

Product-type symbols

Lecture 16: 10 November, 2005

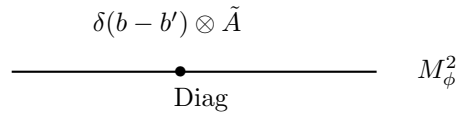
Last time I described the space of $\Psi^m(M/B; E, F)$ of pseudodifferential operators acting on the fibres of a fibration $\phi : M \rightarrow B$. This is defined directly in terms of conormal distributions, as $I^{m'}(M_\phi^2, \text{Diag}; \text{Hom}(E, F) \otimes \Omega_R)$ where M_ϕ^2 is the fibre diagonal, the set of pairs $(m, m') \in M^2$ such that $\phi(m) = \phi(m')$ and Diag is the diagonal of M^2 , so the set of pairs $\{(m, m); m \in M\}$. Such an operator defines a map

$$(L16.1) \quad \Psi^m(M/B; E, F) \ni A : \mathcal{C}^\infty(M; E) \rightarrow \mathcal{C}^\infty(M; F),$$

just as pseudodifferential operators on M do. It therefore has a Schwartz kernel on $M \times M$. This is easily seen to be, in terms of a local trivialization of ϕ

$$(L16.2) \quad K_A(m, m') = \tilde{A}(m, m')\delta(b - b')$$

where \tilde{A} is the conormal distribution defining (and usually identified with) A . Thus there are two submanifolds of M^2 in the picture here, namely M_ϕ^2 and Diag . These are nested as in the simple picture



Thus, in this simplified picture the kernels of elements of $\Psi^m(M/B; E, F)$ are singular all along the bigger submanifold, with a delta-singularity normal to it whereas the elements of $\Psi^m(M; E, F)$ have conormal singularities just at the smaller submanifold, and so are smooth outside it. It is then rather easy to see the following

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EXERCISE 20. Show that the elements of $\Psi^m(M; E, F) \cap \Psi^m(M/B; E, F)$ for a fibration with base and fibre of positive dimension are the fibrewise differential operators of order m (so for instance this intersection is empty if $m \in \mathbb{R} \setminus \mathbb{N}_0$).

For arguments in the proof of the index theorem, and for other reasons too, I want to define a larger class of ‘pseudodifferential operators of product type’ with respect to any fibration which is to include both the fibrewise pseudodifferential operators and the usual pseudodifferential operators on the total space of the fibration. To do this we return to the beginning and use the same pattern of definition as before. Namely, the operators will be defined, through their Schwartz kernels, in terms of a corresponding class of product-type conormal distributions

$$(L16.3) \quad \Psi_{\text{pt}-\phi}^{m,m'}(M; E, F) = I^{m-N, m'-N'}(M^2, M_\phi^2, \text{Diag}; \text{Hom}(E, F) \otimes \Omega_R).$$

Here m is the ‘main order’, m' is the ‘fibre order’ and on the right I am using as yet undefined notation for the conormal distributions with respect to a nested pair of submanifolds; N and N' are dimension shifts as before.

So, to define (L16.3) we wish to define

$$(L16.4) \quad I^{m,m'}(X, Y, Z; E) \subset \mathcal{C}^{-\infty}(X; E)$$

the space of (product-type) conormal distributions (distributional sections of the bundle E) with respect to two embedded submanifolds

$$(L16.5) \quad Z \subset Y \subset X.$$

Here, somewhat confusingly, m is the ‘order at Z ’ whereas m' is the ‘order at Y .’

Following backwards through the previous argument, to define (L16.4) we will want to carefully discuss a model case which we take to be a vector space \mathbb{R}^n with two subspaces. The variables along the smaller manifold Z in (L16.4) are intended to be ‘smooth parameters’ so we can take the smaller subspace to be $\{0\}$ and so consider as the model for a nested pair of submanifolds

$$(L16.6) \quad \{0\} \subset \mathbb{R}_y^k \subset \mathbb{R}_{y,z}^n.$$

Here (y, z) are linear coordinates, with $z = 0$ being the larger of the subspaces so the $y = (y_1, \dots, y_k)$ are coordinates in it.

So now, we want to define

$$(L16.7) \quad I_{\mathcal{S}}^{m,m'}(\mathbb{R}^n, \mathbb{R}^k, \{0\})$$

where the subscript \mathcal{S} is supposed to mean that the elements will have some sort of ‘rapid decay’ at infinity to compensate for the fact that \mathbb{R}^n is not compact. Let me try to motivate the definition a little more. We want these spaces (for appropriate orders) to include both

$$(L16.8) \quad I_{\mathcal{S}}^m(\mathbb{R}^n, \{0\}) \text{ and } \mathcal{S}(\mathbb{R}_y^k; I_{\mathcal{S}}^{m'}(\mathbb{R}^{n-k}, \{0\}))$$

which we defined before, and the latter space being a natural candidate for the space of conormal distributions associated to \mathbb{R}^k with ‘rapid decrease at infinity’.

