CHAPTER 6

Ellipticity

6+.1. Bundles and sections.

Lecture 6: 4 October, 2005

First I want to talk about the basic properties of smoothing operators since to a large extent the study of more operators, particularly elliptic pseudodifferential operators, is ultimately reduced to the study of smoothing 'errors'.

Thus, if X is a compact manifold and E and F are complex vector bundles over X then the space of smoothing operators on X between sections of E and sections of F is

$$\Psi^{-\infty}(X;E,F) = \mathcal{C}^\infty(X^2;\mathrm{Hom}(E,F)\otimes\Omega_R). \tag{L6.1}$$

Here, $\mathrm{Hom}_{(x,x')}(E,F) = \mathrm{hom}(E_{x'},F_x)$ is the 'big' homomorphism bundle. Using the tensor product characterization of homomorphism it can also be identified with the 'exterior' tensor product $\pi_L^*F\otimes\pi_R^*E'$, the tensor product of the pull-back of F from the left fact with the pull-back of the dual of E from the right factor of X. The bundle Ω_R is the 'right density bundle' on X^2, just the pull-back from the right factor of the density bundle. It allows invariant integration.

As operators each $\Psi^{-\infty}(X;E,F)$ defines a linear map $A:\mathcal{C}^\infty(X;E)\longrightarrow \mathcal{C}^\infty(X;F)$ (with which we always identify it) given by

$$Af(x) = \int_X A(x,x')f(x'). \tag{L6.2}$$

Here, the product of $A(x,x')$ and $f(x')$ implicitly includes the action of A as a homomorphism from $E_{x'}$ to F_x. Thus, for fixed x, the integrand is a section of $F_x\otimes\Omega_R$ as a bundle over X in the variable x', i.e. F_x is a trivialized bundle and the integral makes invariant sense.

Basic properties of smoothing operators

- Smoothing operators are characterized (by standard distribution theory) as those continuous linear operators $A:\mathcal{C}^\infty(X;E)\longrightarrow\mathcal{C}^\infty(X;F)$ which extend by continuity to continuous linear operators $A:\mathcal{C}^{-\infty}(X;E)\longleftrightarrow \mathcal{C}^\infty(X;F)$ where $\mathcal{C}^{-\infty}(X;E)$ is the usual space of distributional sections of F over X. I will not use this characterization below, but it is sometimes handy.
- Smoothing operators extend by continuity to compact operators $A:L^2(X;E)\longrightarrow L^2(X;F)$. This is easy to prove using some form of the Ascoli-Arzela theorem which shows that the inclusion $\mathcal{C}^0(X;F)\longrightarrow L^2(X;F)$ is compact, or

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the usual form of Ascoli-Arzela which shows that $\mathcal{C}^1(X;F) \longrightarrow \mathcal{C}^0(X;F)$ is compact, and hence so is $\mathcal{C}^1(X;F) \longrightarrow L^2(X;F)$. From the integral formula (L6.2) it follows that smoothing operators define continuous maps $A : L^2(X;E) \longrightarrow \mathcal{C}^1(X;F)$ the compactness follows. Note that smoothing operators are *not* characterized as the continuous operators $A : L^2(X;E) \longrightarrow \mathcal{C}^\infty(X;F)$. However if an operator has this property and its adjoint, with respect to smooth inner products on the bundles and a smooth density, has the same property, $A^* : L^2(X;F) \longrightarrow \mathcal{C}^\infty(X;E)$ then A is smoothing.

- Now consider the special case $\Psi^{-\infty}(X;E) = \Psi^{-\infty}(X;E,E)$ of operators acting on sections of a fixed bundle. Then $\mathrm{Id} + A$ is *Fredholm* as an operator $A : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;E)$ or $A : L^2(X;E) \longrightarrow L^2(X;E)$. Namely
  (1) The null space is finite dimensional
  (2) The range is closed
  (3) The range has a finite dimensional complement.

