CHAPTER 18

Product-type pseudodifferential operators

Lecture 18: 17 November, 2005

The main application I will make of the product-type conormal distribution, that I discussed last time, is to product-type pseudodifferential operators. Since these operators are associated to a fibration, let me start with a short discussion of the geometry of fibrations.

L18.1. Product-type operators defined. Thus consider a fibration,

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B. \end{tikzcd} \tag{L18.1}$$

If we take the product fibration $M^2 \longrightarrow B^2$ and map the diagonal of B into the product, $B = \mathrm{Diag}_B \longrightarrow B^2$ then pulling back the product gives us the fibre product

$$\begin{tikzcd} Z^2 \arrow[r, dash] & M^2_\phi \arrow[r, hook] \arrow[d, "\phi^2"] & M^2 \arrow[d, "\phi^2"] \\ & B \arrow[r, hook, "\mathrm{Diag}"] & B^2. \end{tikzcd} \tag{L18.2}$$

Since the points of $M^2_\phi \subset M^2$ are exactly those mapped to the diagonal in B^2 under the fibration, we have a pair of embedded submanifolds

$$\mathrm{Diag}_M \subset M^2_\phi \hookrightarrow M^2. \tag{L18.3}$$

Note that M^2_ϕ is often called the fibre diagonal. In local coordinates z, y and z', y near different points in M but above the same point in B with respect to which the fibration is projection onto the second factor,

$$\mathrm{Diag}_M = \{z = z',\ y = y'\} \subset M^2_\phi = \{y = y'\} \subset M^2. \tag{L18.4}$$

Definition 7. The pseudodifferential operators on M of product type with respect to the fibration ϕ and acting between sections of bundle, E and F over M are identified as a space of kernels with the product-type conormal distributions

$$\Psi^{m,m'}_{\phi-\mathrm{pt}}(M; E, F) = I^{\tilde m, \tilde m'}(M^2, M^2_\phi, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \Omega_R). \tag{L18.5}$$

0.7E; Revised: 29-11-2006; Run: November 29, 2006

LEMMA 29. *The product-type pseudodifferential operators act continuously on smooth sections:*

$$\Psi^{m,m'}_{\phi-\mathrm{pt}}(M;E,F) \ni A : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;F). \tag{L18.6}$$

PROOF. This is a direct application of the simple push-forward theorem for product-type conormal distributions, as in the standard case before. Namely the product-type conormal distributions on any manifold X are a module over $\mathcal{C}^\infty(X)$:

$$\mathcal{C}^\infty(X) \cdot I^{m,m'}(X,Y,Z;E) \longrightarrow I^{m,m'}(X,Y,Z;E).$$

If $\pi : X \longrightarrow X'$ is a fibration which is transversal to *both* submanifolds and E is the lift of a bundle from the base then

$$\pi_* : I^{m,m'}(X,Y,Z;\pi^*E\otimes\Omega) \longrightarrow \mathcal{C}^\infty(X';\Omega)$$

is continuous.

In this case we consider the projection $\pi_L : M^2 \longrightarrow M$ as a fibration. For any bundles E and F over M the module product (L18.1) followed by composition in the fibres gives

$$\begin{aligned}\mathcal{C}^\infty(M^2;E)\cdot I^{m,m'}(M^2,M^2_\phi,\mathrm{Diag};\mathrm{Hom}(E,F)\otimes\Omega_R) \longrightarrow \\ I^{m,m'}(M^2,M^2_\phi,\mathrm{Diag};\pi_L^*F\otimes\Omega_R).\end{aligned} \tag{L18.7}$$

The cancellation of left densities, as in the standard case, allows us to interpret (L18.6) as the composition of the maps (L18.1) and (L18.1) (for π_L) with pull-back:

$$\begin{aligned}A : \mathcal{C}^\infty(M;E) \ni u \longmapsto \pi_R^*u \in \mathcal{C}^\infty(M^2;E) \xrightarrow{\cdot A} \\ I^{m,m'}(M^2,M^2_\phi,\mathrm{Diag};\pi_L^*F\otimes\Omega_R) \xrightarrow{(\pi_L)_*} \mathcal{C}^\infty(M;F).\end{aligned} \tag{L18.8}$$

□

L18.2. Symbol maps. The symbol, acting on the space of conormal distributions, associated to the smaller submanifold M^2_ϕ takes values in (conormal) sections of a bundle over the the sphere bundle of the conormal bundle, $SN^*M^2_\phi$.

