CHAPTER 3

Conormality at the zero section

Lecture 3: 15 September, 2005

Next we turn to the case of a real vector bundle $W \longrightarrow Y$ over a compact manifold Y and define the space of conormal distributions on the total space W of the vector bundle with respect to (i.e. only singular at) the zero section 0_W. The latter is a compact embedded submanifold canonically isomorphic to Y.

Before I do this, I want to point out some further properties in the case of the conormal distributions with respect to the origin of a vector space. In particular there is another important invariance property, the proof of which I want to go through. I will also indicate in a simple example how these spaces can be used.

First, these distibutions can be integrated

$$\int_{\mathbb{R}^n} : I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \longrightarrow \mathbb{C}. \tag{L3.1}$$

The integral is well-defined on both distributions of compact support and on $\mathcal{S}(\mathbb{R}^n)$ and we know, using (L2.26), that any $u \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\})$ can be written as a sum $u = \chi u + (1-\chi)u$ of one term of each type. The value of integral is independent of the particular splitting since the definitions agree on the intersection, namely $\mathcal{C}^\infty_c(\mathbb{R}^n)$. In terms of the Fourier transform $a = \mathcal{F}(u)$ the integral can be written explictly:-

$$\int_{\mathbb{R}^n} u(x)dx = a(0). \tag{L3.2}$$

As we shall see this rather trivial observation is decidedly useful later.

For a general vector space we will only get a well-defined map analogous to (L3.1) if we have chosen a volume form, which could be the Lebesgue form for some identification with $\mathbb{R}^n$. I will discuss densities, a better way to do this, later.

So, returning to the case of $\mathbb{R}^n$ recall that we have already shown that

$$z^\alpha D^\beta_z : I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \longrightarrow I^{m+|\beta|-|\alpha|}_{\mathcal{S}}(\mathbb{R}^n, \{0\}). \tag{L3.3}$$

This follows directly from the properties of the Fourier transform. It is also clear that convolution behaves well

$$I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) * I^{m'}_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \subset I^{m+m'+\frac{n}{4}}_{\mathcal{S}}(\mathbb{R}^n, \{0\}). \tag{L3.4}$$

Indeed, the Fourier transform of the convolution is the product of the Fourier transforms so

(L3.5)

$$\widehat{u * v} = \widehat{u}\widehat{v} \in \rho^{-m-m'-\frac{n}{2}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \text{ if } \widehat{u} \in \rho^{-m-\frac{n}{4}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \text{ and } \widehat{v} \in \rho^{-m'-\frac{n}{4}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}).$$

It is also the case that $I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})$ is a $\mathcal{S}(\mathbb{R}^n)$-module, that is multiplication by a Schwartz function maps this space into itself

$$\mathcal{S}(\mathbb{R}^n)\cdot I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})\subset I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\}). \tag{L3.6}$$

Perhaps the obvious way to approach this is the opposite to (L3.5). That is, take the Fourier tranform and then show that

$$\mathcal{S}(\mathbb{R}^n)*\rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n})\subset\rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n}). \tag{L3.7}$$

This is not so hard, and may well be informative. However I will prove it in a slightly different way, using an asymptotic completeness argument.

So we wish to show that if $\phi\in\mathcal{S}(\mathbb{R}^n)$ and $u\in I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})$ then $\phi u\in I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})$. We can simplify this a little by choosing a cutoff function $\chi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ which is identically equal to 1 near the origin and splitting $u=\chi u+(1-\chi)u$ into a compactly supported term and a term in $\mathcal{S}(\mathbb{R}^n)$; then we can ignore the latter since it is in an algebra contained in $I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})$. Now, we can similarly split ϕ into a part supported near, and a part supported away from, the origin. If the latter is supported in the complement of the support of u (now compact) then the product is zero. Thus we are reduced to the special case

$$\mathcal{C}^\infty_c(\mathbb{R}^n)\cdot I^m_{\mathrm{c}}(\mathbb{R}^n,\{0\})\subset I^m_{\mathrm{c}}(\mathbb{R}^n,\{0\}) \tag{L3.8}$$

where the suffix 'c' indicates that supports are compact, as opposed to the Schwartz property at infinity.

