CHAPTER 19

Multiplicativity and excision

Lecture 19: 22 November, 2005

L19.1. Multiplicativity. Last time I set up the following single operator version of multiplicativity but did not complete the proof. The general case is no harder, the notational overhead is just heavier.

PROPOSITION 43. *[Multiplicativity] Consider a 'tower' of compact fibrations*

(L19.1)
$$\begin{tikzcd} S \arrow[r, dash] & \tilde M \arrow[d, "\tilde\phi"] \\ Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B \end{tikzcd}$$

and suppose that $P \in \Psi^0(\tilde M/M;\mathbb{E})$ is an elliptic family with trivial one-dimensional index bundle then for any ellptic family $A \in \Psi^0(M/B;\mathbb{F})$

(L19.2)
$$\operatorname{ind}(A) = \operatorname{ind}(P \otimes A) = \operatorname{ind}(A_P) \in K^0(B)$$

where $P \otimes A$ is the product-type family

(L19.3)
$$P \otimes A = \begin{pmatrix} P & 0 \\ A_{\operatorname{null}(P)} & P^* \end{pmatrix}$$

and $A_P \in \Psi^0(\tilde M/B;\mathbb{E}\otimes\mathbb{F})$ is any family with symbol

(L19.4)
$$\begin{pmatrix} \chi_1\sigma_0(P) & -\chi_2\sigma_0(A)^* \\ \chi_2\sigma(A)) & \chi_1\sigma_0(P)^* \end{pmatrix}$$

where χ_1, χ_2 is a partition of unity on $S^*(\tilde M/B)$ subordinate to the cover by the complements of $\tilde\phi^*(S^*(M/B))$ and $S^*(\tilde M/M)$ for some choice of connection on $\tilde\phi$.

PROOF. I set this up last time in the single operator case, where the bottom fibration just has one fibre and B is a point. Formally the general case is not very different. Thus, by assumption, the null space of the family P is a trivial line bundle over M. We can make A act between sections of $E_+ \otimes \tilde\phi^*F_+$ and $E_+ \otimes \tilde\phi^*F_-$ by considering it as the composite

(L19.5)
$$A_{\operatorname{null}(P)} = \pi_{\operatorname{null}(P)} A \pi_{\operatorname{null}(P)} : \Psi^{0,-\infty}_{\tilde\phi\text{-pt}}(\tilde M/B; E_+ \otimes \mathbb{F})$$

where I am using $\mathbb{F}$ to stand for the 'superbundle' $(F_+, F-)$ and $E_+ \otimes \mathbb{F}$ stands for $(E_+ \otimes F_+, E_+ \otimes F_-)$. Then (L19.3) is a well-defined family of product-type operators (it is a family over B, and of product-type with respect to $\tilde{\phi}$.)

We know by direct computation that its index is $\operatorname{ind}(A)$, at least if A has been stabilized to have a smooth null bundle. Namely, the null space of $P \otimes A$ consists of pairs $(u, v) \in \mathcal{C}^\infty(\tilde{M}; E_+ \otimes F_+) \oplus \mathcal{C}^\infty(\tilde{M}; E_+ \otimes F_-)$ satisfying

$$
\begin{aligned}
&Pu = 0 \Longrightarrow u \in \mathcal{C}^\infty(M; E_+ \otimes \operatorname{null}(P)),\\
&A_{\operatorname{null}(P)} u + P^* v = 0 \Longrightarrow Au = 0,\ v = 0.
\end{aligned}
\tag{L19.6}
$$

In the second line we use the fact that the range of P^* is orthogonal to the null space of P so the two terms must vanish separately. Then $Au = 0$ just recovers the null space of A. For the adjoint

$$
(P \otimes A)^* = \begin{pmatrix} P^* & A^*_{\operatorname{null}(P)} \\ 0 & P \end{pmatrix}
\tag{L19.7}
$$

we similarly conclude that (u', v') in the null space implies that $u = 0$, $v \in \mathcal{C}^\infty(M; E_- \otimes \operatorname{null}(P))$ and then $u \in \operatorname{null}(A^*)$.

