F_t^* u = d F_t^* v \in d\, C^\infty(X;\Lambda^{k-1})$ for $t > 0$.

Similarly, we can define

$$r: H^k_{Ho,A}(X) \longrightarrow H^k_{dR,A}(X)$$

$$u \longmapsto [F_t^* u], \quad t > 0 \text{ small}.$$

using the fact that $H^k_{dR,A}(X) \subset C^\infty(X^\circ;\Lambda^k)$. Since the composite $r \circ i$ is the identity it suffices to see that r is injective, but this follows from the fact that ~~...~~ $i \circ r$ is close to the identity.

Next consider what happens for $k \geq \frac{n}{2} + 1$. We always have $C^\infty(X;\Lambda^k) \hookrightarrow L^2_g$ and

$$H^k_{dR,\pi}(X) \longrightarrow H^k_{Ho,A}(X), \quad k \ge \frac{n}{2}+1$$

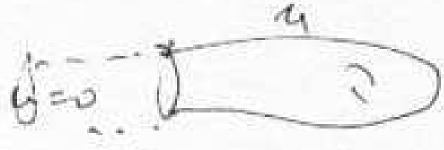
so again the first question is injectivity. The vanishing of $[u]$ in the range means $u = dv$, $v \in D_A^{k-1}$.

In case ~~$k > \frac{n}{2}+1$~~ as this means ~~(in case $N=1$)~~

$$v_n \to v \text{ in } L^2, \quad dv_n \to u \text{ in } L^2$$

so $\langle u,u\rangle = \langle u, dv\rangle = \lim \langle u, dv_n\rangle = 0$,

so injectivity is obvious. To get surjectivity we want to 'contract the other way'. We know (at least for $N=1$) that $u_t \in x^{-\frac{n}{2}+k+\delta}H^\infty_b(X;\Lambda^k X)$, $d u_n \in x^{-\frac{n}{2}+(k-1)+\delta}H^\infty_b(X;\Lambda^{k-1}X)$ are conormal, $\delta > 0$ (in fact $\delta < \frac12$). Since $k \ge \frac{n}{2}+1$ this implies both are continuous at u_t is at least C^1. So if we extend u across ∂X as zero to $\bar u$,



$$d\bar u = 0.$$

$X_{-\varepsilon}$

This tells us that $u \in H^k_{Ho,A}(X)$ does represent a relative class. More precisely look at

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the condition $du = 0$ which, near the boundary, becomes

$$d_t u_t = 0 \quad -d_t u_n + x\partial_x u_t = 0$$

(where I have dropped the x^k nonvanishing terms). The decay at the boundary is enough so that

$$w = \varphi \int_0^x u_n(x, \cdot)\, dx$$

makes sense, with φ a cut off near $x = 1$. Then

$$dw = \varphi'\, dx \wedge \int_0^x u_n\, dx$$

$$+ \varphi \int_0^x d_t u_n\, dx \;+\; \varphi\, dx \wedge u_n$$

$$= \varphi'\, dx \wedge \int_0^? u_n\, dx \;+ \varphi u.$$

In particular, $u - dw \in \dot{C}^\infty(X; \Lambda^k)$ as desired.

Thus we can define

$$r : H^k_{H_0, X}(X) \longrightarrow H^k_{dR, R}(X) \qquad k \geq \tfrac{n}{2} + \tfrac{1}{2}$$

$u \mapsto [u - w]$. Again it is an isomorphism since dw projects into $d\mathcal{D}_A$.

Finally then the 'middle' case. Here, what is to say $k=\frac{n}{2}$ or $k=\frac{n}{2}+\frac{1}{2}$ we cannot forget smooth forms into $H^k_{H_0,A}(X)$, since they are not L^2. Thus we only have

$$H^k_{dR,R}(X) \longrightarrow H^k_{H_0,A}(X) \qquad k=\frac{n}{2},\frac{n+1}{2}.$$

Going the other way, we can still extract any for dX so we do get

$$H^k_{H,A}(X) \longrightarrow H^k_{dR,A}(X).$$

Now the claim is that this composite is the map ~~induced~~ induced by inclusion $i_{R,A}$. This is already the same argument. If $u\in H^k_{dR,R}$ whose to zero in $H^k_{dR,A}$, $u=d\varphi$, $\varphi\in C^\infty(X;\Lambda^{k-1})$. Now $k=\frac{n}{2}-1$ or $\frac{n}{2}-\frac{1}{2}$, so this does forget into L^2_g, so $u\mapsto 0$ in $H^k_{H_0,A}$. Thus we do get

$$\mathrm{Im}\left(i_R : H^k_{dR,R}(X)\to H^k_{dR,A}(X)\right)\hookrightarrow H^k_{H,A}.$$

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Injectivity follows from the fact that we can arrange (using our retraction) that $u \in C^\infty_c(X^\circ; \Lambda^k)$ and then $[u] = [F_t^* u]$ and $F_t^* \tilde{u} = d\varphi$ if $u \mapsto 0$.

<u>Exercise</u> Complete the last step of the proof, to show that the map $u \longmapsto u - dw$ is injective.

I have made rather heavy work of this proof, it would be better to define some intermediate objects, in particular compute the cohomology groups for forms with <u>conormal</u> coefficients and various weights.

<u>Definition</u> The signature of a/compact manifold with boundary of dimension $4k$ is the signature of the quadratic form

(sig) $$H^{2k}_{Ho,g}(X) \times H^{2k}_{Ho,g}(X) \ni (\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta$$

for any of these metrics. [This is really a theorem!]

