CHAPTER 9

Homotopy invariance of the index

Lecture 9: 18 October, 2005

Let me first improve a little on the parametrix constructed in the case of an elliptic pseudodifferential operator.

Proposition 18. *If $A \in \Psi^m(X; E, F)$ is elliptic then there exists $B \in \Psi^{-m}(X; F, E)$ such that*

$$BA = \mathrm{Id}_E - \pi,\ AB = \mathrm{Id}_F - \pi' \tag{L9.1}$$

where $\pi \in \Psi^{-\infty}(X; E)$ is projection onto the null space of A and π' is projection onto the null space of B which is a complement to the range of A. Choosing inner products and smooth densities one can further arrange that $\pi^* = \pi$ and $(\pi')^* = \pi'$.

Proof. We know already, as a consequence of the assumption of ellipticity, that there exists a parametrix $B_0 \in \Psi^{-m}(X; F, E)$ such that $B_0 A = \mathrm{Id} - R$, $AB_0 = \mathrm{Id} - R'$ with R and R' smoothing operators on the appropriate bundles, E and F. Since the finite rank smoothing operators are dense in the smoothing operators, we can find a finite rank operator R_F such that $\tilde{R} = R - R_F$ has L^2 norm less than one. Thus $(\mathrm{Id} - \tilde{R})^{-1}$ exists as a bounded operator on $L^2(X; E)$ and is of the form $\mathrm{Id} - \tilde{S}$ with $\tilde{S} \in \Psi^{-\infty}(X; E)$. Composing on the right with this operator,

$$B'A = (\mathrm{Id} - \tilde{S})(\mathrm{Id} - R) = (\mathrm{Id} - \tilde{R})^{-1}(\mathrm{Id} - \tilde{R} - R_F) = \mathrm{Id} - (\mathrm{Id} - \tilde{S})R_F = \mathrm{Id} - S_F,\ B' = [(\mathrm{Id} - \tilde{S})B_0], \tag{L9.2}$$

where $S_F \in \Psi^{-\infty}(X; E)$ also has finite rank. On the null space of S_F, which has finite codimension, A is injective, since B' inverts it. It also follows from (L9.2) that the null space of A is contained in the null space of $\mathrm{Id} - S_F$, which is finite dimensional. Thus we may choose a finite dimensional subspace $U \subset \mathcal{C}^\infty(X : E)$ which complements $\mathrm{null}(S_F) + (A)$ in $\mathcal{C}^\infty(X; E)$. Setting $D = \mathrm{null}(S_F) + U$ it follows that

$$\mathcal{C}^\infty(X; E) = D + \mathrm{null}(A) \tag{L9.3}$$

and that $A : D \longrightarrow A(D) = A(\mathrm{null}(S_F) + A(U) \subset \mathcal{C}^\infty(X; F)$ is injective. Let $V \subset \mathcal{C}^\infty(X; F)$ be a complement to $A(D)$; thus V is finite-diemnsional and in terms of this, and the decomposition (L9.3),

$$A = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \tag{L9.4}$$

Then we may simply define B to be the inverse of A on $A(D)$ and to be zero on V. Note that B differs from B', which inverts A on $A(\mathrm{null}(S_F))$ by a finite rank

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smoothing operator and $BA = \mathrm{Id} - \pi$ where π is the projection onto $\mathrm{null}(A)$ which vanishes on D and that $AB = \mathrm{Id} - \pi'$ where π' is the identity on V and vanishes on $A(D)$. Thus we have arrived at (L9.1).

If we give E and F Hermitian inner products and choose a positive smooth density on X then we may consider the effect of replacing D in the discussion above by $\mathrm{null}(A)^\perp$. Certainly $\mathrm{null}(A)^\perp \cap D$ has finite codimension in $\mathrm{null}(A)^\perp$ and the same codimension in D. We may replace D by $\mathrm{null}(A)^\perp$ in the discussion above and choose V to be $A(\mathrm{null}(A)^\perp)^\perp$. This ensures that $\pi^* = \pi$ and $(\pi')^* = \pi'$. □

Now observe that for a finite rank, smoothing, projection $\mathrm{Tr}(\pi)$ is equal to its rank. Thus, with B the 'generalized inverse' of Proposition 18 we find that

$$\mathrm{ind}(A) = \mathrm{Tr}(\pi) - \mathrm{Tr}(\pi') = \mathrm{Tr}(\mathrm{Id}_E - BA) - \mathrm{Tr}(\mathrm{Id}_F - AB). \tag{L9.5}$$