Proof. The null space is

$$\mathrm{null}(\mathrm{Id} + A) = \{u \in L^2(X;E); u + Au = 0\} \tag{L6.3}$$

so for any element $u \in \mathrm{null}(\mathrm{Id} + A)$ it follows that $u = -Au \in \mathcal{C}^\infty(X;E)$. Thus the unit ball $\{u \in \mathrm{null}(\mathrm{Id} + A); \|u\| = 1\}$ is precompact in $L^2(X;E)$ and hence compact (since it is closed). It is a standard theorem that any Hilbert space with a compact unit ball is finite dimensional so proving (1) for L^2. The null space on $\mathcal{C}^\infty(X;E)$ is the same as the null space on $L^2(X;E)$ so this is also finite dimensional.

To see that the range is close, suppose $f_n \in L^2(X;E)$ and $f_n \to f$ in $L^2(X;E)$ and $f_n = (\mathrm{Id} + A)u_n$ for $u_n \in L^2(X;E)$. We can assume that $u_n \perp \mathrm{null}(\mathrm{Id} + A)$ and then we wish to show that $u_n \to u$ in $L^2(X;E)$ which implies that $f = (\mathrm{Id} + A)u$. So, suppose first that the sequence $\|u_n\|$ is unbounded. Passing to a subsequence, and relabelling, we may suppose that $\|u_n\| \to \infty$. Thus $v_n = u_n/\|u_n\|$ has unit norm and $(\mathrm{Id} + A)v_n = f_n/\|u_n\| \to 0$ in $L^2(X;E)$. Passing to a subsequence we may assume that $v_n \rightharpoonup v$ converges weakly (by the weak compactness of the unit ball in a Hilbert space). Then $v_n = Av_n + f_n$ must converge strongly, since A is a compact operator. Thus $v_n \to v$ with $\|v\| = 1$ and $v \in \mathrm{null}(\mathrm{Id} + A)$ which is a contradiction, since $u_n \perp \mathrm{null}(\mathrm{Id} + A)$ implies $v \perp \mathrm{null}(\mathrm{Id} + A)$. So in fact the assumption was false and $\|u_n\|$ is necessarily bounded. Then the same argument shows that on an subsequence $u_n \rightharpoonup u$ and hence $u_n = Au_n + f_n \to Au + f$ converges strongly and (2) follows.

Recall that the adjoint of a bounded operator is defined if one has a smooth (sesquilinear) inner product on the fibres of E and a smooth positive density ν on X – one needs these really to fix the inner product on $L^2(X;E)$,

$$\langle u, v\rangle = \int_X \langle u(x), \rangle_{E_x} d\nu(x) \tag{L6.4}$$

by

$$\langle Au, v\rangle = \langle u, A^* v\rangle \ \forall \ u, v \in L^2(X;E). \tag{L6.5}$$

In the case of a smoothing operator (and in fact in general) it follows that the kernel of A^* is $A^*(x', x)$ in terms of $*$ acting on $\mathrm{Hom}(E,E)$. Thus $A^* \in \Psi^{-\infty}(X;E)$ is also a smoothing operator.

Directly from the definition of the adjoint, the orthcomplement of the range of any bounded operator is always the null space of A^*

$$\langle Au, v\rangle = 0 \ \forall \ u \in L^2(X;E) \Longleftrightarrow A^*v = 0. \tag{L6.6}$$

Thus $\operatorname{null}(\operatorname{Id} + A^*)$ is a complement to the range of $\operatorname{Id} + A$ which is therefore finite dimensional, provign (3).