Now, let us replace ϕ by its Taylor series expansion, to high order and with remainder, about the origin

$$\phi(z)=\sum_{|\alpha|\le N}c_\alpha z^\alpha+\sum_{|\alpha|=N+1}\phi_\alpha(z)z^\alpha,\ \phi_\alpha(z)\in\mathcal{C}^\infty(\mathbb{R}^n). \tag{L3.9}$$

If you recall, this is proved by radial integration. Now, multiplying $\phi(z)$ by another cutoff $\chi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ which is equal to 1 in a neighbourhood of the support of u (so $\chi u=u$) we find that

$$\phi(z)u=\sum_{|\alpha|\le N}c_\alpha z^\alpha u+\sum_{|\alpha|=N+1}\phi^{(N)}_\alpha(z)z^\alpha u,\ \phi^{(N)}_\alpha\in\mathcal{C}^\infty_c(\mathbb{R}^n). \tag{L3.10}$$

The advantage of doing this is that we know about all the terms in the first sum, namely $z^\alpha u\in I^{m-|\alpha|}_{\mathcal{S}}(\mathbb{R}^n,\{0\})$. Similarly the remainder terms are of the form

$$\sum_{|\alpha|=N+1}\phi^{(N)}_\alpha(z)u^{(N)}_\alpha,\ u^{(N)}_\alpha\in I^{m-N-1}_{\mathrm{c}}(\mathbb{R}^n,0). \tag{L3.11}$$

On the other hand from the estimates I did last time, we know that if $N>m+n+p$ then

$$u^{(N)}_\alpha\in\mathcal{C}^p_{\mathrm{c}}(\mathbb{R}^n),\ \forall\ |\alpha|=N+1. \tag{L3.12}$$

After multiplying by a smooth function of compact support this remains true. Note that in the first sum in (L3.10) the term of order α is fixed once $N\ge|\alpha|$. Thus, by asymptotic completeness we can find one element $v\in I^m_{\mathcal{S}}(\mathbb{R}^n,\{0\})$ such that

$$v-\sum_{|\alpha|\le N}c_\alpha z^\alpha u\in I^{m-N-1}_{\mathcal{S}}(\mathbb{R}^n,\{0\})\ \forall\ N. \tag{L3.13}$$

Combining this with (L3.10) and (L3.11), with the same estimate on the regularity at the origin for the difference in (L3.13) we conclude that

$$\phi u - v \in \mathcal{S}(\mathbb{R}^n) + \mathcal{C}^p_c(\mathbb{R}^n)\ \forall\ p \tag{L3.14}$$

and hence $\phi u - v \in \mathcal{S}(\mathbb{R}^n)$ which proves (L3.8) and hence (L3.6).

Let me make an immediate application of this to a 'baby' problem which is intended to illustrate how we can use these conormal distributions. Observe that the constants are in the space $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ so

$$\delta \in I_{\mathcal{S}}^{-\frac{n}{4}}(\mathbb{R}^n, \{0\}) \tag{L3.15}$$

just to make you think of an example.

Now, combining (L3.3) and (L3.6) we see that if P is a differential operator with Schwartz coefficients

$$P = \sum_{|\alpha|\le k} p_\alpha(z) D_z^\alpha,\ p_\alpha(z) \in \mathcal{S}(\mathbb{R}^n) \tag{L3.16}$$

then

$$P : I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \longrightarrow I^{m+k}_{\mathcal{S}}(\mathbb{R}^n, \{0\}). \tag{L3.17}$$

The project is to try to partially invert this map by showing that

$$\text{Given } f \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\})\ \exists\ u \in I^{m-k}_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \\ \text{s.t. } Pu = f + g,\ g \in \mathcal{S}(\mathbb{R}^n) \text{ provided } P \text{ is elliptic at } 0 \tag{L3.18}$$

Of course I have not said what the condition of ellipticity means, but we will find out in the proof. We only 'partially' solve the problem in the sense that there is a Schwartz error, but at least we can remove the singularity.