Proposition 19. *For any parametrix, $B \in \Psi^{-m}(X; F, E)$ of an elliptic element $A \in \Psi^m(X; E, F)$*

$$\mathrm{ind}(A) = \mathrm{Tr}(\mathrm{Id}_E - BA) - \mathrm{Tr}(\mathrm{Id}_F - AB). \tag{L9.6}$$

Proof. Denote the generalized inverse of Proposition 18, for which (L9.5) holds, as $\tilde{B}$. Then for a parameterix as in the statement, $C = B - \tilde{B} \in \Psi^{-\infty}(X; F, E)$ and $B_t = \tilde{B} + tC$ is a smooth family of parametrices for $t \in [0, 1]$ with $B_0 = \tilde{B}$ and $B_1 = B$. Thus it suffices to show that the right side in (L9.6) is constant in t. Since

$$\frac{d}{dt}\left(\mathrm{Tr}(\mathrm{Id}_E - B_t A) - \mathrm{Tr}(\mathrm{Id}_F - AB_t)\right) = \mathrm{Tr}(AC) - \mathrm{Tr}(CA) = 0 \tag{L9.7}$$

since

Lemma 18. *For any $C \in \Psi^{-\infty}(X; F, E)$ and $A \in \Psi^m(X; E, F)$*

$$\mathrm{Tr}(AC) = \mathrm{Tr}(CA). \tag{L9.8}$$

Proof. If $C_i \longrightarrow C$ in $\Psi^{-\infty}(X; F, E)$ then $AC_i \longleftrightarrow AC$ and $C_i A \longrightarrow CA$ in $\Psi^{-\infty}(X; F)$ and $\Psi^{-\infty}(X; E)$ respectively. Since we may choose the C_i to be of finite rank, it suffices to prove (L9.8) for finite rank smoothing operators. Since the identity is linear in C it is enough to consider the case where C has rank 1, $Cf = v(f)w$ where $v(f) = \int_X v \cdot f$ for some $v \in \mathcal{C}^\infty(X; F' \otimes \Omega_X)$ and $w \in \mathcal{C}^\infty(X; E)$ is fixed. Then AC and CA are also of rank 1 (or 0)

$$AC(f) = v(f)Aw,\ CA(g) = v(Ag)w \tag{L9.9}$$

and

$$\mathrm{Tr}(AC) = \int_X v \cdot Aw = \mathrm{Tr}(CA). \tag{L9.10}$$

□

□

From this we deduce that

Proposition 20. *The index is a (smooth) homotopy invariant of elliptic operators.*

PROOF. Consider a smooth family of elliptic operators $A_t \in \mathcal{C}^\infty([0,1]; \Psi^m(X; E, F))$, (the argument works equally well if we just assume continuity in t). Then, as shown above, we may construct a smooth family of parametrices $B_t \in \mathcal{C}^\infty([0,1]; \Psi^{-m}(X; F, E)$. Thus $B_t A_t - \mathrm{Id}_E$ and $A_t B_t - \mathrm{Id}_F$ are both smooth families of smoothing operators. It follows from (L9.6) that the index itself depends smoothly on $t \in [0,1]$. However it takes integer values and so is constant. $\square$

Since the index is homotopy invariant we can change the lower order terms freely and leave the index unchanged. Thus $\mathrm{ind}(A)$ actually only depends on $\sigma(A) \in \mathcal{C}^\infty(S^*X; \hom(E,F))$ since it two operators A, A' have the same symbol then $(1-t)A + tA'$ has constant symbol and hence remains elliptic, so the $\mathrm{ind}(A) = \mathrm{ind}(A')$.

In fact even some of the information in the symbol is irrelevant for the index and to state the index theorem we eliminate this extraneous data by passing to a topological object.

PROPOSITION 21. *For any compact manifold Y (of positive dimension) and any bundle G over Y*

$$K^{-1}(X) = [X; G^{-\infty}(Y; G)] \tag{L9.11}$$

the set of smooth homotopy classes of smooth maps, is an Abelian group naturally independent of the choice of Y and G.