The range of $\operatorname{Id} + A$ is closed in $\mathcal{C}^\infty(X;E)$ by essentially the same argument. Namely if $(\operatorname{Id} + A)u_n = f_n \to f$ in $\mathcal{C}^\infty(X;E)$ then (since the null spaces on $L^2(X;E)$ and $\mathcal{C}^\infty(X;E)$ are the same) we may assume that $u_n \in \mathcal{C}^\infty(X;E)$ and $u_n \to u$ in $L^2(X;E)$ by the discussion above. Then $u_n = -Au_n + f_n \to u$ in $\mathcal{C}^\infty(X;E)$. It also follows that the range of $\operatorname{Id} + A$ has finite codimension in $\mathcal{C}^\infty(X;E)$, in fact $\operatorname{null}(\operatorname{Id} + A^*)$ is still a complement (in the algebraic sense that

$$(\operatorname{Id} + A)\mathcal{C}^\infty(X;E) + \operatorname{null}(\operatorname{Id} + A^*) = \mathcal{C}^\infty(X;E). \tag{L6.7}$$

In fact we know that the left side is a closed subspace of the right, so if they were not equal then there would be a non-trivial distributional section $v \in \mathcal{C}^{-\infty}(X;E)$ such that $\langle v, (\operatorname{Id} + A)u\rangle = 0$ for all $u \in \mathcal{C}^\infty(X;E)$ and $v(w) = 0$ for all $w \in \operatorname{null}(\operatorname{Id} + A^*)$. However the first condition is just $v + Av = 0$ as a distribution, but then $v = -Av$ and $A : \mathcal{C}^{-\infty}(X,E) \longrightarrow \mathcal{C}^\infty(X;E)$ so together these imply $v = 0$. $\square$

Now consider differential opertors, $P \in \operatorname{Diff}^k(X;E,F)$. These are operators $P : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;F)$ which are given everywhere locally, in terms of local coordinates and trivializations of the bundles, by a finite sum of derivatives composed with a matrix

$$P = \sum_{|\alpha| \le k} p_\alpha(x) D_x^\alpha. \tag{L6.8}$$

We say that such an operator is elliptic if the leading part of this sum

$$\sum_{|\alpha| = k} p_\alpha(x)\xi^\alpha \text{ is invertible for each } \xi \in \mathbb{R}^n \setminus \{0\} \tag{L6.9}$$

and for each x (i.e. is invertible as an $N \times N$ matrix).

The sum in (L6.9) makes invariant sense as a section over $T^*X \setminus \{0\}$ of the pull-back from the base of the bundle $\hom(E,F)$. To see this we simply have to give an invariant definition of its value at a point of T^*X! Choose such a point, $\Xi \in T^*_{\bar{x}}X$. Thus, near $\bar{x} \in X$ we may choose $f \in \mathcal{C}^\infty(X)$, real valued, such that $df(\bar{x}) = \Xi$. Now, given an element $\bar{u} \in E_{\bar{x}}$ choose $u \in \mathcal{C}^\infty(X;E)$ such that $u(\bar{x}) = \bar{u}$. Then, for $t \in \mathbb{R}$,

(L6.10)

$$P(ue^{itf}) = e^{itf}U(t,x),\ U(t,x) \in \mathcal{C}^\infty(\mathbb{R}\times X;F),\ U(t,\bar{x}) = t^k\sigma_k(P)(\bar{x}, df(\bar{x}) + O(t^{k-1}.$$

We can use (L6.8) to see this. Thus, $U(t)$ must be a polynomial of degree at most k in t and the leading term, of order k, at $\bar{x}$ is just

$$\sum_{|\alpha| = k} p_\alpha(\bar{x})(df(\bar{x})^\alpha \tag{L6.11}$$

which is just (L6.9). Thus in fact the principal symbol of a differential operator of order m, defined locally by (L6.9) is in fact a well defined section

$$\sigma_k(P) \in \mathcal{C}^\infty(T^*X; \hom(E,F)) \text{ is a fibre-polynomial of degree } k. \tag{L6.12}$$