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Proof What, you say a definition should not require a proof? This is a sign of just how disorganized these notes are! Of course I do have to show that the signature of (sig) is independent of the metric and, more seriously, that it is even defined – well that is clear since it forms are L^2. In fact the same definition works on $H^{2k} = i(H^{2k}_{dR,R}(X) \to H^{2k}_{dR,c}(X))$ since it is symmetric ($2k$ is even) and if $\alpha \in C^\infty(X;\Lambda^{2k})$ and $\beta = d\varphi$, $\varphi \in C^\infty(X;\Lambda^{2k})$ we see that

$$\alpha \wedge d\varphi = d(\alpha \wedge \varphi) - d\alpha \wedge \varphi$$

so the integral vanishes if $d\alpha = 0$.

In fact we can restate (sig) in terms of $*$

as

$$H^{2k}_{Ho,g}(X) \times H^{2k}_{Ho,g}(X) \ni (\alpha, \beta) \mapsto \int_X \langle \alpha, *\beta \rangle \, dg.$$

Since $n = 4k$, $*^2 = (-1)^{2k(4k-2k)} = \mathrm{Id}$ is an identified on $H^{2k}_{Ho,g}$ and the signature is the

pos. ev. $-$ # neg ev.

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I had planned to go through the proof of the signature formula on an oriented $4k$-dimensional manifold without boundary

$$\text{(L)} \qquad \operatorname{sgn}(X) = \int_X L$$

and then discuss its generalization to the case $\partial X \neq \emptyset$:

$$\text{(B)} \qquad \operatorname{sgn}(X) = \int_X L - \eta$$

due to Atiyah, Patodi & Singer. However, I am running short of time to go through the family extension combination behind (L). Instead I will concentrate on η as the signature defect. I want to do this by going through K-theory!

Let me go back to the the index in the general case. Recall that $A \in \Psi^m(X;E,F)$ elliptic pseudodifferential operator. Recall was elliptic if its principal symbol $\sigma_m(A) \in C^\infty(S^*X; N^m \otimes \hom(E,F))$ was invertible. Then we showed that it was Fredholm with a generalized inverse $B \in \Psi^{-m}(X;F,E)$ such that $AB - \mathrm{Id}$ and $BA - \mathrm{Id}$ are smoothing operators. Later then I showed that

$$\operatorname{ind}(A) = \dim \operatorname{null}(A) - \dim \operatorname{null}(A^*) = \operatorname{Tr}([B,A]).$$

We can deduce quite a few things from this formula.

Exer Show that if $A:[0,1]\ni t\longmapsto \Psi^m(X;E,F)$ is a continuous family of elliptic operators then it has constant index.

Suppose for the sake of definiteness that $\operatorname{ind}(A)\geq 0.$ Since this means $\dim \operatorname{null}(A)\geq \dim\operatorname{null}(A^*)$ and $\operatorname{null}(A^*)$ is a finite dimensional complement to $\operatorname{Ran}(A)$, we can find $R\in\Psi^{-\infty}(X;E,F)$, finite rank, such that $A+R$ is surjective. In fact we can do a little more.

Lemma Given a finite dimensional subspace $N\subset C^\infty(X;E)$ of dimension equal to the index of $A\in\Psi^m(X;E,F)$ elliptic, $\exists\, R\in\Psi^{-\infty}(X;E,F)$ such that

$$\operatorname{null}(A+R)=N.$$

Proof. For clarity, choose an hermitian fibre metric on E and a positive smooth density on X so that the L^2 norm is defined by

$$\int_X |u|^2\nu \qquad u\in C^\infty(X;E).$$

Then choose orthonormal basis $e_1, \dots, e_I$ of N and $e_1, \dots, e_k, f_{k+1}, \dots, f_I$ of null(A) where $k = \dim (N \cap \text{null}(A))$ (is most likely zero). Then define R to be the smoothing operator with kernel

$$R = \sum_{i > k} Ae_i \cdot (f_i - e_i)^* \nu \in C^\infty(X^2; \mathrm{Hom}(E,F) \otimes \Omega_R)$$

where ν^* is the action of ν through the inner product. It should be that $A + R$ is surjective with null space N.

Now, with N fixed look at all such perturbations

$$\mathcal{P}_{A,N} = \{ R \in \Psi^{-\infty}(X; E, F) \,;\, A + R \text{ is surjective \& has null space } N \}.$$

The lemma shows that $\mathcal{P}_A \neq \emptyset$. If $A' \in \mathcal{P}_A$ then A' has a generalized inverse B which is surjective & has range a complement to N; $B \in \Psi^{-m}(X, F, E)$.

<u>Lemma</u> For fixed $N \subset C^\infty(X; E)$ with $\dim N = \mathrm{ind}(A)$, $A \in \Psi^m(X; E, F)$ elliptic,

$$\text{(P)} \quad \mathcal{P}_{A,N} \cong \{ E \in \Psi^{-\infty}(X; F) \,;\, (\mathrm{Id} + E)^{-1} \in \mathrm{Id} + \Psi^{-\infty}(X; F) \text{ exists} \}.$$

<u>Proof</u> Choose $A' \in \mathcal{P}_{A,N}$ with generalized inverse B as described above. Then for a general element $\tilde{A} \in \mathcal{P}_{A,N}$,

$$\tilde{A} \circ B = \mathrm{Id} + \tilde{E}, \quad \tilde{E} \in \Psi^{-\infty}(X;F)$$

is invertible. Indeed, $\tilde{A} = A' + R$ for some $R \in \Psi^{-\infty}(X;F)$ so $\tilde{A}\circ B = A' \circ B + R \circ B = \mathrm{Id} + \tilde{E}$, $\tilde{E} \in \Psi^{-\infty}(X;F)$ and $\tilde{A} \circ Bu = 0$ implies $Bu \in N$, but $N \cap \mathrm{Ran}(B) = \{0\}$ so $u = 0$ and $\mathrm{Id} + \tilde{E}$ must be injective. It follows that it is invertible with inverse $\mathrm{Id} + \tilde{E}'$, $\tilde{E}' \in \Psi^{-\infty}(X;F)$