PROOF. We know that we may deform a smooth map $F : X \longleftrightarrow G^{-\infty}(Y; G)$ to be of the form $\mathrm{Id} - \tilde{F}$ with $\tilde{F}$ of uniformly finite rank, i.e. acting on a fixed finite-dimensional subspace of $\mathcal{C}^\infty(Y; G)$. Choosing a basis of this space, this reduces the map to $\tilde{F} : X \longrightarrow M(N, \mathbb{C})$, $\mathrm{Id} - \tilde{F} \in \mathcal{C}^\infty(X; \mathrm{GL}(N, \mathbb{C})$.

Consider especially the case $Y = \mathbb{S}$, $G = \mathbb{C}$. Then we may identify $M(N, \mathbb{C})$, the algebra of $N \times N$ matrices, with the operators on finite Fourier series

$$\begin{gathered} M(N, \mathbb{C}) \ni \{a_{jk}\}_1^N \longmapsto a(\theta, \theta') = \frac{1}{2\pi} \sum_{j,k=1}^{N} a_{jk} e^{ij\theta} e^{-ik\theta'}, \\ a\left(\sum_{p=1}^{N} u_p e^{ip\theta}\right) = \sum_k \left(\sum_l a_{kl} u_l\right) e^{ik\theta}. \end{gathered} \tag{L9.12}$$

Combined with the discussion above, this allows us to deform F to the finite rank perturbation $\tilde{F}$ and then embed into $G^{-\infty}(\mathbb{S})$:

$$[X; G^{-\infty}(Y, G)] \longmapsto [X; G^{-\infty}(\mathbb{S})]. \tag{L9.13}$$

Note that the homotopy class of the image is independent of the basis chosen, since $\mathrm{GL}(N, \mathbb{C})$ is connected. Similarly, it does not depend on N, increasing it results in a homotopic map.

This construction is reversible, so proving the first part of the proposition.

So, this is just a consequence of the possibility of finite rank approximation. In standard topological approaches $K^{-1}(X)$ is defined simiply by the stabilization of maps in $\mathrm{GL}(N, \mathbb{C})$, we 'avoid' this by passing to $G^{-\infty}$. Note that $G^{-\infty}$ is like $\mathrm{GL}(N, \mathbb{C})$, as non-commutative as can be. Nevertheless $K^{-1}(X)$ is an Abelian group with the product induced by the product in $G^{-\infty}$. Namely, after retracting both $F_i \in \mathcal{C}^\infty(X; G^{-\infty})$ to $\tilde{F}_i \in \mathcal{C}^\infty(X; \mathrm{GL}(N, \mathbb{C}))$ we may embed $\mathrm{GL}(N, \mathbb{C})$ as the upper left corner in $\mathrm{GL}(2N,\mathbb{C})$ as 2×2 matrices with entries in $M(N,\mathbb{C})$ (stabilized by the identity in the lower right corner) and then we may rotate using

$$\begin{aligned}\text{(L9.14)}\quad &\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\tilde F & 0\\ 0 & \mathrm{Id}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\\ &\qquad = \begin{pmatrix}\tilde F & 0\\ 0 & \mathrm{Id}\end{pmatrix} \text{ at } \theta=0,\ \begin{pmatrix}\mathrm{Id} & 0\\ 0 & \tilde F\end{pmatrix} \text{ at } \theta=\pi/2.\end{aligned}$$

This allows us to deform $\tilde F_2$ until it commutes with $\tilde F_1$. Thus the product is commutative. □

Lemma 19. *For any compact manifold with corners we may embed* $K^{-1}(X)\longmapsto K^{-1}(X\times\mathbb{S})$ *as the subgroup of homotopy classes of* $\mathbb{S}$*-constant maps and then* $K^{-1}(X\times\mathbb{S})$ *splits as a direct sum of groups*

$$K^{-1}(X\times\mathbb{S}) = K^{-1}(X)\oplus K^{-2}(X) \tag{L9.15}$$

where

$$K^{-2}(X) = [X\times\mathbb{S}, X\times\{1\}, G^{-\infty}(Y;G),\mathrm{Id}] \tag{L9.16}$$

may be identified as the homotopy classes of pointed maps.

Note that the identification (L9.15) can be seen at the level of maps as

$$[f]\longmapsto [f_1]+[f_2],\ f_1(x,\theta)=f(x,1),\ f_2(x,\theta)=f(x,1)^{-1}f(x,\theta) \tag{L9.17}$$

which is clearly an isomorphism at the level of maps.

