CHAPTER 4

Conormality at a submanifold

Lecture 4: 20 September, 2005

Last time I defined the space of conormal distributions at the zero section of a real vector bundle and checked the basic properties. These include invariance under bundle transformations and diffeomorphism of the base. The next step is to transfer the defintion to a general embedded submanifold. As noted at the end of last lecture, to do this we need a more general invariance result. To make a change of pace I will do this locally rather than globally. There is no particularly compelling reason for this, I just felt it was time to make sure we could 'see' what is happening.

Thus consider a trivial vector bundle over $\mathbb{R}^n$, $W = \mathbb{R}^n \times \mathbb{R}^k$. We have not really defined the conormal distibutions with respect to $\mathbb{R}^n \times \{0\}$ 'globally' on $\mathbb{R}^n$, although we could easily do so – and indeed I will need them later. Let me instead consider the space of conormal distributions on $\mathbb{R}^n \times \mathbb{R}^k$ with compact support and in fact supported in some bounded open set $N \subset \mathbb{R}^n \times \mathbb{R}^k$ which meets $\mathbb{R}^n \times \{0\}$ (so that we are not just looking at smooth functions). Since N is bounded we can choose a large constant so that $N \subset [-\pi, \pi]^n \times \mathbb{R}^k$ and then we may think of it as a subset of a trivial bundle over the torus

$$N \subset \mathbb{T}^n \times \mathbb{R}^n, \ \mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n. \tag{L4.1}$$

So, by definition the conormal distributions supported in N are just the fibre inverse Fourier transforms of classical symbols, the elements of

(L4.2)

$$\mathrm{I}^m_{\mathcal{S}}(W, O_W) = \mathcal{F}^{-1}_{\text{fib}}\left(\mathcal{C}^\infty(W'; \Omega_{\text{fib}}(W') \otimes N_{-m'})\right), \ W = \mathbb{T}^n \times \mathbb{R}^n, \ m' = m - \frac{n}{4} + \frac{k}{4},$$

to which we simply add the condition that

$$\operatorname{supp}(u) \subset N. \tag{L4.3}$$

The main invariance result I will prove is

PROPOSITION 8. *If $F : N' \longrightarrow N$ is a diffeomorphism, between open subsets of $\mathbb{R}^n \times \mathbb{R}^k$ both satisfying* (L4.1)*, and which satisfies*

$$\begin{cases} F(p) = p \\ F_* = \mathrm{Id} \ \text{ on } N'_p(\mathbb{R}^n \times \{0\}) \end{cases} \quad \forall \ p \in N' \cap (\mathbb{R}^n \times \{0\}) \tag{L4.4}$$

then

$$u \in \mathrm{I}^m_{\mathcal{S}}(W, O_W) \text{ and } \operatorname{supp}(u) \Subset N \Longrightarrow F^* u \in \mathrm{I}^m_{\mathcal{S}}(W, O_W). \tag{L4.5}$$

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Proof. As discussed last time, we will use Moser's method which depends on the construction of a 1-parameter family of such diffeomorphism.

Lemma 11. *If $F : N' \longrightarrow N$ is as in Proposition 8 then for some open $N'' \subset N'$ with $N'' \cap (\mathbb{R}^n \times \{0\}) = N' \cap (\mathbb{R}^n \times \{0\})$ there is a smooth 1-parameter family of maps $F_s : N'' \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$, $s \in [0,1]$, which are diffeomorphisms onto their ranges and satisfy $F_0 = \mathrm{Id}$, $F_1 = F\big|_{N''}$ and (L4.4) for each t.*

Proof. The assumptions on the diffeomorphism F imply that

$$F(x,z) = (x + \sum_{z_j} G_j(x,z), z + \sum_{jk} z_j z_k H_{jk}(x,z)),\ (x,z) \in N'. \tag{L4.6}$$

Indeed, the first restriction on the components realizes the condition $F(x,0) = (x,0)$ and the second correspond to the requirement that the Jacobian $\partial_z \partial_z F(x,0) = \mathrm{Id}$. Then we can simply set

$$\begin{aligned} F_s(x,z) = (X(s), Z(s)),\ X_i(s) = x_i + s\sum_{z_j} G_{ij}(x,z), \\ Z_p(s) = z_p + s\sum_{jk} z_j z_k H_{pjk}(x,z)),\ (x,z) \in N'' = N' \cap |z| < \epsilon \end{aligned} \tag{L4.7}$$

where choosing $\epsilon > 0$ small enough ensures, by the inverse function theorem, that all the maps are diffeomorphisms onto their images. $\square$

Recall that for any smooth function (and hence by continuity also for distributions) the chain rule becomes

$$\frac{d}{ds} F_s^* v_s = F_s^* (\frac{d}{ds} v_s + V(s) v_s) \tag{L4.8}$$

for a smooth vector field V_s. Indeed the vector field is just

$$\frac{dX_i}{ds}\partial_{X_i} + \frac{dZ_p}{ds}\partial_{Z_p} \tag{L4.9}$$

where the coefficients should be treated as functions of $(X(s), Z(s))$. It follows from (L4.7) that

$$V(s) = \sum_k Z_k V_k,\ V_k \text{ smooth and tangent to } Z = 0, \tag{L4.10}$$

which is to say the zero section.