The space on the right in (P) is a group

$$G^{-\infty}(X;E) = \{\mathrm{Id} + E;\ E \in \Psi^{-\infty}(X;E) \text{ invertible},$$
$$(\mathrm{Id} + E)^{-1} = \mathrm{Id} + E',\ E' \in \Psi^{-\infty}(X;F)\}$$

<u>Proposition</u> If X is a compact (connected) manifold without boundary (actually with boundary is okay too) and E is any complex vector bundle over X then

$$G^{-\infty}(X;E) \hookrightarrow \Psi^{-\infty}(X;E)$$

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is open and dense in the C^∞ topology and is isomorphic to a fixed group, such as

$$G^{-\infty} = \Big\{ a_{ij} : \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{C},\ \sum i^p j^q |a_{ij}| < \infty$$

$$\forall\, p,q,\ (\mathrm{Id} + a)^{-1} = \mathrm{Id} + b \text{ exists} \Big\}.$$

Proof. It is convenient to use the eigenbasis for a self-adjoint elliptic (pseudo) differential operator of positive order. Say an [illegible] $|\nabla|_g^2$ for some metric g. The eigensections form a complete orthonormal basis

$$e_i \in C^\infty(X;E), \quad \text{so}$$

$$u \in C^\infty(X;E) \Longleftrightarrow u = \sum_{i=1} c_i e_i,$$

$$\sum_i |c_i| i^p < \infty \quad \forall p.$$

Then $A \in \Psi^{-\infty}(X;E)$ may be identified with an infinite matrix

$$a_{ij} = \overline{\langle e_i, A e_j \rangle}$$

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The $e_i \otimes e_j^* \cdot \nu$ form an orthonormal basis for $C^\infty(X; \hom(E) \otimes \Omega)$ and this is equivalent to the 'Fourier' expansion of the kernel, i.e.

$$A \in \Psi^{-\infty}(X;E) \iff \sum_{i,j} i^{p} j^{q} |a_{ij}| < \infty \quad \forall p, q.$$

Moreover this is an isomorphism of algebras, so the result follows.

Thus these groups are all 'the same'.

<u>Definition</u> For any compact manifold (without corners) the K-space is defined as

$$K^*(Y) = [Y \times S^1; G^{-\infty}(X;E)]$$

the homotopy classes of smooth maps into $G^{-\infty}(X;E)$.

Note that even though these groups are now clearly infinite dimensional, there is no problem understanding

... The $G^{-\infty}(X;E) \subset C^\infty(X^2; \hom(E) \otimes$

so a C^∞ map into $G^{-\infty}(X;F)$ is just a C^∞ section of

$$F \in C^\infty(Y\times S^1\times X^2;\ \hom(E)\otimes\Omega)$$

which happens always to have invertible values (when we add the identity). So we can take homotopy to mean smooth homotopy :-

$$F_0 \sim F_1 \iff \exists\, F \in C^\infty([0,1]\times Y\times S^1\times X^2;\ \hom(F)\otimes\Omega)$$

s.t. $F(0,\cdot)=F_0$, $F(1,\cdot)=F_1$ and
$\left(\mathrm{Id}+F(t,y,\theta)\right)^{-1}$ exists $\forall\ t,y,\theta$.

Going back to $A\in\Psi^m(X;E,F)$ which is elliptic, we see that

$\mathcal{P}_{A,N}$ is a principal $G^{-\infty}(X;F)$-space
(if $\mathrm{ind}(A)=\dim N\geq 0$)

__Exer__ Go through the ~~then~~ case $-\mathrm{ind}(A)\leq 0$ to show a similar $G^{-\infty}(X;E)$-space can be constructed.

XVI/8 If we pick a point, $1 \in \mathbb{S}^1$ we can consider two subgroups of $K^{*}(Y)$:

$$K^{-1}(Y) = \{[F];\ F : Y \to G^{-\infty}(X;E) \text{ is constant on } \mathbb{S}^1\}$$

$$K^{-2}(Y) = \{[F];\ F: Y \times \mathbb{S}^1 \to G^{-\infty}(X;E) \text{ and } F(y,1) = \mathrm{Id}\ \forall y \in Y\}.$$

Lemma $K^{*}(Y) = K^{-1}(Y) + K^{-2}(Y)$ is an abelian group.

Proof For a general $[F] \in K^{*}(Y)$ define

$$F_{-2}(y,\theta) = F(y,1)$$

so $[F_{-2}] \in K^{-2}(Y)$, being constant on the circle, and $F_{-1} = F F_{-2}^{-1}$ is the identity at $(y,1)$, so $[F_{-1}] \in K^{-1}(Y)$. The group law should clearly be

$$[F] + [G] = [FG]$$

using the group composition in $G^{-\infty}$.

Exa Show that $K^{-1}(Y) \cap K^{-2}(Y) = \{0\} = [\mathrm{Id}]$.

Why is the 'product' abelian? This comes from our ability to approximate by finite rank operators. This is easiest to see in the matrix $G^{-\infty}$ model.

Let Π_N be the projection onto say the span of the first N eigen sections for our posit elliptic self-adj operator, $\pi_N : C^\infty(X, E) \to C^\infty(X, E)$. Then for any compact subset $K \subset \Psi^{-\infty}(X, E)$ $\exists N$, sufficiently large, such that

$$\forall A \in K, \quad \| A - \Pi_N A \pi_N \|_{L^2} < \tfrac{1}{2}.$$

So given $F : Y \times S' \longrightarrow G^{-\infty}(X; E)$ we can choose N so large that

$$F_t = (1-t) F + t \, \Pi_N F \Pi_N$$

is a smooth map $F_\cdot : [0,1] \times Y \times S' \to G^{-\infty}(X; E)$.