Now recall that we defined pseudodifferential operators in terms of conormal distributions

$$\Psi^m(X;E,F) = I^{m'}(X^2,\mathrm{Diag};\mathrm{Hom}(E,F)\otimes\Omega_R) \tag{L6.13}$$

and showed that the acted on smooth sections

$$A : \mathcal{C}^\infty(X;E) \longleftrightarrow \mathcal{C}^\infty(X;F),\ Au(x) = \int_X A(x,x')u(x'). \tag{L6.14}$$

We also showed, locally, that differentiation of a conromal distribution gives a conormal distribution with the order increased by one. Since we also know that conormal distributions form a $\mathcal{C}^\infty$ module, it follows that
(L6.15)

$$P(x,D_x) : I^{m'}(X^2,\mathrm{Diag};\mathrm{Hom}(E,F)\otimes\Omega_R) \longrightarrow I^{m'+k}(X^2,\mathrm{Diag};\mathrm{Hom}(E,F)\otimes\Omega_R).$$

This in fact shows that

$$\mathrm{Diff}^k(X;E,F)\Psi^m(X;F,E) \subset \Psi^{m+k}(X;F). \tag{L6.16}$$

Now, consider what happens to the symbol of $A \in \Psi^m(X;F,E)$ under this action on the left by a differential operator. The symbol can be computed locally near a point of the diagonal and in terms of any normal fibration. In particular we can choose the normal fibration to be the 'right fibration with fibres given by the constancy of the second variable x'. That is a local fibre of the normal fibration (in local coordinates and with respect to a local trivialization of the bundles) is just $x' = \bar{x}$ is constant. Thus $P(x,D_x)$ just acts by differentiation on the fibre so the kernel of PA on this fibre is

$$P(x,D_x)A'(x-\bar{x}',\bar{x}') \tag{L6.17}$$

where the left variable has been shifted so that it vanishes at $\bar{x}'$, i.e. where the diagonal meets the fibre, and $A(x-\bar{x}',\bar{x}')$ is the kernel of A on this fibre. Now, it follows from (L6.8) that any lower order terms in P can only raise the order at most to $m+k-1$. Since we know that multiplication by $x_j - \xi'_j$ lowers the oder by 1 (since it vanishes at the singular point) we see that the symbol of PA, modulo lower order terms, is just

$$\sigma_k(P)(\bar{x},\xi)\sigma_m(A). \tag{L6.18}$$

Now, since we are assuming that P is elliptic everywhere, in particular $\sigma_k(\bar{x},\xi)$ is a homogeneous polynomial which does not vanish outside the origin. From the earlier discussion of this in the case of conormal distributions at a point, we know that we can solve the problem

$$PA = \mathrm{Id}_F + R,\ A \in \Psi^{-k}(X;F;E),\ B \in \Psi^{-\infty}(X;F) \tag{L6.19}$$

provided of course that $P \in \mathrm{Diff}^k(X;E,F)$ is elliptic.

Proposition 11. *If $P \in \mathrm{Diff}^k(X;E,F)$ is elliptic then there exists $A \in \Psi^{-k}(X;F,E)$ such that*
(L6.20)

$$P\circ A = \mathrm{Id}_F + R_F,\ R_F \in \Psi^{-\infty}(X;F),\ A\circ P = \mathrm{Id}_E + R_E,\ R_E \in \Psi^{-\infty}(X;E)$$

from which it follows that $P : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;F)$ is Fredholm.

PROOF. From the existence of a right parametrix, $A \in \Psi^{-k}(X; F, E)$, satisfying the first condition in (L6.20) we can conclude that the range is closed and of finite codimension. Indeed the range of P certainly contains the range of PA and this is equal to the range of $\mathrm{Id} + R_F$. This, as we know, is a closed subspace of $\mathcal{C}^\infty(X; F)$ of finite codimension, so any subspace of $\mathcal{C}^\infty(X; F)$ containing it must also be closed and of finite codimension.

