CHAPTER 24

Index theorems and applications

Lecture 24: 13 December, 2005

L24.1. Inadequacies and extensions. Today I will sketch some conjectural extension of the theorem index theorem I talked about last time and some other applications of the things I have been talking about. In fact I can list a few extra lectures, or topics within lectures, that I would like to do or have done. Starting with the latter category are a couple of topics that I feel I have covered somewhat inadequately (but I will likely put something in the addenda to the notes).

- The discussion of stabilization.
- Isotropic calculus and proper coverage of Bott/Thom/Todd – respectively elements, isomorphism(s) and class. In particular at this stage I have not really described the Todd class at all. Give an oriented real vector bundle $V \longrightarrow M$, say over a compact manifold M, the Thom isomorphism in cohomology is the identification

$$H^*_{\mathrm{c}}(V) \longrightarrow H^*(M) \tag{L24.1}$$

given by fibre integration – it is always an isomorphism. On the other hand given a *complex* vector bundle, which we can also denote V, there is an extension of the Bott isomorphism (which is the case that V is trivial)

$$K_{\mathrm{c}}(V) \longrightarrow K(M). \tag{L24.2}$$

Both for the compactly supported K-theory and the K-theory of the base there are Chern character maps – as we have discussed. This gives a diagramme

$$\begin{tikzcd} K_{\mathrm{c}}(V) \arrow[r, "\mathrm{Thom}"] \arrow[d, "\mathrm{Ch}"'] & K(M) \arrow[d, "\mathrm{Ch}"] \\ H^{\mathrm{evn}}_{\mathrm{c}}(V) \arrow[r, "\mathrm{Thom}", "\times\,\mathrm{Td}"'] & H^{\mathrm{evn}}(M). \end{tikzcd} \tag{L24.3}$$

The problem here is that if we simply take the Thom isomorphisms top and bottom then the diagramme does not commute. This is not totally surprising, since the maps Thom isomorphism are defined under different conditions. To get an isomorphism we have to follow the Thom isomorphism on the bottom, in cohomology, by multiplication by a characteristic class. This is the Todd class, Td, of V. The class that appears in the

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Atiyah-Singer formula is the Todd class of M which is by definition the Todd class of the complexified Tangent bundle of M.
- The computation of $\mathrm{Ch}(P_A)$, the Chern class of the the 'extension' of the original elliptic family. This is very close to the relative Chern character that I have discussed and I will certainly try to put it in somewhere. The Todd class comes out quite naturally from this computation.
- General product-tye index theorem in K-theory.
- Product-type index theorem in cohomology.
- Odd families and determinants.
- Determinant line bundle.
- Gerbes.

I will briefly describe what I think is going on as regards the index theorem for product type operators and quickly indicate how to define the determinant line bundle and gerbes – I don't think I will have time to mention the odd index theorem and determinants, although that is very closely related to the eta invariant and so things I have been talking about. It is possible that I will be motivated enough to write out some more ficticious lectures.

L24.2. Product-type K-theory. So, to talk about the formulation of an index theorem, in K-theory, associated to product-type pseudodifferential operators, let me recall the Atiyah-Singer index theorem, again. This is really at least two theorems. In K-theory it states that for any fibration of compact manifolds

$$\begin{tikzcd} M \arrow[d, "\phi"] \\ B \end{tikzcd} \tag{L24.4}$$

there are two different maps in K-theory which are equal

$$K^0_{\mathrm{c}}(T^*(M/B)) \underset{\mathrm{ind_t}}{\overset{\mathrm{ind_a}}{\longrightarrow}} K^0(B). \tag{L24.5}$$

The top map, the analytic index, is defined by identifying elements of the K-group as triples $(\mathbb{E}, a)$ in which a may be identified as the symbol of a family of elliptic operators, $A \in \Psi^0(M/B; \mathbb{E})$ and then $\mathrm{ind_a}([(\mathbb{E}, a)])$ is the image of the (stabilized) index bundle in $K^0(B)$. The other map is defined via 'geometric trivialization', in which the fibration is embedded as a subfibration of a product fibration $\mathbb{S}^N \times B$.