Then $[F] = [F_1]$ where $\Pi_N F_1 = F_1 \Pi_N = F_1$.

XVI/10 Given two elements, $[F], [G] \in K^0(Y)$ we can choose the same N for both, so $[F] = [F_1]$, $[G] = [G_1]$.

Now consider $\pi_{2N}(\mathrm{Id} - \Pi_N)$. This is the projection onto the eigensections numbers $N+1, \dots, 2N$. This gives us an isomorphism: $S^{\#+} e_i = S e_{N+i}$, $i = 1, \dots, N$, $S^{\#+} e_j = 0$ otherwise, and $S \in \Psi^{-\infty}(X; E)$. Consider the "rotation"

$$S_\tau = \left[\begin{array}{c|c} \cos\tau & -S^- \sin\tau \\ \hline S^+ \sin\tau & \cos\tau \end{array}\right]$$

where the "blocks" are spanned by $\{e_1, \dots, e_N\}\{e_{N+1}, \dots, e_{2N}\}$ and $S^- e_{N+i} = e_i$, $i = 1 \dots N$ and zero otherwise.

$$\tilde{G}_\tau = S_{-\tau} G_1 S_\tau$$

defines a homotopy from G_1, at $\tau = 0$, to $\tilde{G}$ at $\tau = \pi/2$ where $\tilde{G}$ is just G acting on $e_{N+1}, \dots, e_{2N}$.

Then $[G_1] = [\tilde{G}]$ but $F_1 \tilde{G} = \tilde{G} F_1$.

Now, suppose $A_y \in \Psi^m(X;E,F)$ is a smooth *family* of elliptic operators, i.e. is a map $Y \longrightarrow \Psi^m(X;E,F)$ for some compact manifold Y. The *families index problem* is thus: Is it possible to find a smooth map

$$\text{(FI)} \quad \begin{cases} R : Y \longrightarrow \Psi^{-\infty}(X;E,F) \ \text{s.t.} \\ A_y + R_y \ \text{is invertible} \ \forall y \ ? \end{cases}$$

We shall see, I hope, that there is a well-defined element

$$\operatorname{Ind}(A_\cdot) \in K^{-1}(Y) \ \text{s.t.}$$

$$\operatorname{Ind}(A_\cdot) = 0 \iff R \text{ in (FI) exists.}$$

How to construct and then compute $\operatorname{Ind}(A_\cdot)$?

We can ask for a slightly weaker condition than (FI). Namely, if $N \subset C^\infty(X;E)$ is *fixed* we can ask for

$$\text{(RFI)} \quad \begin{cases} R : Y \longrightarrow \Psi^{-\infty}(X;E,F) \ \text{s.t.} \\ A_y + R_y \ \text{is surjective onto} \ \text{[illegible]} \ \text{space } N. \end{cases}$$

17. Lecture XVII, 4 November, 2003

Next lecture will be Thursday November 13 (and it will be a short one!)

Last time I constructed a principal bundle associated to any family $A \in \mathcal{C}^\infty(Y; \Psi^k(X; E, F))$ of elliptic pseudodifferential operators on a compact manifold without boundary, X, ofr a compact parameter space Y. The *structure group* of this bundle is $G^{-\infty}(X; E)$ or $G^{-\infty}(X; F)$ as the numerical index of the family if negative or positive. Assuming for the same of definiteness that $\# - \operatorname{ind}(A) \geq 0$, the bundle

$$\begin{tikzcd} G^{-\infty}(X; F) \arrow[r] & \mathcal{P}_{A,N} \arrow[d] \\ & Y \end{tikzcd} \tag{1}$$

has fibre at $y \in Y$

$$\mathcal{P}_{A,N,y} = \left\{B \in \Psi^{-\infty}(X; E, F); A_y + B \text{ has null space exactly } N\right\} \tag{2}$$

where $N \subset \mathcal{C}^\infty(X; E)$ is fixed but is chosen arbitrarily with dimension equal to $\# - \operatorname{ind}(A.)$.

Essentially by definition, this bundle is trivial if and only if there exists a smooth map $E \in \mathcal{C}^\infty(Y; \Psi^{-\infty}(X; E, F))$ such that $A_y + E_y$ is surjective for all $y \in Y$ and has null space N.

Exercise 29. Check this carefully, starting from the definition of triviality of a principal bundle.

Thus, the triviality of the principal bundle, together with the vanishing of the numerical index is *precisely* the obstruction to 'perturbative invertibility'.

Also recall that I *defined*

$$K^{-1}(Y) = [Y; G^{-\infty}] \tag{3}$$

$$K^{-2}(Y) = [Y \times \mathbb{S}, Y \times \{1\}; G^{-\infty}, \mathrm{Id}] \tag{4}$$

i.e. $K^{-1}(Y)$ is the set of homotopy classes of smooth maps into $G^{-\infty}$ (for any model) and $K^{-2}(Y)$ is similarly the set of homotopy classes of smooth maps from $Y \times \mathbb{S}$ into $G^{-\infty}$ taking $Y \times \{1\}$ to Id. We can also thing of (4) as

$$K^{-2}(Y) = [Y; \mathcal{L}G^{-\infty}, \mathrm{Id}] \tag{5}$$

where $\mathcal{L}G^{-\infty}$ is the loop group:

$$\mathcal{L}G^{-\infty} = \left\{F : \mathbb{S} \longrightarrow G^{-\infty}; F(1) = \mathrm{Id}\right\}. \tag{6}$$

The definitions (3) and (4) depend on the fact, which is the essential nature of *Bott Periodicity* that

$$\Pi_j(G^{-\infty}) = \begin{cases} \{0\} & j \text{ even} \\ \mathbb{Z} & j \text{ odd.} \end{cases} \tag{7}$$

Exercise 30. Assuming (7) show that

$$\Pi_j(\mathcal{L}G^{-\infty}) = \begin{cases} \mathbb{Z} & j \text{ even} \\ \{0\} & j \text{ odd} \end{cases} \tag{8}$$

where the higher homotopy groups can be considered as maps into the component of the identity.