As in the discussion above, we do not get to our goal immediately, but we proceed by steps. Suppose $u_0 \in I^{m-k}(\mathbb{R}^n, \{0\})$ then we know from (L3.17) that $Pu \in I^m(\mathbb{R}^n, \{0\})$ but we can get more information about the 'leading singularity'. Namely, the part of the sum in (L3.16) over $|\alpha| < k$ maps u into $I^{m-1}(\mathbb{R}^n, \{0\})$. Similarly, any part of the coefficients which vanishes at the origin has a factor of z_j in it and so, even after k differentiations, this part maps into $I^{m-1}(\mathbb{R}^n, \{0\})$ as well. Thus

$$Pu_0 = \sum_{|\alpha|=k} p_\alpha(0) D_z^\alpha u_0 + f',\ f' \in I^{m-1}(\mathbb{R}^n, \{0\}). \tag{L3.19}$$

Taking the Fourier transform of this 'leading term' we get

$$p_k(0,\xi)\widehat{u_0}(\xi) \in \rho^{-m-k+\frac{n}{4}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}),\ p_k(0,\xi) = \sum_{|\alpha|=k} p_a\alpha(0)\xi^\alpha. \tag{L3.20}$$

So we want to solve

$$p_k(0,\xi)\widehat{u_0}(\xi) = \widehat{f}(\xi) \tag{L3.21}$$

just as though we were solving a constant coefficient operator. Of course in general this does not have a smooth solution because of the zeros of $p_k(0,\xi)$. We say that

$$P \text{ is elliptic at } 0 \text{ if } p_k(0,\xi) \neq 0 \text{ in } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{L3.22}$$

Even assuming this we cannot quite solve (L3.21) since (unless we are in the completely trivial case where $k = 0$) $p_k(0, \xi)$ does vanish at the origin. However we can choose

$$\widehat{u_0}(\xi) = \frac{(1 - \chi(\xi))\widehat{f}(\xi)}{p_k(0, \xi)} \in \rho^{-m-k+\frac{n}{4}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}). \tag{L3.23}$$

where $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ is equal to 1 near zero. Since p_k is homogeneous of degree k (and non-vanishing) $(1 - \chi)/p_k \in \rho^k\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ from which (L3.23) follows. Moreover then we get

$$p_k(0, \xi)\widehat{u_0} = \widehat{f} + g, \ g \in \mathcal{S}(\mathbb{R}^n). \tag{L3.24}$$

Inserting this in (L3.19) we have made progress, namely we have shown that

$$\text{Given } f \in I^m(\mathbb{R}^n, \{0\}) \ \exists \ u_0 \in I^{m-k}(\mathbb{R}^n, \{0\}) \text{ s.t.} \quad Pu = f + f'', \ f'' \in I^{m-1}(\mathbb{R}^n, \{0\}) \tag{L3.25}$$

provided P satisfies the ellipticity condition (L3.22).

Now we can proceed by induction. Namely the order m in (L3.25) is arbitrary and the inductive statement is that we have constructed

$$u_j \in I^{m-k-j}(\mathbb{R}^n, \{0\}), \ j = 0, \dots, l \quad \text{s.t. } P(\sum_{j=0}^{l} u_j) = f - f_{l+1} \text{ where } f_{l+1} \in I^{m-l-1}(\mathbb{R}^n, \{0\}). \tag{L3.26}$$

Then we use (L3.25) with m replaced by $m - l - 1$ and f replaced by f_{l+1} to construct u_{l+1} and then define $f_{l+2} = f_{l+1} - Pu_{l+1} \in I^{m-l-2}_{\mathcal{S}}(\mathbb{R}^n, \{0\})$. This proves the inductive statement for all l.