To extend this to the product-type case we consider a compound fibration of compact manifolds

$$\begin{tikzcd} M \arrow[d, "\psi"] \arrow[dd, bend left, "\phi=\Phi\psi"] \\ Y \arrow[d, "\Phi"] \\ B. \end{tikzcd} \tag{L24.6}$$

Here the choice of notation indicates that it is the overall fibration which is analogous to (L24.4). Let me denote by $\Psi^{0,0}\psi - \mathrm{pt}(M/B; \mathbb{E})$ the space of product-type pseudodifferential operators for the compound fibration. For each point $b \in B$ this is an operaor on the fibre of ϕ which is of product type with respect to the fibration ψ (over $\Phi^{-1}(b)$. Given what we have done above, the definition should be fairly self-evident. Such an operator has two 'symbols' and ellipticity means invertibility of both

$$\begin{aligned} \sigma(A) &\in \mathcal{C}^\infty([S^*(M/B), S^*(X/B)]; \hom(\mathbb{E})) \\ \beta(A) &\in \Psi^0(\pi^* M/S^*(X/B); \mathbb{E}) \end{aligned} \tag{L24.7}$$

where, with some abuse of notation, $\pi : S^*(X/B) \longrightarrow X$ and the fibration in the second case is the pull-back of $M \longrightarrow X$ to $S^*(X/B)$. In this case the analytic index is well defined by stabilization of any elliptic family, $\mathrm{ind_a}(A) \in K^0(B)$, and only depends on $\sigma(A)$ and $\beta(A)$.

There is every reason (meaning I have not checked the details) to think that we can define an adequate replacement for $K^0_{\mathrm{c}}(T^*(M/B))$ in this setting. Namely, if we consider all invertible pairs (L24.7), subject to the compatibility condition that the symbol of β is the restriction of σ to the boundary, and then impose an equivalence condition corresponding to bundle isomorphism, stabilization and homotopy, then we arrive at an Abelian group which I will denote $K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)$. There will also be an odd version of this, $K^{-1}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)$. Of course the basic idea is that the analytic index descends to this space and defines

$$\mathrm{ind_a} : K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B) \longrightarrow K^0(B). \tag{L24.8}$$

Assuming this construction does work we will get a map generated by σ,

$$K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B) \longrightarrow K^0_{\mathrm{c}}(T^*(M/X). \tag{L24.9}$$

This comes from evaluating the symbol in the 'vertical' directions of the fibration. The manifold with boundary on which σ is defined fibres over $S^*(M/X)$ and this implies that the homotopy class of σ, ignoring β and the compatibility condition, is actually determined by the image in (L24.9). The families index theorem for the pull-back of fibration of M over X to $S^*(X/B)$ gives rise to a second map

$$\mathrm{ind} : K^0_{\mathrm{c}}(T^*(M/X) \longrightarrow K^0(S^*(X/B) \tag{L24.10}$$

which vanishes on the image of (L24.9), expressing the fact that the symbol *is* the symbol of a family of invertible operators there – namely the $\beta(A)$. The space $K^{-1}(S^*(X/B))$ can be identified with[1] homotopy classes of maps into a $G^{-\infty}$. We can take this to be smooth families β which therefore give a pair of symbols (L24.7) with true symbolic part the identity. Putting all this together we arrive at the *conjectural* 6-term exact sequence

$$\begin{tikzcd} K^{-1}(S^*(X/B)) \arrow[r, "\iota"] & K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \arrow[r, "\sigma"] & K^0_{\mathrm{c}}(T^*(M/X)) \arrow[d, "\mathrm{ind}"] \\ K^{-1}_{\mathrm{c}}(T^*(M/X)) \arrow[u, "\mathrm{ind}"] & K^{-1}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \arrow[l, "\sigma"] & K^0(S^*(X/B)) \arrow[l, "\iota"] \end{tikzcd} \tag{L24.11}$$

The analytic index should be compatible with this complex (it is also defined on $K^{-1}(S^*(X/B))$ and it should be possible to define a toplogical index by embedding and arrive at a (conjectural at this stage) extension of (L24.5):

$$K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \xrightarrow[\mathrm{ind_t}]{\mathrm{ind_a}} K^0(B). \tag{L24.12}$$

[1]In fact was defined here as

QUESTION 1. Is there some more traditional realization of the group $K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B))$ – the 6-term exact sequence (L24.11) should serve as a guide to what this might be.