Exercise 31. What are $K^{-1}(\mathbb{S}^n)$ and $K^{-2}(\mathbb{S}^n)$?

I will prove the first part of (7). The first claim is

Lemma 7. *$G^{-\infty}$ is connected.*

Proof. This is a direct consequence of the fact that for any smoothing operators $A \in \Psi^{-\infty}(X;E)$ the family $\operatorname{Id} + zA$ is invertible for $A \in \mathbb{C} \setminus D$ where $D \subset \mathbb{C}$ is discrete (i.e. countable and without points of accumulation). We can either use the Fredholm determinant to prove this or proceed directly. Fix a value $\bar{z}$ of z. If $\operatorname{Id} + zA$ is invertible then we know it has a bounded inverse as an operator on $L^2(X;E)$ and by the openness of the set of invertible operators (i.e. convergence of the Neumann series) it remains invertible for $|z-\bar{z}|\|A\|\|(\operatorname{Id} - \bar{z}A)^{-1}\| < 1$. Thus D is closed. If $\operatorname{Id} + \bar{z}A$ is not invertible then we use finite rank approximation to write $A = A_1 + A_2$ where A_1 is a finite rank smoothing operator and A_2 has small norm, for instance $|\bar{z}|\|A_2\|_{L^2} < \frac{1}{2}$. Then $\operatorname{Id} + zA_2$ has inverse $\operatorname{Id} + B(z)$ for $|z - \bar{z}| < \epsilon$ with $B(z)$ holomorphic with values in the smoothing operators and we are reduced to considering

$$(\operatorname{Id} + B(z))(\operatorname{Id} + zA) = \operatorname{Id} + A'(z),\ \ A'(z) = (\operatorname{Id} + B(z))zA_1$$

Thus $A'(z)$ has finite rank, at most the rank of A_1, and is holomorphic near z so $\operatorname{Id}_N + A'(z)$ is invertible for $0 < |z - \bar{z}| < \epsilon'$ for some $\epsilon > 0$ and D is discrete.

Thus if $\operatorname{Id} + A$ is invertible then it can be connected to the identity by an inverible family $\operatorname{Id} + z(s)A$. $\square$

Exercise 32. Use such a finite rank approximation to define the Fredholm determinant $\det(\operatorname{Id} + A)$ as an entire function of A, extending the usual definition for finite rank operators, such that $(\operatorname{Id} + A)^{-1}$ exists if and only if $\det(\operatorname{Id} + A) \neq 0$ with the usual multiplicative and differential properties

$$\begin{gathered}\det((\operatorname{Id} + A)(\operatorname{Id} + B)) = \det(\operatorname{Id} + A)\det(\operatorname{Id} + B),\\ \frac{d}{dz}\det(\operatorname{Id} + zA) = \det(\operatorname{Id} + zA)\operatorname{Tr}(\operatorname{Id} + zA)^{-1}A) \text{ where } \det(\operatorname{Id} + zA) \neq 0.\end{gathered} \tag{9}$$

The second, and more substantial, part of (7) is

Proposition 11. *If $F : \mathbb{S} \longrightarrow G^{-\infty}(X;E)$ is a smooth loop then*

$$w(F) = \frac{1}{2\pi i}\int_{\mathbb{S}} \operatorname{Tr}\left(F(\theta)^{-1}\frac{dF(\theta)}{d\theta}\right) d\theta \in \mathbb{Z} \tag{10}$$

there exists a smooth map

$$\tilde{F} : [0,1] \times \mathbb{S} \longrightarrow G^{-\infty}(X;E) \text{ with } \tilde{F}(0) = F \text{ and } \tilde{F}(1) = (\operatorname{Id} - \pi) + z^{w(F)}\pi \tag{11}$$

where π is a projection of rank one.

We start off with a simple case, where the family is actually affine.

Lemma 8. *If A, B are $N \times N$ complex matrices and $A + zB$ is invertible on $|z| = 1$ then, for $|z| = 1$, it is homotopic to $(\operatorname{Id} - \pi) + z\pi$ where π is a projection of rank $w(A + zB)$.*

Proof. If A is not invertible, we may perturb the family slightly and so deform it to $(A + t\operatorname{Id}) + zB$ where the constant term is invertible. Then, using the connectedness of $\mathrm{GL}(N)$, which follows from the proof above, we may deform away the constant term and replace the family by $\operatorname{Id} + zB'$, $B' = (A + t\operatorname{Id})^{-1}B$. On the circle $z = e^{i\theta}$,

$dz = ie^{i\theta}d\theta$ so for a family which is holomorphic near the circle the integral in (10) can be written as a contour integral

$$w(F) = \frac{1}{2\pi i}\int_{|z|=1} \operatorname{Tr}\left(F(\theta)^{-1}\frac{dF(z)}{dz}\right)dz \tag{12}$$

In this case $dF/dz = B$ and the integral, without the trace, becomes

$$M = \frac{1}{2\pi i}\int_{|z|=1}(\operatorname{Id}+zB')^{-1}B'dz \tag{13}$$

which is in fact the projection onto the span of the generalized eigenspaces of B for outside the unit circle (and with null space the span of those inside). We don't need all of this information but we do need to see that M is a projection (or perhaps better to say an idempotent, $M^2 = M$). Indeed the square can be written as the double integral

$$M^2 = \frac{1}{(2\pi i)^2}\int_{|z|=1}\int_{|z'|=1+\epsilon}(\operatorname{Id}+zB')^{-1}(\operatorname{Id}+z'B')^{-1}(B')^2dzdz' \tag{14}$$

for $\epsilon > 0$ small (using Cauchy's theorem). Now the resolvent identity can be written

$$(\operatorname{Id}+zB')^{-1}(\operatorname{Id}+z'B')^{-1}B' = (z'-z)^{-1}\left((\operatorname{Id}+zB)^{-1} - (\operatorname{Id}+z'B')\right),\ z \neq z'. \tag{15}$$

Inserting this into (14), one of the integrals can be carried out for each term. Indeed the second is holomorphic in $|z| \le 1$ so integrates to zero, while for the first has a simple pole in z' at $z' = z$ and so the z' integral may be replaced by the residue which is just M.