LEMMA 30. *For any fibration,* (L18.1)*, the sphere bundle of the conormal bundle to the fibre diagonal, may be naturally identified as the pull-back*

$$SN^*(M^2_\phi) = \pi^*(M^2_\phi) \text{ where } \pi : S^*B \longrightarrow B, \tag{L18.9}$$

as a fibration over S^*B with fibre Z^2 giving a commutative diagramme

$$\begin{tikzcd} Z^2 \arrow[r, dash] & SN^*(M^2_\phi) \arrow[r, "\pi"] \arrow[d] & M^2_\phi \arrow[r, hook] \arrow[d, "\phi^2"] & M^2 \arrow[d, "\phi^2"] \\ & S^*B \arrow[r, "\pi"] & B \arrow[r, hook, "\mathrm{Diag}"] & B^2. \end{tikzcd} \tag{L18.10}$$

PROOF. At any point $p \in M^2_\phi$ the conormal fibre in M^2, $N^*_pM^2_\phi$ is the space of differentials of functions vanishing on M^2_ϕ. Since M^2_ϕ is the preimage of Diag_B under the product fibration, this is just $(\phi^2)^*N^*_{\phi(p)}\mathrm{Diag}_B = T^*_{\phi(p)}B$. The same is therefore true of the spherical quotient, $SN^*_pM^2_\phi$ which is therefore identified with the pull-back of the fibration, $SN^*_pM^2_\phi = \pi^*_pS^*B$. In local coordinates this is just saying that $N^*_pM^2_\phi$ is spanned by the dy_j in terms of product coordinates. □

For any fibration manifold $SN^*(\mathrm{Diag}_M) = S^*M$ as usual where the identification comes from pull-back from the left factor of M.

LEMMA 31. *For a fibration, the boundary of the relative compactification of* $N^*\,\mathrm{Diag}_M$ *with respect to the subbundle* $N^*_{\mathrm{Diag}_M} M^2_\phi$ *may be identified with the blow-up* $[S^*M, \phi^*S^*B]$ *and this fibres*

$$\gamma : [S^*M, \phi^*S^*B] \longrightarrow S^*(M/B) \tag{L18.11}$$

with fibre, over $b \in B$, *modelled on* $\overline{T^*_b B}$; *the boundary of* $[S^*M, \phi^*S^*B]$ *is naturally identified as*

$$\partial[S^*M, \phi^*S^*B] = \pi^*S^*(M/B), \tag{L18.12}$$

the pull-back to S^*B *of the bundle* $S^*(M/B)$ *over* B.

PROOF. This follows from the earlier discussion of the relative compactificaiton of a vector bundle U with respect to a subbundle V. Namely, the 'main' boundary component of the relative compactification of U with respect to V may be identified with blow-up $[SU, SV]$, that this fibres over SU/V with fibres modelled on $\overline{V_p}$ (at any point p) and has boundary naturally diffeomorphic $\partial[SU, SV] \equiv \pi^*_V S(U/V)$ to the pull-back of $S(U/V)$ to SV. In the present case the base is M, $U = T^*M$ and $V = \phi^*T^*B$, the pull-back to M of the cotangent bundle to the base. Thus, $SU = S^*M$ and $SB = \phi^*S^*B$ and the 'main' boundary face, $[S^*M, \phi^*S^*B]$, fibres over the 'vertical sphere bundle' $S^*(M/B)$ with fibre modelled on the fibres of $\overline{T^*B}$ with the boundary, which is to say the corner of ${}^V\overline{U}$, being the pull-back to S^*B of $S^*(M/B)$. $\square$

With these reinterpretations of the manifolds on which the symbols of product-type conormal distributions are defined we may reinterpret the general symbol maps in the case of pseudodifferential operators to give

$$\begin{aligned} \sigma_{0,0} : \Psi^{0,0}_{\phi-\mathrm{pt}}(M; E, F) &\longrightarrow \mathcal{C}^\infty([S^*M, \phi^*S^*B]; \hom(E, F)), \\ \beta_{0,0} : \Psi^{0,0}_{\phi-\mathrm{pt}}(M; E, F) &\longrightarrow \Psi^0(\pi^*M/S^*B; E, F). \end{aligned} \tag{L18.13}$$