To examine the null space we need the second condition in (L6.20). First we try to construct an element $A' \in \Psi^{-k}(X; F, E)$ satisfying this condition without worrying whether it is related to A. To do so, note that we may take adjoints and the condition becomes

$$P^* \circ (A')^* = \mathrm{Id} + R_E^*. \tag{L6.21}$$

From the local discussion above we see that for differential operators,

$$\sigma_k(P^*) = (\sigma_k(P))^* \tag{L6.22}$$

so P is elliptic if and only if P^* is elliptic. Thus we may apply the same construction as above to find $(A')^* \in \Psi^{-k}(X; E, F)$, satifying (L6.21) and then A' is a right parametrix. From this we conclude that the null space of P is finite dimensional, since it is contained in the null space of $\mathrm{Id} + R_E$.

So, it only remains to see that there is an element $A \in \Psi^{-k}(X; F, E)$ which is simultaneously a left- and a right-parametrix. Consider the left parametrix just constructed. From the identity for the right parametrix, and associativity of products, it satisfies

(L6.23)
$$A' = A'(PA - R_F) = (A'P)A - A'R_F = A + R_E A - A'R_F = A + S,\ S \in \Psi^{-\infty}(X; F, E).$$

Thus the left and right parametrices differ by a smoothing operator, either of them is a two-sided parametrix. □

In fact, and such elliptic operator has a 'generalized inverse'. If we choose inner products and densities so that the orthocomplement of the range of P may be identified with the null space of P^* and the orthocomplement of the null space of P may be identified with the range of P^* then there is a unique operator $A : \mathcal{C}^\infty(X; F) \longrightarrow \mathcal{C}^\infty(X; E)$ which vanishes on the null space of P^* has range exactly the range of P^* and which is a two-sided inverse of P as a map from the range of P^* to its own range. In fact, as we shall see next time, this is a pseudodifferential operator (i.e. differs from a parametrix A by a smoothing operator).

L6.2. Hodge theory. Next I want to remind you how the Fredholm properties of elliptic operators on $\mathcal{C}^\infty$ spaces lead to *Hodge theory*, either for the usual exterior differential complex or some other elliptic complex (such as the Dolbeault complex).

On a compact manifold, consider the exterior form bundle ΛX. Thus $\Lambda^k_x X$ is totally antisymmetric part of the k-fold tensor power of $T^*_x X$. Then, as is well-known (and this is really the reason for the definition)

$$d : \mathcal{C}^\infty(X; \Lambda^p X) \longrightarrow \mathcal{C}^\infty(X; \Lambda^{p+1} X),\ d^2 = 0 \tag{L6.24}$$

where we may think of $d : \mathcal{C}^\infty(X; \Lambda^* X) \longrightarrow \mathcal{C}^\infty(X; \Lambda^*)$ as the direct sum of these operators or write it out as a complex

$$\cdots \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^p X) \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^{p+1} X) \xrightarrow{d} \cdots. \tag{L6.25}$$

The definition of the symbol of a differential operator in (L6.10) can be applied since

$$d(e^{itf}u) = e^{itf}\,(itdf \wedge u + du) \Longrightarrow \sigma_1(d)(x,\xi) = i\xi \wedge . \tag{L6.26}$$

In particular of course $\sigma_1(d)^2 = 0$, but that follows directly from the property for d.

If we consider a general differential complex, so a sequence of differential operators $P_i \in \mathrm{Diff}^k(X; E_i, E_{i+1})$ (the orders can be taken to be different but it makes for heavier algebra) such that $P_{i+1} \circ P_i = 0$. Such a complex is said to be *elliptic* if

$$\cdots \xrightarrow{\sigma_k(P_{i-1})(x,\xi)} E_{i,x} \xrightarrow{\sigma_k(P_i)(x,\xi)} E_{i+1,x} \xrightarrow{\sigma_k(P_{i+1})(x,\xi)} \cdots \quad \text{is exact } \forall\ (x,\xi) \in T^*X \setminus 0_X. \tag{L6.27}$$

The deRham complex (L6.25) is elliptic in this sense, since for any $0 \neq \xi \in T^*_xX$ the elements $\alpha \in \Lambda^p_xX$ satisfying $\xi \wedge \alpha = 0$ are exactly those which are of the form $\xi \wedge \beta$ for some $\beta \in \Lambda^{k-1}X$ – to see this simply introduce coordinates in which $\xi = dx_1$ and decompose forms accordingly.