Furthermore M and B' commute, since B' commute with $(\operatorname{Id}+zB')^{-1}$ and

$$(\operatorname{Id}+zB')^{-1}M \text{ is holomorphic in } |z| \ge 1,\ (\operatorname{Id}+zB')^{-1}(\operatorname{Id}-M) \text{ is holomorphic in } |z| \le 1. \tag{16}$$

This involves an argument similar to that above, to prove the first write

$$\begin{aligned}(\operatorname{Id}+zB')^{-1}M &= \frac{1}{2\pi i}\int_{|s|=1}(\operatorname{Id}+zB')^{-1}(\operatorname{Id}+sB')^{-1}B'ds\\ &= \frac{1}{2\pi i}\int_{|s|=1}(s-z)^{-1}\left((\operatorname{Id}+zB')^{-1} - (\operatorname{Id}+sB')^{-1}\right)ds.\end{aligned}$$

Here the first term vanishes (for $|z| > 1$), by Cauchy's theorem, and the second is holomorphic. The other case is similar.

Finally, we conclude that under the deformation $B'_t = t(\operatorname{Id}-M)B' + M(tB' + 2(1-t))$ $(\operatorname{Id}+zB'_t)^{-1}$ remains holomorphic near $|z| = 1$ and results in a family as desired. $\square$

18. Lecture XVIII, 13 November, 2003

Handwritten notes: Pages 1-11

19. Lecture XIX, 18 November, 2003

Handwritten notes: Pages 1-10

References

[1] J. Brüning and R. Seeley, *The expansion of the resolvent near a singular stratum of conical type*, J. Funct. Anal. **95** (1991), no. 2, 255–290. MR **93g:**58146

[2] Jeff Cheeger, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), no. 5, 2103–2106. MR **80k:**58098

[3] ———, *On the Hodge theory of Riemannian pseudomanifolds*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Amer. Math. Soc., Providence, R.I., 1980, pp. 91–146. MR **83a:**58081

[4] ———, *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** (1983), no. 4, 575–657 (1984). MR **85d:**58083

[5] Jeff Cheeger, Mark Goresky, and Robert MacPherson, *L^2-cohomology and intersection homology of singular algebraic varieties*, Seminar on Differential Geometry, Princeton Univ. Press, Princeton, N.J., 1982, pp. 303–340. MR **84f:**58005

[6] M. Goresky, G. Harder, and R. MacPherson, *Weighted cohomology*, Invent. Math. **116** (1994), no. 1-3, 139–213. MR **95c:**11068

[7] Mark Goresky and Robert MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162. MR **82b:**57010

[8] ———, *Intersection homology. II*, Invent. Math. **72** (1983), no. 1, 77–129. MR **84i:**57012

[9] Edith A. Mooers, *Heat kernel asymptotics on manifolds with conic singularities*, J. Anal. Math. **78** (1999), 1–36. MR **2000g:**58039

[10] R. Seeley, *Conic degeneration of the Dirac operator*, Colloq. Math. **60/61** (1990), no. 2, 649–658. MR **92b:**58222

[11] Robert Seeley, *Conic degeneration of the Gauss-Bonnet operator*, J. Anal. Math. **59** (1992), 205–215, Festschrift on the occasion of the 70th birthday of Shmuel Agmon. MR **94e:**58135

[12] ———, *The resolvent expansion for the signature operator on a manifold with a conic singular stratum*, Journées "Équations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1996), École Polytech., Palaiseau, 1996, pp. Exp. No. XIX, 8. MR **97j:**58149

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu

Eta invariant : Lecture 18

Last time I discussed the principal-bundle approach to the families index. Today I want to continue the discussion of the old families index. Recall that I stated (but did not really prove) that

$$G^{-\infty}(X;E) = \{A \in \Psi^{-\infty}(X;E);\ (Id+A)^{-1} = Id + B,\ B \in \Psi^{-\infty}(X;E)\}$$

is a classifying space for odd K-theory. For the moment I want to concentrate on $\pi_1(G^{-\infty})$, or really $H^1(G^{-\infty})$. The latter is generated by the Fredholm determinant

$$\det : Id + \Psi^{-\infty}(X,E) \longrightarrow \mathbb{C}$$

$$G^{-\infty}(X;E) = \{A \in \Psi^{-\infty}(X;E);\ \det(Id+A) \neq 0\}$$

(1)
$$c_1 = \frac{1}{2\pi i}\, d \log \det$$
$$= \frac{1}{2\pi i} \operatorname{Tr}((Id+A)^{-1} dA).$$

Thus $[c_1]$ spans $H^1(G^{-\infty})$.

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The corresponding loop group

$$\mathcal{L}G^{-\infty} = \{F : \mathbb{S}^1 \longrightarrow G^{-\infty};\ F(1) = \mathbb{1}\}$$

is a classifying group for even K-theory. Let me "fatten" $\mathcal{L}G^{-\infty}$ a little by looking at

$$\Psi_s^{-\infty}(X;E) = \mathcal{S}(\mathbb{R};\Psi^{-\infty}(X;E))$$
$$\equiv \mathcal{S}(\mathbb{R}\times X^2;\ \mathrm{Hom}(E)\otimes\Omega_R)$$

and the corresponding group

$$G_s^{-\infty}(X;E) = \{A \in \Psi_s^{-\infty}(X;E);\ (\mathrm{Id}+A)^{-1} = \mathrm{Id}+B,\ B \in \mathcal{S}(\mathbb{R};\Psi^{-\infty})\}.$$

Lemma $G_s^{-\infty}(X;E)$ is also a classifying group for even K-theory.