For operators of double order other than $0, 0$ we need to add appropriate 'homogeneity bundles' to the symbol maps

$$\begin{aligned} \sigma_{m,m'} : \Psi^{m,m'}_{\phi-\mathrm{pt}}(M; E, F) &\longrightarrow \mathcal{C}^\infty([S^*M, \phi^*S^*B]; \hom(E, F)) \otimes N_{m.m'}, \\ \beta_{m',m} : \Psi^{m,m'}_{\phi-\mathrm{pt}}(M; E, F) &\longrightarrow \Psi^{m'}(\pi^*M/S^*B; E, F \otimes N_{-m}). \end{aligned} \tag{L18.14}$$

These two maps are therefore separately surjective and have joint range the compatible subset

$$\sigma_{m,m'}(\beta_{m',m}(A)) = \sigma_{m,m'}(A)\big|_{\partial[S^*M, \phi^*S^*B]}. \tag{L18.15}$$

I will generally call $\sigma_{m,m'}(A)$ the 'usual symbol' since it is a fairly obvious extension of the standard symbol map. On the other hand I will call $\beta_{m',m}$ the 'base symbol'. This may be a rather contrarian name, since the base symbol is actually a family of fibre-wise pseudodifferential operators. However, these depend on the cotangent varriables in the base and this is why I think of it as the 'base' sybmol – it looks like the symbol of an operator on the base except that it takes values in pseudodifferential operators on the fibres instead of simply bundle homomorphisms.

L18.3. Composition. Also very much as in the standard case, the composite of two pseudodifferential operators, as maps (L18.6) is again a pseudodifferential operator of product type.

PROPOSITION 35. *For any fibration of compact manifolds* (L18.1) *and any three bundles E, F and G over M,*

$$\Psi^{m_1,m_1'}_{\phi-\mathrm{pt}}(M;F,G)\circ\Psi^{m_2,m_2'}_{\phi-\mathrm{pt}}(M;E,F)\subset\Psi^{m_1+m_2,m_1'+m_2'}_{\phi-\mathrm{pt}}(M;E,G) \tag{L18.16}$$

and the symbol maps are both homomorphisms, i.e. map products to products

$$\begin{aligned}\sigma_{m_1+m_2,m_1'+m_2'}(AB)=&\sigma_{m_1,m_1'}(A)\sigma_{m_2,m_2'}(B),\\ &A\in\Psi^{m_1,m_1'}_{\phi-\mathrm{pt}}(M;F,G),\ B\in\Psi^{m_2,m_2'}_{\phi-\mathrm{pt}}(M;E,F)\end{aligned} \tag{L18.17}$$

$$\beta_{m_1'+m_2'}:(AB)=\beta_{m_1'}(A)\beta_{m_2'}(B)\in\Psi^{m_1'+m_2'}(\pi^*M/S^*B;E,G\otimes N_{m_1+m_2}).$$

PROOF. This is basically the same as in the standard case – I did not go through it carefully in the lecture, but it is written out below in the addenda. ☐

L18.4. Ellipticity. If both symbols are invertible then A is said to be fully elliptic and then (in fact iff) it has a parametrix.

PROPOSITION 36. *If $A\in\Psi^{m,m'}_{\phi-\mathrm{pt}}(M;E,F)$ is fully elliptic in the sense that $\sigma_{m,m'}(A)$ has an inverse in $\mathcal{C}^\infty([S^*M,\phi^*S^*B];\hom(F,E))\otimes N_{-m,-m'}$ and $\beta_{m',m}(A)$ has an inverse in $\Psi^{-m'}(M/B;F,E\otimes N_{-m})$ then there exists $B\in\Psi^{-m,-m'}_{\phi-\mathrm{pt}}(M;F,E)$ such that*

$$A\circ B=\mathrm{Id}_F-R',\ B\circ A=\mathrm{Id}_E-R,\ R\in\Psi^{-\infty}(M;E),\ R'\in\Psi^{-\infty}(M;F). \tag{L18.18}$$

PROOF. This is a good opportunity to review the construction of a parametrix for an elliptic operator in the standard case, since the argument is almost precisely the same. ☐

Homotopy invariance of the index follows as before. Namely, if A_t is a smooth (in $t\in[0,1]$) family of elliptic operators then we can find a smooth family of parametrices B_t up to smoothing errors. The arguments leading to the formula

$$\mathrm{ind}(A_t)=\mathrm{Tr}(\mathrm{Id}_E-BA)-\mathrm{Tr}(\mathrm{Id}_F-AB) \tag{L18.19}$$

carry over directly to this more general setting and show that the index is smooth and integer-valued, hence constant.