We also 'know' (I only did it in the single operator case in fact) that a homotopy of totally elliptic product-type pseudodifferential operators has constant index in K-theory of the base; I will add this to the addenda. So we proceed to deform the family (L19.3) but keeping total ellipticity. Recall that the family $P \otimes A$ is totally elliptic because it symbol and base symbol are respectively

$$
\begin{aligned}
\sigma_{0,0}(P \otimes A) &= \begin{pmatrix} \sigma_0(P) & 0 \\ 0 & \sigma_0(P)^* \end{pmatrix}\\
\beta_0(P \otimes A) &= \begin{pmatrix} P & 0 \\ \sigma_0(A)\pi_{\operatorname{null}(P)} & P^* \end{pmatrix}.
\end{aligned}
\tag{L19.8}
$$

Now, we have the partition of unity χ_1, χ_2 on $S^*(\tilde{M}/B)$ in which χ_2 is supported near the lift of $\overline{T^*(M/B)}$ under $\tilde{\phi}$. This means that $\chi_2\sigma_0(A)$ is a well-defined symbol in a neighbourhood of the 'non-commutative' front face – on the fibres of which it is constant. Take an element $\tilde{A} \in \Psi^0(\tilde{M}/B; E_+ \otimes \mathbb{F})$ which has symbol $\chi_2\sigma_0(A)$ and consider the curve of operators

$$
\begin{pmatrix} P & -t\tilde{A}^* \\ (1-t)A_{\operatorname{null}(P)} + t\tilde{A} & P^* \end{pmatrix} \in \Psi^{0,0}_{\tilde{\phi}\text{-pt}}(\tilde{M}/B; \mathbb{E} \otimes \mathbb{F}).
\tag{L19.9}
$$

The claim is that this remains elliptic. Its usual symbol is just

$$
\begin{pmatrix} \sigma_0(P) & -t\chi_2\sigma_0(A)^* \\ t\chi_2\sigma(A) & \sigma(P)^* \end{pmatrix}.
\tag{L19.10}
$$

The crucial property of this (Clifford) tensor product matrix is that it is invertible because the 'diagonal' part is invertible. Consider an element (α, α') of the null space. Note that $\sigma(P)$ and $\sigma(A)$ commute, because the act on different factors of the tensor product, so

$$
\begin{aligned}
&\sigma_0(P)\alpha - t\chi_2\sigma_0(A)^*\alpha' = 0,\ t\chi_2\sigma(A)\alpha + \sigma(P)^*\beta = 0 \Longrightarrow\\
&\qquad \alpha = t\chi_2\sigma_0(A)^*\sigma_0(P)^{-1}\alpha' = -t^2\chi_2^2\sigma_0(A)^*\sigma_0(A)(\sigma(P)^*)^{-1}\sigma_0(P)^{-1}\alpha.
\end{aligned}
\tag{L19.11}
$$

Thus the null space is trivial, because of the invertibility of $\sigma_0(P)$ and hence the operator is 'symbolically' elliptic. The non-commutative, or base symbol is

$$\begin{pmatrix} P & -t\chi_2\sigma_0(A) \\ (1-t)\sigma_0(A)_{\mathrm{null}(P)} + t\sigma_0(A) & P^* \end{pmatrix} \tag{L19.12}$$

since (as a fibre family) P is its own base symbol. We know this to be invertible for $t = 0$ and for $t > 0$ a similar argument applies. The symbol preserves the decomposition coming from the null space of P. On it, it is invertible because it is actually constant in t. Off the null space of P it is invertible because of the invertibility of P and an argument just like (L19.11) but now using P instead of its symbol. Note that P and $\sigma_0(A)$ commute because the latter is fibre constant for $\tilde{\phi}$ and acts on a different bundle in the tensor product. Thus we arrive at the operator with $t = 1$. Now choose an element

$$\tilde{P} \in \Psi^0(\tilde{M}/B; \mathbb{F} \otimes \mathbb{E}_+) \text{ with } \sigma_0(\tilde{P}) = \chi_1\sigma_0(P) \tag{L19.13}$$

and perform the homotopy

$$\begin{pmatrix} (1-t)P + t\tilde{P} & -\tilde{A}^* \\ \tilde{A} & (1-t)P^* + t\tilde{P}^* \end{pmatrix} \in \Psi^{0,0}_{\tilde{\phi}\text{-pt}}(\tilde{M}/B; \mathbb{E} \otimes \mathbb{F}). \tag{L19.14}$$