To prove the proposition, consider u as in (L4.5). We will choose a curve of distributions supported very close to $N'' \cap (\mathbb{R}^n \times \{0\})$ and such that

$$\frac{d}{ds} u(s) + V(s) u(s) \in \mathcal{C}^\infty,\ u(1) = u. \tag{L4.11}$$

Recall that we have shown above that the action of any smooth vector field tangent to the zero section leaves the order of a conormal distribution unchanged and multiplying by any Z_k lowers it. Thus

$$V(s) : \{u \in \mathrm{I}^m_{\mathcal{S}}(W, O_W); \mathrm{supp}(u) \Subset F_s(N'')\} \longrightarrow \mathrm{I}^{m-1}_{\mathcal{S}}(W, O_W). \tag{L4.12}$$

So in fact it is easy to solve (L4.11) iteratively. Just make a first choice of $u_0 = u$ which is constant. This means that we have the initial step for the inductive hypothesis

$$\text{(L4.13)}\quad \frac{d}{ds}u_{(N)}(s)+V(s)u_{(N)}(s)=f_{N+1}(s)\in \mathrm{I}^{m-N-1}_{\mathcal{S}}(W,O_W),\\ u_{(N)}(s)=u_0(s)+\cdots+u_N(s),\ u_j(1)=0,\ j>1.$$

Supposing we have solved it to level N, setting

$$\text{(L4.14)}\qquad u_{N+1}(s)=\int_s^1 f_{N+1}(s')ds'\Longrightarrow \frac{d}{ds}u_{N+1}(s)=-f_{N+1}s+f_{N+2}$$

gives the inductive hypothesis at the next level. Taking an asymptotic sum

$$\text{(L4.15)}\qquad u(s)\sim\sum_j u_j(s)\text{ gives (L4.11).}$$

Notice that I have not bothered talking about the supports here, but they can be arranged to be arbitrarily close to the compact set $\operatorname{supp}(u)\cap(\mathbb{R}^n\times\{0\})$ by making additional smooth errors.

This completes the proof of Proposition 8 since $\frac{d}{ds}F_s^*u(s)$ is smooth in all variables and hence

$$\text{(L4.16)}\quad F^*u=F_1^*u(1)=F_0^*u(0)+v=u(0)+v\in \mathrm{I}^m_{\mathcal{S}}(W,O_W)\text{ since }v\in\mathcal{C}^\infty_c(N'').$$

□

We can easily apply this local result to obtain a more global one along the lines that I mentioned last time.

Proposition 9. *Let W be a real vector bundle over a compact manifold Y and suppose that $f:N\longrightarrow N'$ is a diffeomorphism between open neighbourhoods of the zero section 0_W with the properties* (L3.53) *and* (L3.54) *(so it fixes each point of the zero section and has differential projecting to the identity on the normal space to the zero section at each point) then*

$$\text{(L4.17)}\qquad u\in I^m_{\mathcal{S}}(W,0_W)\ \text{with}\ supp(u)\Subset N'\Longrightarrow f^*u-u\in I^{m-1}_{\mathcal{S}}(W,0_W).$$

and in particular

$$\text{(L4.18)}\qquad \sigma_m(f^*u)=\sigma_m(u)\in\mathcal{C}^\infty(\mathbb{S}W';N_{-m'}\otimes\Omega W'),\ m'=m-\frac{d}{4}+\frac{n}{4}.$$

Proof. Each point of Y has a neighbourhood in Y over which W is trivial and Proposition 8. Thus, taking a partition of unity ϕ_j of a neighbourhood of $0_W=Y$ in W with each element supported in such a set we may apply Proposition 8 to f and $\phi_j u$ on each set. Since $u-\sum\limits_j(\phi_j u)$ is smooth and $f^*(\phi_j u)-\phi_j u$ is conormal, and of order $m-1$, for each j we deduce the global form (L4.17).