REMARK 1. This suggests a harder index problem, which I hope to come back to before the end of the semester, namely what is the (families) index of A of product-type; it depends only on the (invertible) joint symbol $\sigma_{m,m'}(A)$, $\beta_{m',m}(A)$. Of course it is also the case that full ellipticity is quite a strong condition, since it requires the invertibility of a family of operators. On the other hand the index theorem in the standard case gives us a good hold on invertibility, after smoothing purturbation.

L18.5. Subalgebras. For the application to the index of ordinary pseudodifferential operators we need three important inclusions (see (L18.11)). The first is of the fibrewise operators.

Proposition 37. *For any fibration of compact manifolds*

(L18.20)
$$\Psi^{m'}(M/B;E,F)\subset\Psi^{0,m'}_{\phi-\mathrm{pt}}(M;E,F),\ \sigma_{m,m'}(A)=\gamma^*\sigma_{m',0}(A),\ \beta_{0,m'}(A)=A.$$

Proof. This is just the corresponding inclusion of conormal distributions discussed last time

(L18.21)
$$I^{m'}(Y,Z;E\otimes\Omega_Y)\ni u\hookrightarrow u\cdot\delta_Y\in I^{0,m'}(X,Y,Z;E\otimes\Omega_X)$$

in which a (conormal) distribution on Y, with respect to Z, is extended to X as a 'Dirac delta' in the normal variables. Locally (for the fibration case) this is rather obvious, since in product coordinates z,y and z',y (the same in the base, but possibly near different points in the fibre)

(L18.22)
$$\Psi^{m'}(M/B;E)\ni A=A(y,z,z')\in I^m(Z^2;E\otimes\Omega_Z)\longrightarrow$$
$$\delta(y-y')A(y,z,z')\in I^{\tilde m,\tilde 0}(M^2;M^2_\phi,\mathrm{Diag};E).$$

As usual the densities take care of themselves (which one needs to check of course) and the symbol behaves as indicated in (L18.21). Namely, the base symbol comes from the (local) Fourier transform in y so recovers the operator and the usual symbol comes from the full Fourier transform on y,z which is constant in the dual to y. □

Similarly the inclusion of the standard pseudodifferential operators corresponds to the inclusion
$$I^m(X,Z;E)\longleftrightarrow I^{m,m}(X,Y,Z;E)$$
for any embedded submanifold $Z\subset Y$.

Proposition 38. *For any fibration of compact manifolds*

(L18.23)
$$\Psi^m(M;E,F)\subset\Psi^{m,m}_{\phi-\mathrm{pt}}(M;E,F),\ \sigma_{m,m}(A)=\sigma_m(A),\ \beta_{m,m}(A)=\sigma_m(A)\big|_{\phi^*S^*B}.$$

Thus in the second case the 'base symbol' is just the ordinary symbol – so acts as a bundle isomorphism on the fibres.

Perhaps the most important inclusion for us is that of pseudodifferential operators on the base. For any bundle E over M we may view $\mathcal{C}^\infty(M;E)$ as an infinite-dimensional bundle over B, it could be denoted $\mathcal{C}^\infty(M/B;E)$, with fibre isomorphic to $\mathcal{C}^\infty(Z;E\big|_Z)$. Suppose we have a family of smoothing projections, hence of finite rank,

(L18.24)
$$\pi\in\Psi^{-\infty}(M/B;E),\ \pi^2=\pi(\text{ and }\pi^*=\pi\text{ if you want.})$$

Then the range of π is a finite dimensional bundle which sits inside $\mathcal{C}^\infty(M/B;E)$.

Proposition 39. *If $\pi_1\in\Psi^{-\infty}(M/B;E)$ has range isomorphic to a bundle $\tilde E$ over B and $\pi_2\in\Psi^{-\infty}(M/B;F)$ has range isomorphic to $\tilde F$ over B then*

(L18.25)
$$\Psi^m(B;\tilde E,\tilde F)\ni A\longrightarrow\pi_2A\pi_1\in\Psi^{-\infty,m}_{\phi-\mathrm{pt}}(M;E,F),$$
$$\sigma_{-\infty,m}(\pi_FA\pi_1)=0\ \textit{(by definition)},\ \beta_{m,-\infty}(\pi_2A\pi_1)=\pi_2\sigma(A)\pi_1.$$

Proof. This corresponds to the general inclusion for product-type conormal distributions