Finally we use asymtotic completeness, which shows that there exists one fixed $u \in I^{m-k}_{\mathcal{S}}(\mathbb{R}^n, \{0\})$ such that

$$u - \sum_{j=0}^{l} u_j \in I^{m-k-l-1}_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \ \forall \ l \tag{L3.27}$$

and from this (L3.18) follows.

This argument is a model for quite a few arguments below.

Now, what I really want to do today is to define conormal distributions on a vector bundle. What we need here is the invariance under linear transformations, which we have already checked. However we also want to be able to write things in an invariant form and to do so it is convenient to use the language of densities.

Recall that given a vector space over the reals there are many 'associated' vector spaces. The dual W', tensor powers and in particular exterior powers – the totally antisymmetric parts of the tensor powers. If $\dim W = n$ then the maximal (non-trivial) exterior power is $\Lambda^n W$. Its elements are n-multilinear and totally antisymmetric forms

$$\mu : (W')^{\times n} = W' \times W' \times \dots W' \longrightarrow \mathbb{R}$$

where multilinearity is linearity in each of the n variables separately and antisymmetry reduces to oddness under the exchange of any neighbouring pair of variables. This is a 1-dimensional vector space and by standard linear algebra its dual is canonically isomorphic to $\Lambda^n(W')$. That is, μ can be identified (canonically so we use the same name) with a linear map

$$\mu : \Lambda^n(W') \longrightarrow \mathbb{R}. \tag{L3.28}$$

The fundamental property of these forms is that on $\mathbb{R}^n$, for the action of $\mathrm{GL}(n, \mathbb{R})$,

$$(G^*\mu)(w_1, \dots, w_n) = \mu(Gw_1, \dots, Gw_n) = \det G \cdot \mu(w_1, \dots, w_n) \tag{L3.29}$$

in terms of (3). Of course it has to be a multiple since the space is 1-dimensional.

Now, in place of (L3.28) we can consider more general maps

(L3.30)
$$\Omega^t W = \{\nu : \Lambda^n(W') \setminus \{0\} \longrightarrow \mathbb{R},\ \nu(cv) = |c|^t \nu(v)\ \forall\ c \in \mathbb{R} \setminus \{0\},\ v \in \Lambda^n(W')\}.$$

Instead of being linear these are absolutely homogeneous of degree t. If $t = 0$ we just have constants but in all cases, for each $t \in \mathbb{R}$, these are linear spaces of dimension 1. In the special case $t = 1$ we just use the notation ΩW. Observe that if $\mu \in \Lambda^n W$ then $|\mu| \in \Omega W$ and any element is equal to $\pm|\mu|$ for some such μ. Thus the only real difference between $\Lambda^n W$ and ΩW is to do with orientation. Anyway, it follows from this observation that in the case of $\mathbb{R}^n$,

$$G^*\nu = |\det G|\nu \tag{L3.31}$$

in terms of the same action of $\mathrm{GL}(n, \mathbb{R})$ on $\Lambda^n \mathbb{R}^n$. This is the reason densities are important, because they transform in such a way that integration becomes invariant (for the moment under linear transformations).

Let us apply this discussion directly to the Fourier transform. For Schwartz functions on $\mathbb{R}^n$

$$\widehat{G^* u}(\zeta) = \int e^{-iz\cdot\zeta} u(Gz) dz,\ G \in \mathrm{GL}(n, \mathbb{R}). \tag{L3.32}$$

Changing the variable of integration from z to $y = Gz$ we 'know' (from integration theory) that

$$\widehat{G^* u}(\zeta) = \int e^{-iy\cdot(G^{-1})^t\zeta} u(y) dy |\det G|^{-1} \tag{L3.33}$$

where I have used the definition of the transpose to write $G^{-1}y \cdot \zeta = y \cdot (G^{-1})^t \zeta$. Thus

$$\widehat{G^* u}(\zeta) = |\det G|^{-1}.((G^{-1})^t)^* \hat{u} \tag{L3.34}$$

The action via the transpose of the inverse is exactly what we expect on the dual space but there is an extra factor of the determinant, admittedly just a constant but annoying nevertheless! We can remove this and get complete invariance by redefining the Fourier transform as a density

$$\mathcal{F}u(\zeta) = \hat{u}(\zeta)|d\zeta| \in \mathcal{S}(\mathbb{R}^n; \Omega). \tag{L3.35}$$