The invariance of the symbol, (L4.18), follows immediately from (L4.17). □

This result in turn allows us to define the space $I^m(X,Y)$ of conormal distributions associated with (only singular at) an embedded closed submanifold of a compact manifold. To do so we need an appropriate form of

Theorem 1. *[Collar Neighbhourhood Theorem] Let $Y\subset X$ be a closed embedded submanifold of a compact manifold (so Y is a closed subset and for each point $y\in Y$ there exist local coordinates on X based at y in which Y meets the coordinate patch in the set given by the vanishing of the last $d-k$ coordinates) then there* *are an open neighbourhood D of Y in X and D' of the zero section of the normal bundle, NY, to Y in X and a diffeomorphism $f : D \longrightarrow D'$ such that*

$$\begin{gathered} f\big|_Y \text{ is the natural identification of } Y \text{ with } 0_{NY} \\ f_* \text{ induces the natural identification of } N_yY \text{ with } N_yY \ \forall\ y \in Y. \end{gathered} \tag{L4.19}$$

Perhaps in this form the theorem requires a little more explanation. First the normal bundle has, as I said early, fibre at a point $y \in Y$ the quotient

$$N_yY = T_yX/Y_yY. \tag{L4.20}$$

If X is a given a Riemannian structure then we may identify this quotient witht he metric normal space and write

$$T_yX = T_yY \oplus N_yY \tag{L4.21}$$

but in general there is no natural way of embedding NY as a subbundle of T_YX. However, once we have a smooth map $f : D \longrightarrow D'$ which maps a neighbourhood of Y in X to a neighbourhood of the zero section of NY, and maps each $y \in Y$ to its image point in 0_{NY} then

$$f_* : T_yX \longrightarrow T_y(NY).$$

Since we are assuming that f maps Y onto 0_{NY} as 'the identity' it must map T_yY to $Y_y(0_{NY}) = T_yY$ as the identity and hence projects to a map on the quotients

$$f_* : N_yY \longrightarrow T_y(NY)/T_y0_{NY} = N_yY \tag{L4.22}$$

where we can identify the normal space to the zero section unambiguously with the fibre for any vector bundle. Thus the second condition is that this map should also be the identity.

Proof. I will not give a complete proof of the Collar Neighbourhood Theorem in this form. Suffice it to say that the standard approach is to use geodesic flow map for a Riemann metric on X. Using the embedding of NY in T_YX coming from (L4.21) one can check that the restriction of the exponential map to a small neighbourhood of the zero section of the normal bundle gives a diffeomorphism onto a neighbourhood of Y and the inverse of this satisfies the two conditions. □

For our application, the uniqueness part is also important. Namely given two local diffeomorphism f_i, $i = 1, 2$, both as in the theorem, the composite $f = f_2 \circ f_1^{-1}$ is a diffeomorphism of one neighbourhood of the zero section of NY to another and it necessarily satisfies both (L3.53) and (L3.54). This means that the definition we have been working towards makes good sense.

Definition 3. If $Y \subset X$ is a closed embedded submanifold of a compact manofold then

$$\begin{gathered} I^m(X, Y) = \big\{u \in \mathcal{C}^{-\infty}(X); u = u_1 + u_2,\ u_2 \in \mathcal{C}^\infty(X) \text{ and} \\ u_1 = f^*v,\ v \in I^m_{\mathcal{S}}(NY, 0_{NY}),\ \operatorname{supp}(v) \subset D' \text{ for some diffeomorphism as in (L4.19)}\big\}. \end{gathered} \tag{L4.23}$$

Now, many properties of the $I^m(X, Y)$ now follow directly from the properties already stablished for the $I^m_{\mathcal{S}}(W, 0_W)$. First the inclusion for these spaces gives immediately

$$I^{m-1}(X, Y) \subset I^m(X, Y). \tag{L4.24}$$

This inclusion is important because it is captured by the symbol. Since this is rather important in the sequel, let me state this formally.

LEMMA 12. *The symbol map on $I^m(NY, 0_{NY})$ induces a symbol map on $I^m(X,Y)$ and this gives a short exact sequence*

$$I^{m-1}(X,Y) \hookrightarrow I^m(X,Y) \xrightarrow{\sigma_m} \mathcal{C}^\infty(SN^*Y; N_{-m'} \otimes \Omega_{\text{fib}}),\\ m' = m - \frac{d}{4} + \frac{n}{4},\ d = \dim Y,\ n = \operatorname{codim} Y. \tag{L4.25}$$

So what are the important properties of these distributions?