The 'commutative' symbol of this is

$$\begin{pmatrix} (1-t)\sigma_0(P) + t\chi_1\sigma_0(P) & -\chi_2\sigma_0(A)^* \\ \chi_2\sigma(A) & (1-t)\sigma(P)^* + t\chi_2\sigma_0(P)^* \end{pmatrix}$$

which remains invertible everywhere. Similarly the 'non-commutative' symbol is

$$\begin{pmatrix} (1-t)P & -\sigma_0(A) \\ \sigma_0(A)_{\mathrm{null}(P)} & (1-t)P^* \end{pmatrix}$$

which is invertible because of the invertibilty of $\sigma_0(A)$. Thus the family remains elliptic throughout the deformation and we finally arrive at

$$\begin{pmatrix} \tilde{P} & -\tilde{A}^* \\ \tilde{A} & \tilde{P}^* \end{pmatrix} \in \Psi^0(\tilde{M}/B; \mathbb{E} \otimes \mathbb{F}) \tag{L19.15}$$

which is an elliptic family in the usual sense with symbol (L19.4). Thus (L19.2) follows. □

Corollary 7. *For an iterated fibration* (L19.1)*, if* $b \in K^0_c(T^*(\tilde{M}/M))$ *has* $\mathrm{ind}(b) = 1 \in K^0(M)$ *then tensor product gives a commutative diagramme*

$$\begin{tikzcd} K^0_c(T^*(M/B)) \arrow[rr, "\otimes b"] \arrow[dr, "\mathrm{ind}"'] & & K^0_c(T^*(\tilde{M}/B)) \arrow[dl, "\mathrm{ind}"] \\ & B & \end{tikzcd} \tag{L19.16}$$

where for $[(\mathbb{E}, a)] \in K^0_c(T^*(M/B))$, $b \otimes [(\mathbb{E}, a)]$ *is represented by*

$$\begin{pmatrix} \chi_1 b & -\chi_2 a \\ \chi_2 a & \chi_1 b^* \end{pmatrix} \tag{L19.17}$$

where $b = [(\mathbb{F}, b)]$.

Proof. The main thing to check is that the top map in (L19.16) is well-defined, using (L19.17). This is straightforward. □

L19.2. Excision.

PROPOSITION 44. *[Excision.] Let $M_i \longrightarrow B$ be fibrations of compact manifolds and suppose $i_j : E \hookrightarrow M_j$, $j = 1, 2$, are inclusions of a non-compact manifold as an open subset giving a commutative diagramme,*

(L19.18)
$$\begin{tikzcd} M_1 \arrow[dr, "\phi_1"'] & U \arrow[l, hook', "i_1"'] \arrow[r, hook, "i_2"] & M_2 \arrow[dl, "\phi_2"] \\ & B & \end{tikzcd}$$

then the diagramme

(L19.19)
$$\begin{tikzcd} & K^0_c(T^*(M_1/B)) \arrow[dr, "\mathrm{ind}"] & \\ K^0_c(T^*(U/B)) \arrow[ur, "(i_1)_*"] \arrow[dr, "(i_2)_*"'] & & K^0(B) \\ & K^0_c(T^*(M_2/B)) \arrow[ur, "\mathrm{ind}"'] & \end{tikzcd}$$

is commutative.

PROOF. The main issue is to understand the maps $(i_j)_*$ induced by the inclusions. A representative of an element of $K^0_c(T^*(U/B))$ is a triple (E_+, E_-, a) where $E_\pm$ are bundles over U and a is an isomophism between the lifts of them outside a compact subset $K \subset T^*(U/B)$. The fact that the two fibrations are the same on U means that $T^*(U/B)$ is a well-defined bundle over U, identified by the i_j with $T^*)i_j(U)(M_j/B)$. The image, K', of K under projection to U is compact and a is therefore defined over the whole of the bundle $T^*_{U\setminus K'}(U/B)$. We can use the restriction of a to the zero section to identify the two bundles E_+ and E_- over $U \setminus K''$, where K'' is the image of a slightly larger compact subset of $T^*(U/B)$ which contains K' in its interior and having done this use the fibre homogeneity of the bundle to give a homotopy between a and a' which is now the identity isomorphism between E_+ and E_- in $U \setminus K''$. Now, recall from the definition that E_+ and E_- are in any case supposed to be trivial outside a compact set, so we may replace (E_+, E_-, a) by a representative in which $E_\pm$ are trivial outside a compact subset of U and $a = \mathrm{Id}$ outside such a set. Of course a need not be the identity outside a compact subset of $T^*(U/B)$. Then the maps are given by extending $E_\pm$ and a trivially outside U to give well-defined maps