Proof. The map induced by (L9.17) gives an isomorphism (L9.15) since under homotopy of f both f_1 and f_2 undergo homotopies within their respective classes of maps, constant and pointed. □

There are other useful representations of $K^{-2}(X)$. One that will occur later corresponds to maps which are not only 'pointed' in the sense that $f(x,1)=\mathrm{Id}$ but are flat at this submanifold, that is they differ from the constant, identity, map by a map into $\Psi^{-\infty}(Y;G)$ which vanishes to infinite order at $X\times\{1\}$. Namely, if $F: X\times\mathbb{S}\longrightarrow \Psi^{-\infty}(Y;G)$ defines $\mathrm{Id}+F: X\times\mathbb{S}\longrightarrow G^{-\infty}(Y;G)$ and $F(x,1)=0$ then if $\phi\in\mathcal{C}^\infty(\mathbb{S})$ has $0\le\phi\le 1$ and $\phi(\theta)=1$ in $|\theta-1|\le\epsilon$, $\phi(\theta)=0$ if $|\theta-1|>2\epsilon$ for $\epsilon>0$ small enough,

$$\mathrm{Id}+(1-\rho)F: X\times\mathbb{S}\longrightarrow G^{-\infty}(Y;G) \tag{L9.18}$$

is homotopic to $\mathrm{Id}+F$.

Definition 5. The (smooth, flat, pointed) loop group, $G^{-\infty}_{(1)}(Y;G)$, of $G^{-\infty}(Y;G)$ is the space of Schwartz maps

$$G^{-\infty}_{(1)}(Y;G) = \left\{a\in {}^{\mathrm{b}}S(\mathbb{R};\Psi^{-\infty}(Y;G)) \text{ s.t. } (\mathrm{Id}+a(t))\in G^{-\infty}(Y;G)\ \forall\ t\in\mathbb{R}\right\}. \tag{L9.19}$$

Lemma 20. *For any compact manifold* Y *and complex vector bundle* G *over* Y, $G^{-\infty}_{(1)}(Y;G)$ *is a topological group with the topology inherited from* ${}^{\mathrm{b}}S(\mathbb{R};\mathcal{C}^\infty(Y^2;\mathrm{Hom}(G)\otimes\Omega_Y)$.

PROOF. Since we already know that $G^{-\infty}(Y;G)$ is a topological group, this is straightforward. In fact $G^{-\infty}_{(1)}(Y;G)$ is an open subset of ${}^{\mathrm{b}}S(\mathbb{R};\mathcal{C}^\infty(Y^2;\mathrm{Hom}(G)\otimes\Omega_Y)$, since invertibility in $G^{-\infty}(Y;G)$ is the same as invertibility on $L^2(Y;G)$. Composition and inversion are continuous, since the are continuous on $G^{-\infty}(Y;G)$. $\square$

The smooth map $\mathbb{R}\ni t\longrightarrow \exp\left(i\frac{t}{1+t^2)^{\frac12}}\pi\right)$ identifies the complement of 1 in $\mathbb{S}$ with $\mathbb{R}$. Using this and the deformation above, we may identify

$$K^{-2}(X)=[X;G^{-\infty}_{(1)}(Y;G)], \tag{L9.20}$$

since this is just a restatement of flatness at the submanifold $X\times\{1\}$.

Thus, essentially by definition, $G^{-\infty}(Y;G)$ and $G^{-\infty}_{(1)}(Y;G)$ are classifying spaces for odd and even K-theory, respectively. Later I will reinterpret $G^{-\infty}_{(1)}(Y;G)$ as the 'symbol group' for elliptic Toeplitz operators on the circle (stabilized by having values in the smoothing operators on Y). This will lead to an exact *classifying sequence* for K-theory of the form

$$G^{-\infty}(\mathbb{S}\times Y;G)\longrightarrow *\longrightarrow G^{-\infty}_{(1),0}(Y;G) \tag{L9.21}$$

where $*$ is a contractible group (a group of invertible Toeplitz perturbations of the identity) and the extra '0' on the loop group means the component of the identity, on which the index vanishes. This is closely related to Bott periodicity. The sequence in (L9.21) is essentially the symbol sequence for Toeplitz operators, as a subalgebra of the pseudodifferential operators on the circle.

9+. Addenda to Lecture 9