L24.3. Product-type cohomology. In the standard case, the second form of the Atiyah-Singer theorem, or if you prefer 'the Atiyah-Singer formula', gives Chern character of the index bundle in terms of the Chern character of the symbol. So here we expect to get a Chern character

$$\mathrm{Ch}_{\psi-\mathrm{pt}}: K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \longrightarrow H^{\mathrm{evn}}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \tag{L24.13}$$

where both the map and the image space are yet to be determined. One thing we expect is that the 6-term exact sequence will be replicated at this level and in fact will correspond to a natural transformation (i.e. be functorial) from the K-theory complex to the 'cohomological complex':

$$\begin{tikzcd}
H^{\mathrm{odd}}(S^*(X/B)) \arrow[r, "\iota"] & H^{\mathrm{evn}}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \arrow[r, "\sigma"] & H^{\mathrm{evn}}_{\mathrm{c}}(T^*(M/X)) \arrow[d, "\mathrm{ind}"] \\
H^{\mathrm{odd}}_{\mathrm{c}}(T^*(M/X)) \arrow[u, "\mathrm{ind}"] & H^{\mathrm{odd}}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \arrow[l, "\sigma"] & H^{\mathrm{evn}}(S^*(X/B)) \arrow[l, "\iota"]
\end{tikzcd} \tag{L24.14}$$

with the maps from (L24.11) all being the corresponding Chern characters
(L24.15)

$$\begin{tikzcd}
K^{-1}(S^*(X/B)) \arrow[rr, "\iota"] \arrow[dr, "\mathrm{Ch}_{\mathrm{odd}}"] & & K^0_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \arrow[rr, "\sigma"] \arrow[d, "\mathrm{Ch}_{\psi-\mathrm{pt}}"] & & K^0_{\mathrm{c}}(T^*(M/X)) \arrow[dl, "\mathrm{Ch}_{\mathrm{evn}}"] \arrow[ddd, "\mathrm{ind}"] \\
& H^{\mathrm{odd}}(S^*(X/B)) \arrow[r, "\iota"] & H^{\mathrm{evn}}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \arrow[r, "\sigma"] & H^{\mathrm{evn}}_{\mathrm{c}}(T^*(M/X)) \arrow[d, "\mathrm{ind}"] & \\
& H^{\mathrm{odd}}_{\mathrm{c}}(T^*(M/X)) \arrow[u, "\mathrm{ind}"] & H^{\mathrm{odd}}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \arrow[l, "\sigma"] & H^{\mathrm{evn}}(S^*(X/B)) \arrow[l, "\iota"] & \\
K^{-1}_{\mathrm{c}}(T^*(M/X)) \arrow[uuu, "\mathrm{ind}"] \arrow[ur, "\mathrm{Ch}_{\mathrm{odd}}"] & & K^{-1}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \arrow[ll, "\sigma"] \arrow[u, "\mathrm{Ch}_{\psi-\mathrm{pt}}"] & & K^0(S^*(X/B)) \arrow[ll, "\iota"] \arrow[ul, "\mathrm{Ch}_{\mathrm{evn}}"]
\end{tikzcd}$$

So, what should $H^k_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B))$ be? The anticipated form of the Chern character is the guide here. Let me try to be a little abstract here and consider a more general setting of a manifold with boundary Z with a fibration $\psi : \partial Z \longleftrightarrow Y$ of its boundary. In the present setting, $Z = [\overline{T^*(X/B)}, S^*X]$ with the 'old' boundary removed (because we really want to consider relative cohomology, i.e. with compact supports, as far as this part of the boundary is concerned.) Thus, we will not assume that Z is compact but we do assume that Y is compact. Then consider pairs

$$(u, \tau) \in \mathcal{C}^\infty_c(X; \Lambda^k) \times \mathcal{C}^\infty(Y; \lambda^j) \tag{L24.16}$$

where $k = j + d$, d being the fibre dimension of the fibration ψ. This pair is closed if $du = 0$ as a smooth form and

$$\iota^*_{\partial Z} u = \psi^* d\tau. \tag{L24.17}$$

The pair is exact if there exists $(u', \tau') \in \mathcal{C}^\infty_c(Z; \Lambda^{k-1}) \times \mathcal{C}^\infty(Y; \Lambda^{j-1})$ such that $du' = u$ and $\iota^*_{\partial Z} u' = \psi^* \tau'$.

Note that this definition is modelled on relative cohomology.

Exercise 21. Check that in the case that $\psi : \partial Z \longrightarrow \partial Z$ on recovers the compactly supported cohomology of $\operatorname{int} Z = Z \backslash \partial Z$ which is the same as $H^k_{\mathrm{c}}(Z, \partial Z)$.

Problem 1. Show that this cohomology is well-defined and gives a 6-term exact sequence as in (L24.14). Try to identify the cohomology with intersection cohomology of the stratified space Z/ψ in which the boundary of Z is 'smashed' to Y.