Now, from the discussion above this transforms in precisely the correct way so that we have a map which is independent of the choice of linear coordiantes

$$\mathcal{F} : \mathcal{S}(W) \longrightarrow \mathcal{S}(W'; \Omega W') \tag{L3.36}$$

where the image space is just the space of Schwartz functions valued in $\Omega W'$ (which is just a 1-dimensional vector space.)

As a consequence of this we can now identify, independent of the choice of basis

$$I^m(W, O_W) = \mathcal{F}^{-1}\left(\rho^{-m-\frac{n}{4}} \mathcal{C}^\infty(\overline{W'}; \Omega W')\right). \tag{L3.37}$$

Then

$$I^{m-1}(W,O_W) \longrightarrow I^m(W,O_W) \xrightarrow{\sigma_m} \mathcal{C}^\infty(\mathbb{S}W'; N_{-m-\frac{n}{4}} \otimes \Omega W') \tag{L3.38}$$

is a short exact sequence.

One advantage of this definition, or the coordinate version for $\mathbb{R}^n$, is that we can immediately see what it means for such a distribution to 'depend smoothly on parameters.' Said another way, these spaces come with topologies, since the Fourier transform is used as an isomorphism, we can use the topology (uniform convergence of all derivatives on compact sets) on $\mathcal{C}^\infty(\overline{W'})$ to give a toplogy on

$$\rho^{-m}\mathcal{C}^\infty(\overline{W'}; \Omega W') = \mathcal{C}^\infty(\overline{W'}; N_{-m} \otimes \Omega W')$$

for any m and hence we have a topology on $I^m(W, 0_W)$.

So, if Y is a compact manifold, what is $\mathcal{C}^\infty(Y; I^m(W, O_W))$ for a vector space W? It is a space of distributions on $Y \times W$ which is identified by Fourier transform with $\mathcal{C}^\infty(Y; \rho^{-m-\frac{n}{4}}\mathcal{C}^\infty(\overline{W'}; \Omega W'))$. Again the definition of the topology just given means that we remove the weight factor and take a basis of the vector space and so identify this with $\mathcal{C}^\infty(Y; \mathcal{C}^\infty(\mathbb{S}^{n,1}))$. Now, it is a standard analytic fact that (for any manifolds)

$$\mathcal{C}^\infty(Y; \mathcal{C}^\infty(\mathbb{S}^{n,1})) = \mathcal{C}^\infty(Y \times \mathbb{S}^{n,1}) = \mathcal{C}^\infty(Y \times \mathbb{S}^n)\big|_{Y \times \mathbb{S}^{n,1}} \tag{L3.39}$$

is just the space of smooth functions on the product manifold, itself a compact manifold with boundary. Or, backing up a little with the identifications it is the same thing as

$$\rho^{-m-\frac{n}{4}}\mathcal{C}^\infty(Y \times \overline{W'}; \Omega W'). \tag{L3.40}$$

For reasons that might seem trivial compared to the resulting annoyance, we identify this space, as a space of distributions on $Y \times W$ with

$$I^m_{\mathcal{S}}(Y \times W, Y \times \{0\}) = \mathcal{F}^{-1}\left(\rho^{-m+\frac{d}{4}-\frac{n}{4}}\mathcal{C}^\infty(Y \times \overline{W'}; \Omega W')\right), \quad d = \dim Y,\ n = \dim W. \tag{L3.41}$$

Here $\mathcal{F}$ is to be interpreted as in (L3.36).

Now, suppose that rather than a product with a vector space, W is a smooth real vector bundle over the compact manifold Y. We want to define $I^m_{\mathcal{S}}(W, O_W)$ so that it reduces to (L3.41) in case the bundle is trivial.