(1) Each element of $I^m(X,Y)$ is smooth outside Y and

$$\bigcap_k I^{m-k}(X,Y) = \mathcal{C}^\infty(X). \tag{L4.26}$$

(2) Invariance:- If $F : X' \longrightarrow X$ is a diffeomorphism then

$$F^* : I^m(X,Y) \longrightarrow I^m(X', F^{-1}(Y)),\ \sigma_m(F^*u) = F^*\sigma_m(u) \tag{L4.27}$$

where you need to check the sense in which F^* induces an isomorphism of the conormal bundles N^*Y in X and $N^*(F^{-1}(Y))$ in X'.

(3) Action of differential operators. If $P \in \operatorname{Diff}^k(X)$ (which I have not really defined) then

$$P : I^m(X,Y) \longrightarrow I^{m+k}(X,Y),\ \sigma_{m+k}(Pu) = \sigma_k(P)\big|_{N^*Y}\sigma_m(u). \tag{L4.28}$$

(4) Asymptotic completeness. If $u_k \in I^{m-k}(X,Y)$ then there exists $u \in I^m(X,Y)$ such that

$$u - \sum_{k<N} u_k \in I^{m-N}(X,Y),\ \forall\ N. \tag{L4.29}$$

4+. Addenda to Lecture 4

4+.1. Listing the properties. Let me briefly summarize, again, the properties of the conormal distributions as I have defined them above and outline proofs. For the momemnt we only have 'generalized functions'. For each $m \in \mathbb{C}$ (I have mostly been treating m as real but this is not usd anywhere) and any embedded closed submanifold of a compact manifold, $Y \subset X$, we have defined

$$I^m(X,Y) \subset \mathcal{C}^{-\infty}(X) = (\mathcal{C}^\infty(X;\Omega))'. \tag{4+.30}$$

This is Definition 3 in terms of conormal distributions with respect to the zero section of a vector bundle (in this case the normal bundle to Y in X). The definition in that case is (L3.50) as the inverse fibre Fourier transform of 'symbols' on the radial compactification of the dual bundle. It follows from the inclusion for the symbol spaces that if $k \in \mathbb{N}$ then

$$I^{m-k}(X,Y) \subset I^m(X,Y),\ \bigcap_k I^{m-k}(X,Y) = \mathcal{C}^\infty(X). \tag{4+.31}$$

Asymptpotic completeness of the symbol spaces shows that if $u_k \in I^{m-k}(X,Y)$ then there exists $u \in I^m(X,Y)$ such that

$$u - \sum_{k\le N} u_k \in I^{m-N}(X,Y)\ \forall\ N. \tag{4+.32}$$

The main thing that distinguishes conormal distributions is that their leading singularities are describeable by the principal symbol map which gives a short exact sequence for each m

(4+.33)

$$I^{m-1}(X,Y) \longrightarrow I^m(X,Y) \xrightarrow{\sigma_m} \mathcal{C}^\infty(SN^*Y; N_{m'} \otimes \Omega_{\text{fib}}),\ m' = m - \frac{1}{4}\dim X + \frac{1}{4}\dim X,$$

N_{-m} is the bundle of functions homogeneous of degree m' on N^*Y (the normal bundle to Y in X) and Ω_{fib} is the bundle of densities on the fibres of N^*Y.

Exercise 16. Show that the density bundle on X, restricted to Y, can be decomposed

$$\Omega_Y X = \Omega Y \otimes \Omega_{\text{fib}} NY \tag{4+.34}$$

where $\Omega_{\text{fib}} NY$ is the 'normal density bundle to Y, so is the 'absolute value' of the maximal exterior power of the conormal bundle to Y. (The notation is to indicate that this is the usual normal bundle on the fibres of NY made into a bundle over Y.) So if $0 < \mu \in \mathcal{C}^\infty(Y; \Omega NY)$ is a positive smooth 'normal density' on Y (and such always exists) then

$$u_\mu : \mathcal{C}^\infty(X; \Omega X) \ni \nu \longmapsto \int_Y (\nu/\mu) \in \mathbb{C} \text{ (or } \mathbb{R}) \tag{4+.35}$$

is a well-defined distibution. Show that this 'delta' section is an element of $I^{-\frac{1}{4}\operatorname{codim} Y}(X,Y)$ and compute its symbol (in terms of μ.)