Such an elliptic complex is 'almost exact' in the sense that the cohomology (originally called the hypercohomology) of the complex is finite dimensional.

PROPOSITION 12. *If*

$$\cdots \xrightarrow{P_{i-1}} \mathcal{C}^\infty(X; E_i) \xrightarrow{P_i} \mathcal{C}^\infty(X; E_{i+1}) \xrightarrow{P_{i+1}} \cdots \tag{L6.28}$$

is an elliptic complex of differential operators of order k then the range of each P_i is closed in $\mathcal{C}^\infty(X; E_{i+1})$ and

$$\mathrm{null}(P_i)/P_{i-1}\mathcal{C}^\infty(X; E_{i-1}) \text{ is finite dimensional.} \tag{L6.29}$$

PROOF. Hodge's idea was to choose inner products and densities (well he actually did it in a very algebraic setting) and consider the adjoint complex. Since the adjoint of a product is the product of the adjoints in the opposite order, we get an elliptic complex going the other way

$$\cdots \xleftarrow{P^*_{i-1}} \mathcal{C}^\infty(X; E_i) \xleftarrow{P^*_i} \mathcal{C}^\infty(X; E_{i+1}) \xleftarrow{P^*_{i+1}} \cdots \tag{L6.30}$$

Now each of the operators

$$\Delta_i = P^*_i P_i + P_{i-1}P^*_{i-1} \in \mathrm{Diff}^{2k}(X; E_i) \tag{L6.31}$$

is elliptic. Indeed, its symobl at each point $(x,\xi) \in T^*_xX \setminus \{0\}$ is

$$\sigma_{2k}(\Delta_i) = \sigma_k(P_i)^*\sigma_k(P_i) + \sigma_k(P_{i-1})\sigma_k(P_{i-1})^*. \tag{L6.32}$$

This is a self-adjoint matrix and and element of its null space satisfies

$$\langle \sigma_{2k}(\Delta_i)u, u\rangle = |\sigma_k(P)_iu| + |\sigma_k(P_{i-1})u| = 0 \Longrightarrow \sigma_k(P_{i-1})^*u = 0 = \sigma_k(P_i)u. \tag{L6.33}$$

Since the null space of $\sigma_k(P_{i-1})^*$ is a complement to the range of $\sigma_k(P_{i-1})$, this implies u is zero.

Thus the null space of Δ_i is finite dimensional and its range is closed and has orthocomplement this same null space, by self-adjointness. Again by integration by parts on X, the null space of Δ_i is the intersection of the null spaces of P_i and P^*_{i-1}. It follows that for each i we may decompose

$$\mathcal{C}^\infty(X; E_i) \ni u = u_0 \oplus P_{i-1}v_{i-1} \oplus P^*_i v_{i+1},\ \ P_iu_0 = 0 = P^*_{i-1}u_0 \tag{L6.34}$$

where the decomposition is orthogonal and unique. The range of P_{i-1} must therefore be closed (since the closure in the $\mathcal{C}^\infty$ topology is contained in the closure in L^2). $\square$

Note that the 'Hodge decomposition' (L6.34) is a useful way to encapsulate the consequences of ellipticity for a complex. It shows in particular that (L6.29) can be seen in the stronger form that

$$\operatorname{null}(\Delta_i) \longrightarrow \operatorname{null}(P_i)/P_{i-1}\mathcal{C}^\infty(X; E_{i-1}) \text{ is an isomorphism} \tag{L6.35}$$

which is the *Hodge theorem.*

6+. Addenda to Lecture 6