Let me start with the radial compactification of the real vector bundle W. I will, for just this once, take the 'high road' of associated bundles, but then give a transition-map description.

A real vector bundle over Y is a manifold W with a smooth surjective map $\pi : W \longrightarrow Y$ which is a submersion (has surjective differential at each point), is such that, for each $y \in Y$, $\pi^{-1}(y) = W_y$ has a linear structure (over $\mathbb{R}$) and which is also locally trivial in the sense that Y has a covering by open sets $\mathcal{U}$ such that for each $U \in \mathcal{U}$, there is a diffeomorphism $T_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ giving a commutative diagramme with the projections

(L3.42)
$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "F_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "\pi_U"] \\
& U &
\end{tikzcd}$$

and which is linear on the fibres.

From a vector bundle we can construct a principal bundle. Namely for each $y \in Y$ set

$$P_y = \{T : W_y \longrightarrow \mathbb{R}^n \text{ a linear isomorphism.}\} \tag{L3.43}$$

This is a principal $\mathrm{GL}(n, \mathbb{R})$-space since if $T \in P_y$ then $GT \in P_y$ for each $G \in \mathrm{GL}(n, \mathbb{R})$ and this action of $\mathrm{GL}(n, \mathbb{R})$ is simple and transitive. Putting these spaces together set

$$\begin{tikzcd} P = \bigcup_{y \in Y} P_y \arrow[d, "\pi_P"] \\ Y. \end{tikzcd} \tag{L3.44}$$

The local trivializations (L3.42) of W provide sections over the sets $U \in \mathcal{U}$ of P giving corresponding maps

$$\begin{tikzcd} \pi_P^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U \times \mathrm{GL}(n, \mathbb{R}) \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd} \tag{L3.45}$$

which fix a consistent $\mathcal{C}^\infty$ structure on P.

The vector bundle W can be recovered from the principal bundle P as

$$W = P \times \mathbb{R}^n / \mathrm{GL}(n, \mathbb{R}),\ G(p, v) = (Gp, Gv). \tag{L3.46}$$

In this way we can easily define the radial compactification of W by taking the extension of the $\mathrm{GL}(n, \mathbb{R})$ action to $\mathbb{S}^{n,1}$ and so setting

$$\overline{W} = P \times \mathbb{S}^{n,1} / \mathrm{GL}(n, \mathbb{R}),\ G(p, q) = (Gp, Gq),\ W \hookrightarrow \overline{W} \tag{L3.47}$$

embeds W as the interior of a compact manifold with boundary. Thus we have defined the corresponding 'symbol spaces'

$$\rho^{-m} \mathcal{C}^\infty(\overline{W}; \Omega_{\mathrm{fib}}),\ \Omega_{\mathrm{fib}} = \Omega W. \tag{L3.48}$$

where ρ is as before a defining function for the boundary (which always exists globally).

Thinking in terms of transition maps for local trivializations suppose that U_i, U_j are elements of $\mathcal{U}$ (and a finite number of its elements must cover by the compactness of Y) the two maps (L3.42) combine over $U_{ij} = U_i \cap U_j$, assuming this is non-empty, to give a smooth map

$$h_{ij} : U_{ij} \longrightarrow \mathrm{GL}(n, \mathbb{R}),\ h_{ij}(y) = F_{U_i} \circ F_{U_j}^{-1}. \tag{L3.49}$$

Then the vector bundle can be thought of as the union of the $U_i \times \mathbb{R}^n$ with these identifications over U_{ij}. The fact that the spaces (L3.48) are well defined reduces to the $\mathrm{GL}(n, \mathbb{R})$-invariance of the radial compactification, which of course we used in the 'high road' definition above.