Proof Taking the 1-point compactification $\mathbb{R}\to\mathbb{S}$, $\infty \longmapsto 1$ allows us to identify

$$\Psi_s^{-\infty}(X;E) \cong \{A \in C^\infty(\mathbb{S};\Psi^{-\infty}(X;E));\ A \equiv 0 \text{ at } 1\}$$

Then it is relatively easy to see that $G_s^{-\infty}(X;E) \to \mathcal{L}G^{-\infty}$

as a deformation retract.

Exercise Write this out a bit more carefully!

Now suppose we have a family of self-adjoint elliptic pseudodifferential operators $P_y \in C^\infty(Y; \Psi^1(X;E))$ $P_y^* = P_y$ for some inner product on E and volume form.

<u>Lemma</u> If $P \in \Psi^1(X;E)$ is elliptic and self-adjoint there exists $a \in \Psi_s^{-\infty}(X;E)$ s.t.

$$P + i\tau + a(\tau) \in \Psi^1(X;E) \text{ is}$$

invertible $\forall \tau \in \mathbb{R}$.

<u>Proof</u> Since P is elliptic and self-adjoint and has discrete spectrum $\mathrm{sp}(P) \subset \mathbb{R}$. It follows that $P + i\tau$ is invertible for all $\tau \in \mathbb{R} \setminus \{0\}$ and the only problem is the finite dimensional null space. Let $\Pi_0 \in \Psi^{-\infty}(X;E)$ be projection onto this null space then we choose $\varphi \in C_c^\infty(\mathbb{R})$ s.t.

$$(P + i\tau + \varphi(\tau)\Pi_0)^{-1} \in \Psi^{-1}(X;E) \ \forall \tau \in \mathbb{R}.$$

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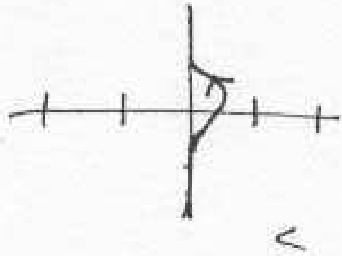
where φ is a curve like this

$i\tau + \varphi(\tau)$:



<u>Exercise</u> Find an explicit φ.

Now we can consider the principal bundle

$$\mathcal{P}_y = \{A \in \Psi_s^{-\infty}(X;E);\ P_y + i\tau + A(\tau)$$
$$\text{is invertible in } \Psi^{-1}(X;E)\ \forall \tau \in \mathbb{R}\}$$

This is a non-trivial principal bundle with structure group $G_s^{-\infty}(X;E)$.

$$(P_y + i\tau + A_1(\tau))^{-1}(P_y + i\tau + A_2(\tau))$$
$$= \mathrm{Id} + B(\tau),\quad B \in \Psi_s^{-\infty}(X;E)$$

if $A_1, A_2 \in \mathcal{P}_y$. Since this ~~classifying~~ structure group is a classifying group for even K-theory we "know" that the obstruction to the triviality of

$$\begin{array}{ccc} G_0^{-\infty}(X;E) & - & P \\ & & \downarrow \\ (PB) & & Y \end{array}$$

is a class $\operatorname{Ind}_0(P) \in K^{-1}(Y)$. To construct the class directly we need a short exact sequence

(M) $\qquad G_S^{-\infty}(X;E) \longrightarrow G \longrightarrow H_0$

where H_0 classifies odd K-theory of G is (weakly) contractible. This always exists, but I will do it explicitly later.

For the moment think about the 1st odd Chern class. We have already seen that $c_1 \in H^1(H_0)$ exists, corresponding to the Fredholm determinant. Then if $\operatorname{Ind}_0(P) \in [Y; H_0]$ can be constructed corresponding to the non-triviality of (PB) we should be able to "find" $\operatorname{Ind}_0^*(P)^* c_1 \in H^1(Y)$ more directly. That is, we should be able to construct

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a 'determinant function' $\tau: Y \to \mathbb{C}^*$ s.t.

$$\mathrm{Ind}_0(P)^* c_1 = \left[\frac{1}{2\pi i}\, d\log\tau\right].$$

Where will this come from? Look at the (unknown to us at the moment) sequence (M). On the left we have classes in all even dimensions, on the right in all odd dimensions and in the middle, nothing. The bottom class, measuring the components, is the winding number (of index)

$$w(F) = \frac{1}{2\pi i} \int_{S} \frac{d}{d\theta} \log\det(F(\theta))\, d\theta$$

$F: S^1 \to G^{-\infty}_1$, Id or $F = \mathrm{Id} + A(t)$, $A \in \Psi^{-\infty}_s(X;E)$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d}{dt} \log\det(\mathrm{Id} + A(t))\, dt$$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{Tr}\left((\mathrm{Id} + A(t))^{-1} \frac{dA}{dt}\right) dt.$$

<u>Lemma</u> The winding number is a group homomorphism

$$w : G_s^{-\infty}(X;E) \twoheadrightarrow \mathbb{Z}.$$

<u>Proof</u> This is the multiplicativity of the determinant:

$$\log\det(F(\tau)G(\tau)) = \log\det F(\tau) + \log\det G(\tau)$$

$$\Rightarrow \quad w(FG) = w(F) + w(G)$$

That w takes values in $\mathbb{Z}$ follows from the properties of the logarithm; that it is surjective requires [illegible] to find a case with $w(F)=1$; e.g. a rotation in 1 variable. $\mathbb{C}$