(L18.26)
$$I^m(X,Y;E)\subset I^{-\infty,m}(X,Y,Z;E).$$

□

I have inserted smoothing operators in (L18.25) 'compressing' the pseudodifferential operator on the base so that it acts on a finite subbundle on the fibres because I felt this was clearer in the application below. One can instead consider an operator on the base as acting on the lifted bundles and then one arrives at

PROPOSITION 40. *For any fibration of compact manifolds there is a natural inclusion*

$$\Psi^m(B;E,F)\subset\Psi^{m,0}(M;\phi^*E,\phi^*F),\ \sigma_{m,0}(A)=\sigma_m(A),\ \beta_{0,m}=\sigma_m(A). \tag{L18.27}$$

L18.6. Connection.

DEFINITION 8. A connection on a fibration is a choice of complementary bundle to $T(M/B)\subset TM$ where

$$T_p(M/B)=\left\{v\in T_pM; v \text{ is tangent to } Z_{\phi(p)}=\phi^{-1}(\phi(p))\right\}. \tag{L18.28}$$

The complement corresponding to a connection is necessarily isomorphic to the lift of the tangent bundle to the base, $\phi^*(TB)$, corresponding to the short exact sequence

$$T(M/B)\longrightarrow TM\longrightarrow\phi^*TB. \tag{L18.29}$$

Thus a connection is a splitting of (L1.2) as a sequence of bundles over M.

L18.7. Tensor product construction. Finally, with this ammuntion (unverified as a lot of it is) we come to the main construction of Atiyah and Singer, at least from this point of view.

PROPOSITION 41. *If $B\in\Psi^0(M/B;E_+,E_-)$ is an elliptic family with trivial index bundle of rank 1 – more specifically which is surjective and has null bundle trivial of rank 1 – then for any elliptic operator $A\in\Psi^0(B;F_+,F_-)$ (having chosen inner products and densities) the operator*

$$P_A=A\otimes B=\begin{pmatrix}B&0\\ \pi_{\mathrm{null}(B)}A\pi_{\mathrm{null}(B)}&B^*\end{pmatrix}\in\Psi^{0,0}_{\phi-\mathrm{pt}}(M;H_+,H_-),\\ H_+=E_+\otimes F_+\oplus E_-\otimes F_-,\ H_-=E_+\otimes F_-\oplus E_-\otimes F_+ \tag{L18.30}$$

is elliptic with

$$\mathrm{ind}(A\otimes B)=\mathrm{ind}(A) \tag{L18.31}$$

and P_A is deformable, through fully elliptic elements of $\Psi^{0,0}_{\phi-\mathrm{pt}}(M;H_+,H_-)$ to an element

$$\tilde A\in\Psi^0(M;H_+,H_-),\ \sigma(\tilde A)=\begin{pmatrix}\chi_1\sigma_0(B)&-\chi_2\sigma_0(A)^*\\ \chi_2\sigma_0(A)&\chi_1\sigma_0(B^*)\end{pmatrix} \tag{L18.32}$$

where $\chi_i\in\mathcal{C}^\infty(S^*M)$ form a partition of unity subordinate to the cover.

The operator P_A can be thought of as the 'Clifford tensor product' of A and B.

How are we going to use this? Given $A\in\Psi^m(B;\mathbb{E})$ (where I will start using 'superbundle' notation, with $\mathbb{E}=(E_+,E_-)$ and B acting between them) then given an embedding $B\hookrightarrow\mathbb{S}^N$ we may take a normal fibration to B. The normal bundle NB is itself is a bundle over B and if we take its 1-point compactification ${}^1\overline{NB}$ we get a fibration over B. The result above is applied to lift A to a pseudodifferential operator on ${}^1\overline{NB}$ with the same index (and the 'same' symbol in the sense of (L18.32). We can actually arrange that the lifted operator is completely trivial near the section 'at infinity' of the 1-point compactification and so extend it to $\mathbb{S}^N$, to be trivial outside the collar neighbourhood of B. This effectively reduces the index problem to $\mathbb{S}^N$, we we can solve it using Bott periodicity.

18+. Addenda to Lecture 18

18+.1. Fredholm condition and ellipticity. In the general mixed order case Sobolev spaces are needed to characterize ellipticity.

Proposition 42. *If $A \in \Psi^{0,0}(M;E,F)$ then A is Fredholm as a map $A : L^2(M;E) \longrightarrow L^2(M;F)$ if and only if it is fully elliptic.*

18+.2. Proof of Proposition 40.