For any differential operator $P \in \operatorname{Diff}^q(X)$ (so $P : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X)$ is a continuous linear operator which is *local*) its symbol $\sigma_q(P)$ is a smooth function on T^*X which is a homogeneous polynomial of degree q on the fibres (defined by the condition

$$P(e^{itf(x)}v(x)) = e^{itf(x)}\left(\sigma_q(tdf)v(x) + O(t^{q-1})\right)\ \forall\ f,\ v \in \mathcal{C}^\infty(X),\ t \in \mathbb{R}) \tag{4+.36}$$

$$P : I^m(X,Y) \longrightarrow I^{m+q}(X,Y),\ \sigma_{m+q}(Pu) = \sigma_q(P)\sigma_m(u). \tag{4+.37}$$

In particular the $I^m(X,Y)$ are $\mathcal{C}^\infty(X)$ modules and they are invariant under diffeomorphisms, so if $f : O \longrightarrow O'$ is a diffeomorphism between open subsets of X, Y and Y' are embedded submanifolds of X and $f(O \cap Y) = O' \cap Y'$ then

(4+.38)

$$f^* : \{u \in I^m(X,Y'); \operatorname{supp}(u) \subset O'\} \longrightarrow I^m(X,Y,\ \sigma_m(f^*u) = (f^*)^*\sigma_m(u)$$

where $f^* : N^*_{O;\cap Y;}Y' \longrightarrow N^*Y$ is the induced map.

Exercise 17. Show that any element of $I^m(X,Y)$ which has support in Y is of the form Pu_μ where u_μ is as in (4+.35) and $P \in \operatorname{Diff}^q(X)$ for some q. What values of m can occur this way?

4+.2. Poincaré forms. Although I have only defined conormal distributions, there is no problem in defining conormal sections of any complex vector bundle E over X (and I will do this next time) giving a space $I^m(X,Y;E)$ with similar properties. In fact I will discuss this in more detail next time. Informally an element of $\mathcal{C}^{-\infty}(X;E)$ is given in terms of any local trivialization of E by a sum over the local basis with distributional coefficients. If these coefficients are in $I^m(X,Y)$ then the distributional section is in $I^m(X,Y;E)$. This tensor-product definition can readily be made rigourous.

Anyway, suppose we have made sense of this already. The 'simplest' sort of conormal distributions are again the 'Dirac delta sections'. One particularly nice example is given by the Poincaré duals of embedded submanifolds. Since this is an opportunity to discuss a little homology, let me do so.

First recall deRham theory in which the spaces of sections of the exterior bundles (exterior powers of the cotangent bundle) over a manifold X are the chain spaces for a (co)homology theory. Namely d gives a complex of differential operators, $d \in \mathrm{Diff}^1(X; \Lambda^k X, \Lambda^{k+1} X)$, $d^2 = 0$

$$\dots \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^{k-1}) \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^k) \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^{k+1}) \xrightarrow{d} \dots. \tag{4+.39}$$

The deRham cohomology groups

$$H^k_{\mathrm{dR}}(X) = \{u \in \mathcal{C}^\infty(X; \Lambda^k); du = 0\}/d\mathcal{C}^\infty(X; \Lambda^{k-1}) \tag{4+.40}$$

are naturally isomorphic (for a compact manifold) to the other 'obvious' cohomology groups – singular, smooth singular or Čech (and as I will discuss later, Hodge).

There are other forms of the deRham groups too. In particular the 'distributional deRham cohomology' is canonically isomorphic to the smooth

$$\{u \in \mathcal{C}^{-\infty}(X; \Lambda^k); du = 0\}/d\mathcal{C}^{-\infty}(X; \Lambda^{k-1}) \equiv H^k_{\mathrm{dR}}(X). \tag{4+.41}$$

Here there is an obvious map from smooth deRham to distributional deRham and this is always an isomorphism. That is, any element of $\mathcal{C}^{-\infty}(X; \Lambda^k)$ which satisfies $du = 0$ is of the form $dv + u'$ with $v \in \mathcal{C}^{-\infty}(X; \Lambda^{k-1})$ and $u' \in \mathcal{C}^\infty(X; \Lambda^k)$ (so of course $du = 0$). This by the way is a consequence of the Hodge theorem proved later (but can be proved more crudely but more directly if you want).

Why care about distributional deRham at all? One reason is the existence of Poincaré dual forms (also sometimes called Leray forms).

Proposition 10. *If $Y \subset X$ is a closed embedded submanifold with an oriented normal bundle then the form given in local coordinates near any point of Y, in which $Y = \{x_{d+1} = \dots = x_n = 0\}$ locally with the correct orientation, by*

$$p_Y = \delta(x_{d+1}) \cdots \delta(x_n) dx_{d+1} \wedge \dots \wedge dx_n \in I^-(X, Y; \Lambda^{n-d}), \ \dim Y = d, \tag{4+.42}$$

is independent of choices, closed and fixes the Poincaré dual class to Y in $H^{n-d}(X)$.