Now we can extend the definition (L3.41) from the product case to the general bundle case by setting

$$I^m_{\mathcal{S}}(W, 0_W) = \mathcal{F}_{\mathrm{fib}}^{-1}\left(\rho^{-m+\frac{d}{4}-\frac{n}{4}} \mathcal{C}^\infty(\overline{W'}; \Omega_{\mathrm{fib}})\right), \\ d = \dim Y,\ n = \dim W. \tag{L3.50}$$

I am being a little casual about the fibrewise Fourier transform but we can see that it all makes sense by the local trivialization approach. In fact the global behaviour in the base is not a big issue. I have done it this way so that the bookkeeping is fairly straightforward.

What bookkeeping? Well, the important property here, that we used repeatedly in the construction at the beginning of the lecture, is the short exact sequence which became (L3.38) in the invariant notation for a vector space. Now we get

$$I^{m-1}_{\mathcal{S}}(W,O_W)\longrightarrow I^m_{\mathcal{S}}(W,O_W)\xrightarrow{\sigma_m}\mathcal{C}^\infty(\mathbb{S}W';N_{-m'}\otimes\Omega W'),\\ m'=m-\frac{d}{4}+\frac{n}{4},\ d=\dim Y,\ n=\dim W_y \tag{L3.51}$$

is exact.

Exercise 14. Check that you do understand what (L3.51) means and how to prove it. In a nutshell, the space $\mathbb{S}W'$ is the boundary of the radial compactification of W' and the surjectivity of the second map corresponds to the fact that every element of $\rho^{-m+\frac{d}{4}-\frac{n}{4}}\mathcal{C}^\infty(Y\times\overline{W'};\Omega_{\text{fib}})$ corresponds to a (unique) conormal distribution by (L3.50). The injectivity on the right is almost by definition and the exactness in the middle is precisely the fact that an element of $\mathcal{C}^\infty(\overline{W'})$ which vanishes on $\mathbb{S}W'$ is an element of $\rho\mathcal{C}^\infty(\overline{W'})$ and conversely.

Finally let me review what we need for the next step, to define $I^m(X,Y)$ where Y is a compact embedded submanifold of a compact manifold X using the Collar Neighbourhood theorem. From the definition above it is immediate, or rather built into the definition that if $g:W\longrightarrow W$ is a bundle isomorphism then

$$g^*:I^m(W,0_W)\longrightarrow I^m(W,0_W). \tag{L3.52}$$

This is true whether g projects to the identity on the base (the usual meaning of a bundle isomorphism) or projects to a non-trivial diffeomorphism of the base.

The point is that this is by no means strong enough for what we want. Indeed we will need to consider a *diffeomorphism* between neighbourhoods of the zero section, N and N', of W but which need not preserve the fibres and even if it does, need not be linear. Of course it will be assumed to map the zero section into itself, otherwise (L3.52) could not possibly hold. Moreover, because of the known invariance under bundle isomorphisms we can assume a bit more. First we shall require that

$$g:0_W\longrightarrow 0_W \text{ is the identity.} \tag{L3.53}$$

Now, this means that at each point $y\in 0_W$ (which is just Y) the tangent space to 0_W is mapped into itself as the identity too. The quotient

$$T_yW/T_y0_W=T_yW_y=W_y \tag{L3.54}$$

is naturally identified with the fibre of W through the point. So it makes sense to add a second condition to (L3.53) on the differential (i.e. the Jacobian) of g at each point of 0_W :

$$g_*:W_y\longrightarrow W_y \text{ is the identity.} \tag{L3.55}$$

Lemma 10. *Any diffeomorphism of a neighbourhood of 0_W onto its image in W which maps 0_W onto itself pointwise can be factorized as $g\cdot h$ where h is a bundle isomorphism and g satisfies* (L3.53) *and* (L3.55).

So, we want to show that (L3.52) holds for g as in (L3.53), (L3.55). Of course it only makes sense to apply g^* to functions or distributions with support in the image set N' but this is no problem since outside any given neighbourhood of 0_W we already know that our conormal distributions are smooth.