So, now to the construction. We want to find

$$\xi : \mathcal{P} \longrightarrow \mathbb{C} \quad \text{s.t.}$$

$$\xi(FP_s) = \xi(P_s) + w(F),$$
$$\forall F \in G_s^{-\infty}(X;E).$$

I will use the following properties of

$$P : \mathbb{R}_\tau \to \Psi^h(X;E) \quad C^\infty$$

i) $P^{-1} : \mathbb{R}_\tau \to \Psi^{-l}(X;E)$ exists,

ii) $\dfrac{d^l}{d\tau^l} P ; \mathbb{R} \to \Psi^{h-l},$

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For any continuous seminorm $\| \ \|_{(k)}$ on $\Psi^{l}(X;E)$

$$(\mathrm{SE}) \qquad \left\| (1+|\tau|)^{j} \frac{d^{l}}{d\tau^{l}} P(\tau) \right\|_{(k,\cancel{\ }\,l+r)} \text{ is bdd}$$

together with the existence of ~~certain~~ provided expansions.
Then we just try to *regularize* the formula for

$w(F)$:

$$w(F) = \frac{1}{2\pi i} \int_{\mathbb{R}} \operatorname{Tr}\left(F^{-1}(\tau) \frac{dF(\tau)}{d\tau} \right) d\tau .$$

The basic problem is that

$$\operatorname{Tr} : \Psi^{-n-1}(X;E) \longrightarrow \mathbb{C}$$

does *not* extend to Ψ^{n} to vanish on commutators.

Instead we regularize in τ, using (SE).

The estimates (SE) mean that ~~for N sufficiently large,~~

$$\left(\frac{d}{d\tau}\right)^{N} \left(F^{-1}(\tau) \frac{dF(\tau)}{d\tau} \right) \in C^{\infty}(\mathbb{R}; \Psi^{-N-1}(X;E))$$

Thus, for $N > \dim X$ we can take the trace and consider

$$\varphi_N(\tau) = \mathrm{Tr}\left(\left(\frac{d}{d\tau}\right)^N F^{-1}(\tau)\frac{dF}{d\tau}(\tau)\right) \qquad N > \dim X$$

Thus $\frac{d}{d\tau}\varphi_N(\tau) = \varphi_{N+1}(\tau)$. Now, if we integrate from the origin N times

$$\psi_N(\tau) = \int_0^{\tau_N} \cdots \int_0^{\tau_1} \varphi_N(\tau)\, d\tau$$

we get a smooth function which still depends on N but clearly

$$\psi_{N'}(\tau) = \psi_N(\tau) + p_{N'N}(\tau),$$

$p_{N'N}(\tau)$ a polynomial.

<u>Claim</u>: For $F(\tau) = A + i\tau + a(\tau) \in \mathcal{P}$, $A \in \Psi^1(X;E)$, elliptic and self-adjoint,

$$\psi_N(\tau) \sim \sum_{j=0}^{\infty} \tau^{L-j}\, a_j^{\pm N} + \log\tau \cdot p_N^{\pm}(\tau)$$

as $\tau \to \pm\infty$ has a <u>complete</u> <u>asymptotic</u> expansion.

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Definition Subject to the validity of this claim we define

$$\xi(F(\cdot)) = \frac{1}{2\pi i} \times \text{coeff of } T^0 \text{ in expansion of}$$

$$\int_{-T}^{T} \varphi_N(\tau_N)\, d\tau_N \text{ as } T \to$$

NB. This does not depend on N, since replacing φ_N by $\varphi_{N'}$ changes the integral by a polynomial

$$\int_{-T}^{T} p(t)\, dt = T\,Q(T)$$

has no constant term as $T \to \infty$. This comes from the 'choice' of symmetric $[-T, T]$ of the interval of integration.

Lemma: ξ has the property

$$\xi(A(z)F(z)) = \omega(A(z)) + \xi(F(z))$$

if $A \in G_s^{-\infty}(X; E)$.

Proof. Expanding out

$$\left(\frac{d}{d\tau}\right)^N \left(F^{-1}(\tau) A^{-1}(\tau) \frac{d}{d\tau}(A(\tau) F(\tau))\right)$$

$$= \left(\frac{d}{d\tau}\right)^N \left(F^{-1}(\tau) \frac{dF}{d\tau}\right) + \left(\frac{d}{d\tau}\right)^N \left(F^{-1}(\tau) A^{-1}(\tau) \frac{dA}{d\tau} F(\tau)\right)$$

observe that the second term is decaying in $\psi_0^{-*}(X; E)$. Thus

$$\xi(A(\tau) F(\tau)) - \xi(F(\tau))$$

$$= \frac{1}{2\pi i} \text{ coeff of } T^0 \text{ in } \int_{-T}^{T} \int_0^{\tau_N} \cdots \int_0^{\tau_1} \left(\frac{d}{d\tau}\right)^N \left(\mathrm{Tr}\left(F^{-1} A^{-1} \frac{dA}{d\tau} F\right)\right) d\tau$$

$$\shortparallel$$

$$\mathrm{Tr}\left(A^{-1} \frac{dA}{d\tau}\right)$$

$$= \frac{1}{2\pi i} \times \text{ coeff of } T^0 \text{ in } \int_{-T}^{T} \mathrm{Tr}\left(A^{-1} \frac{dA}{d\tau}\right) d\tau$$

$$= w(A).$$

since the commutation was just fact for operators of order $-\infty$.

Exercise Show that if F and G are two 'admissible'

elliptic families then $\xi(FG) = \xi(F) + \xi(G)$.

Claim $d\xi = \pi^* \chi$ for $\chi \in C^\infty(Y; \Lambda^1)$, $d\chi = 0$, and

$[\chi] = [c_1] \equiv \mathrm{Ind}_a(P)^* c_1$ in $H^1(Y)$.