(L19.20)
$$(i_j)_* : K^0_c(T^*(U/B)) \longrightarrow K^0_c(T^*(M_j/B).$$

Now, the index is defined by quantizing the 'symbol' a – deformed to be homogeneous of degree 0 outside the zero section of $T^*(M_j/B)$ to a family of pseudodifferential operators. We know that the result is independent of the family chosen with the given symbol, so we may choose the families to be of the form $P_j \in \Psi^0(M_j/B; E_+, E_-)$ and to be equal to the identity outside $M_j \setminus K_j$ for $K = i_j(K)$ the image of a compact subset of U. Thus, $P_j - \mathrm{Id}$ is to have its Schwartz kernel supported in $K_j \times K_j$. Now, in this sense the two families of operators are 'exactly the same'. We only have to make sure that nothing goes wrong in the stabilization process to define the index as the difference of the null and conull bundles. Of course we may start with the 'same' parameterices for the P_j – each being $\mathrm{Id} - Pj'$ where P_j' has kernel support in $K_j \times K_j$ where they are same. The remaining problem here is that I did not do the stabilization procedure fully in the families case. Here I will refer to an alternative stabilization procedure – the relationship between this and the other one (which I did not complete!) will be added to the addenda.

To define the families index we need to stabilize the null space, or the range, to a bundle. One way to do this is to add an auxilliary finite dimensional map. Namely

LEMMA 32. *If $P \in \Psi^m(M/B;\mathbb{E})$, $\mathbb{E} = (E_+, E_-)$, is elliptic then there is a smooth map $S \in \mathcal{C}^\infty(B \times M; \hom(\mathbb{C}^N; E_-))$ such that*

$$\big(P \oplus S\big) : \begin{matrix} \mathcal{C}^\infty(M; E_+) \\ \oplus \\ \mathcal{C}^\infty(B; \mathbb{C}^N) \end{matrix} \longrightarrow \mathcal{C}^\infty(M; E_-) \text{ is surjective.} \tag{L19.21}$$

PROOF. For each point $b \in B$ we know that the range of P has finite codimension. We can therefore find a finite number $v_i \in \mathcal{C}^\infty(Z_b; E_-)$, of smooth sections which span a complement. Extending them to smooth sections of E_- over M (say supported close to b) will mean, by continuity, that the v_j span the range of $P_{b'}$ for b' in a neighbourhood of b. Now, by compactness we may cover B by a finite number of such neighourhoods with corresponding $v_{j,k} \in \mathcal{C}^\infty(M; E_-)$ as k ranges over some finite set. Now, let N be the total number of such sections and let S be the linear map from $\mathbb{C}^N \ni a_{j,k}$ to $\sum\limits_j a_{j,k} v_{j,k}$. The sum $P + S$ is surjective at each point of the base, since it is constructed to be surjective when to the subspace with $a_{j,k} = 0$ for all but one value of k. Now, the fact that $P + S$ is surjective leads, by the same argument as before, to the conclusion that the null spaces form a smooth finite dimensional subbundle of the bundle $\mathcal{C}^\infty(M/B; E_+) \oplus \mathbb{C}^N$ as a bundle over B. The claim (or definition depending on how you look at it) is that

$$\operatorname{ind}(P) = [\operatorname{null}(P + S), \mathbb{C}^N] \in K^0(B). \tag{L19.22}$$

In fact it is easy to see that two choices or S are stabily homotopic – just put all the choices together, maybe refine the covering to a common one such that one each set one of each stabilizations works, and then do an appropriate homotopy]. □

With this 'definition' of the families index, we may complete the proof of excision. Namely the stabilizing sections can always be chosen to have support in K_j and we may take the same stabilization for the two operators P_1 and P_2. □

L19.3. Atiyah-Singer index theorem. Now we can state the first form of the Atiyah-Singer families index theorem – in K-theory.

THEOREM 10. *If $\phi : M \longrightarrow B$ is a fibration of compact manifold then the analytic index map*

$$\operatorname{ind} : K^0(T^*(M/B)) \longrightarrow K^0(B) \tag{L19.23}$$

defined by quantization of symbols, is equal to the topological index map, i.e. can be factored through any embedding of the fibration

$$\begin{tikzcd} M \arrow[rr, "e"] \arrow[dr, "\phi"'] & & \mathbb{R}^n \times B \arrow[dl, "\pi_B"] \\ & B, & \end{tikzcd} \quad \textit{so}$$

(L19.24)
$$\begin{tikzcd} K^0(T^*(M/B)) \arrow[r, "\otimes b"] \arrow[dr, "\mathrm{ind}"'] & K_c^0(T^*(NM/B)) \arrow[r, "i_*"] & K_c(\mathbb{R}^{2N} \times B) \arrow[dl, "\mathrm{ind}"] \\ & K^0(B) & \end{tikzcd} \quad \textit{commutes}$$

where $\otimes b$ is the product with the Bott element for the normal fibration.

19+. Addenda to Lecture 19