Let me check what (L3.53) and (L3.55) mean in local coordinates. If we take a local trivialization of W over some open set $U \subset Y$ and use coordinates y in U and fibre coordinates z in $W_U = U \times \mathbb{R}^n$ then

$$g(y,z) = (y + \sum_j z_j m_j(y,z), z + \sum_{ij} z_i z_j a_{ij}(y,z)) \tag{L3.56}$$

where the m_j and a_{ij} are just some smooth functions. This follows by writing $g(y,z) = (Y(y,z), Z(y,z))$. The fact that $0_W = \{z = 0\}$ is mapped into itself means $Z(y,0) = 0$, the fact that this map on Y is the identity means $Y(y,0) = y$ which gives the first part of (L3.56) and then the part of the Jacobian in (L3.55) is just $\partial Z/\partial z(y,0) = \mathrm{Id}$ which gives the second part of (L3.56).

Now, to show (L3.52) we will use 'Moser's method'. This is based first on the fact that the map in (L3.56) is connected to the identity by a curve of diffeomorphisms (possibly in a smaller neighbourhood of the zero section) of the same type. Locally (and that is all that really matters) this is clear since we can consider

$$g_s(y,z) = (y + s\sum_j z_j m_j(y,z), z + s\sum_{ij} z_i z_j a_{ij}(y,z)). \tag{L3.57}$$

So we want to show that $g_1^* u = v \in I^m_{\mathcal{S}}(W, 0_W)$ if $u \in I^m_{\mathcal{S}}(W, 0_W)$ (and has support sufficiently close to 0_W). Now, the clever idea of Moser (not in this context) is to try to construct a smooth curve

$$u_s \in \mathcal{C}^\infty([0,1]_s; I^m_{\mathcal{S}}(W, 0_W)) \text{ s.t. } u_1 = u \text{ and } \frac{d}{ds} g_s^* u_s = 0. \tag{L3.58}$$

If we could do this (and actually we can) then we conclude that $g_s^* u_s$ is constant, so

$$g^* u = g_1^* u_1 = g_0^* u_0 = u_0 \in I^m_{\mathcal{S}}(W, 0_W). \tag{L3.59}$$

So, why might we expect to be able to do this? Well, the 'trick' here is the identity

$$\frac{d}{ds} g_s^* u_s = g_s^* (\frac{du}{ds} + V_s u_s) \tag{L3.60}$$

where V_s is a vector field determined by g_s. Once we work out what this vector field is, we need to choose u_s to satify, in addition to (L3.58),

$$\frac{du}{ds} + V_s u_s = 0. \tag{L3.61}$$

The remarkable thing is that g_s has disappeared, we only need to consider its 'infinitesmal generator' V_s.

3+. Addenda to Lecture 3

3+.1. Densities. If U is any finite dimensional (complex) vector space we set

$$I^m_{\mathcal{S}}(W, \{0\}; U) = I^m_{\mathcal{S}}(W, \{0\}) \otimes_{\mathbb{C}} U \tag{3+.62}$$

and identify it as the 'space of conormal distributions with values in U.' (Of course you can do this with all distributions, etc).

Exercise 15. Check that the Fourier transform gives an isomorphism

$$cF : I^m_{\mathcal{S}}(W, \{0\}; U) \longrightarrow \rho^{-m'}\mathcal{C}^\infty(\overline{W'}; U \otimes \Omega W'),\ m' = m + \frac{n}{4}. \tag{3+.63}$$

Show further that there is a canonical isomorphism $\Omega U' = (\Omega U)'$ for any vector space, and hence that $\Omega U' \otimes \Omega U \equiv \mathbb{C}$ (or $\mathbb{R}$ if U is real) is canonically trivial. Hence (or directly) show that the integration map (L3.1) gives a linearly-invariant map

$$\int_{\mathbb{R}^n} : I^m_{\mathcal{S}}(W, \{0\}; \Omega W) \longrightarrow \mathbb{C} \tag{3+.64}$$

(as it should).

3+.2. Properties of conormal distributions.

3+.3. The Thom class.

3+.4. Submanifolds and restriction.