The idea of this definition is that the Chern character (L24.13) is supposed to be given by that pair $(u, \tau) = (\mathrm{Ch}(\sigma), \eta(\beta)$ where $\mathrm{Ch}(\sigma)$ is the relative Chern charcter *form* (i.e. the same formula as before but smoothly up to the boundary and η [2] is the eta form as discussed above. The relationship $\iota^* \partial u = \psi^* \tau$ is just what is I showed last in the special case of a product with a circle, but the argument should go over in general when connections are put in.

It is also natural to expect that in the case of a fibration[3], as for $T^*(M/B)$ where the structure is fibrewise, there should be a pushforward map

$$H^{\mathrm{evn}}_{\mathrm{c},\psi-\mathrm{pt}}(T^*(M/B)) \longrightarrow H^{\mathrm{evn}}(B) \tag{L24.18}$$

L24.4. Determinant bundle. Next, a few words about the determinant bundle.

The numerical index is the 0-dimensional part of the Chern character of the index in the standard families case. In the odd case (which I have not discussed) the Chern character maps to odd-degree cohomology and the 1-dimensional part can be thought of as a 'spectral flow' of the phase of a determinant. Back in the usual even setting, the 2-dimensional part of the Chern character corresponds to the determinant bundle, as I will discuss. In the odd case the 3-dimensional part corresponds to the curvature of a gerbe, which I had hoped to get to. The 4-dimensional part of the even Chern character should correspond to a 2-gerbe, but this is not very well understood geometrically.

Back in the usual families case we consider and elliptic family $P \in \Psi^0(M/B; \mathbb{E})$ and for simplicity we assume that the numerical index

$$\# - \operatorname{ind}(P) = \dim \operatorname{null}(P) - \dim \operatorname{null}(P^*) = 0. \tag{L24.19}$$

It is of course constant. This is not strictly necessary but definitely simplifies the construction.

The vanishing of the numerical index means that for each point $b \in B$ the operator $P_b \in \Psi^0(Z_b; \mathbb{E}_b)$ can be perturbed by a smoothing operator to be invertible. This allows us to define a big bundle over B where the fibre at b is

$$\mathcal{P}_b = \{P_b + Q_b; Q_b \in \Psi^{-\infty}(Z_b; \mathbb{E}_b); (P_b + Q_b)^{-1} \in \Psi^0(Z_b; \mathbb{E}^-_b). \tag{L24.20}$$

Not only is each fibre non-empty, but it is a principal space for the action of the group $G^{-\infty}(Z_b; E_{b,-})$. Namely, two elements $P_b + Q_b$, $P_b + Q'_b \in \mathcal{P}_b$ must be related by

$$P_b + Q_b = (\mathrm{Id} + R_b)(P_b + Q'_b), \ R_b \in G^{-\infty}(Z_b; E_{b,-}) \subset \Psi^{-\infty}(Z_b; E_{b,-}). \tag{L24.21}$$

[2] Remember I am not claiming that the normalization is correct (yet).

[3] With oriented fibres

Thus we can think of

$$\begin{array}{c} \mathcal{P} \\ \big\downarrow \\ B \end{array} \tag{L24.22}$$

as a 'principal bundle' although the groups acting on the fibres are actually varying and form a bundle of groups, $G^{-\infty}(M/B;E_-)$. In the discussion earlier on the stabilization of the index bundle I faced[4] the issue of showing that there are 'exhausting' smooth families of projections in $\Psi^{-\infty}(M/B;E_-)$. Using these one can see that the space of components of the sections, $g \in \mathcal{C}^\infty(B;G^{-\infty}(M/B;E_-)$ is actually canonicaly equal to $K^{-1}(B)$. That is, the bundle of groups is at least 'weakly trivial'. Thus (L24.22) does behave very much as a principal bundle.

Recall that the Fredholm determinant is a multiplicative function

$$G^{-\infty}(M/B);E_-) \longrightarrow \mathbb{C}^* \tag{L24.23}$$

defined globallly and invariantly. In this sense it is a 1-dimensional representation of our bundle of groups, or a character if you prefer. As such the principal bundle (L24.22) induces a line bundle (a 1-dimensional vector bundle) over B. The fibre at $b \in B$ is

(L24.24)

$$B_b = (\mathcal{P}_b \times \mathbb{C})/\sim,\ (P_b+Q_b,z) \sim (P_b+Q_b',z') \Longleftrightarrow \text{(L24.21) holds and } z = \det(\mathrm{Id}+R_b)z'.$$

24+. Addenda to Lecture 24

[4]And shelved for some time.

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