Now, the first space in (L16.8) is by definition $\mathcal{F}^{-1}\rho^{-M}(\overline{\mathbb{R}^n})$, in terms of the radial compactification of \mathbb{R}^n , which is a ball (which is where we started). The second space is defined by Fourier (inverse) transform on \mathbb{R}^k so is

$$\mathcal{F}_{\zeta \rightarrow z}^{-1} \left(\mathcal{S}(\mathbb{R}^k; \rho^{-M'} \mathcal{C}^\infty(\overline{\mathbb{R}^{n-k}_\zeta})) \right),$$

To compare these two it is natural to take the Fourier transform in the y variables in the second space as well; since it is just Schwartz in these variables this gives the same space again.

So, assuming that we want to define our new space, (L16.7), as the inverse Fourier transform of some class of ‘symbols’ and we want it to ‘include’ (for appropriate orders) the two older spaces then the symbol space should include

$$(L16.9) \quad \rho^{-M}(\overline{\mathbb{R}^n_{\eta,\zeta}}) \text{ and } \mathcal{S}(\mathbb{R}^k; \rho^{-M'} \mathcal{C}^\infty(\overline{\mathbb{R}^{n-k}_\zeta})).$$

One of the points I am trying to make in this course is that in such circumstances one should look for an appropriate compactification, of \mathbb{R}^n in this case. I already briefly describe the ‘correct’ compactification in the addenda to Lecture 1, when I talked mostly about the radial compactification of vector space. The one I have in mind is the ‘relative’ compactification of a vector space with respect to a subspace. In this case

$$(L16.10) \quad \mathbb{R}^n \hookrightarrow {}^V\overline{W}, \quad W = \mathbb{R}^n, \quad V = \mathbb{R}^{n-k} = \{\eta = 0\} \subset \mathbb{R}^n.$$

Note that we have taken the Fourier transforms, so the symbols are defined on the dual of the original space. So the well-defined subspace is the annihilator of $\mathbb{R}^k_y \subset \mathbb{R}^n$, i.e. $\mathbb{R}^{n-k}_\zeta = \{\eta = 0\} \subset \mathbb{R}^n$.

Let me recall the definition of this compactification from (1+30), changing (and reversing the order of) the variables to fit (L16.10)

$$(L16.11) \quad R_V : W \ni w = (\eta, \zeta) \mapsto (t, s, \eta', \zeta') = \left(\frac{1}{(1 + |\eta|^2)^{\frac{1}{2}}}, \frac{(1 + |\eta|^2)^{\frac{1}{2}}}{(1 + |\eta|^2 + |\zeta|^2)^{\frac{1}{2}}}, \frac{\eta}{(1 + |\eta|^2)^{\frac{1}{2}}}, \frac{\zeta}{(1 + |\eta|^2 + |\zeta|^2)^{\frac{1}{2}}} \right) \in \mathbb{R}^2 \times W.$$

As noted there, the image lies in the compact manifold with corners (product of two half-spheres)

$$(L16.12) \quad {}^V\overline{W} = \{(t, s, \eta', \zeta') \in \mathbb{R}^{2+n}; t^2 + |\eta'|^2 = 1 = s^2 + |\zeta'|^2, t \geq 0, s \geq 0\}.$$

In fact the image is precisely the interior ($s, t > 0$) since the inverse there can be written

$$(L16.13) \quad \eta = \eta'/t, \quad \zeta = \zeta'/st,$$

that is, (L16.11) is a diffeomorphism onto the interior of (L16.12) which is therefore a compactification.

Our symbol spaces will be the same type as before, just ‘Laurent’ functions, meaning smooth functions except for a (possibly non-integral) overall power behaviour at each boundary face. So, we arrive at the basic definition of (model) product-type conormal distributions

$$(L16.14) \quad I_{\mathcal{S}}^{m,m'}(\mathbb{R}^n, \mathbb{R}^k_y \times \{0\}, \{0\}) = \mathcal{F}^{-1}(t^{-M} s^{-M'} \mathcal{C}^\infty({}^V\overline{W})), \\ W = \mathbb{R}^n_{\eta,\zeta}, \quad V = \{\eta = 0\}, \quad M = m, \quad M' = m'.$$

To check that this is consistent with what I claim above, we want to know that

$$(L16.15) \quad u \in I_{\mathcal{S}}^{m,m'}(\mathbb{R}^n, \mathbb{R}^k_y \times \{0\}, \{0\}) \implies u|_{(y,z) \neq 0} \in I^m(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^k \setminus \{0\}),$$

$$(L16.16) \quad I_{\mathcal{S}}^m(\mathbb{R}^n, \{0\}) \subset I^{m,m+}(\mathbb{R}^n, \mathbb{R}^k, \{0\}),$$

$$(L16.17) \quad \mathcal{S}(\mathbb{R}^k; I^{m'}(\mathbb{R}^{n-k}, \{0\})) \subset I^{-\infty, m'}(\mathbb{R}^n, \mathbb{R}^l, \{0\}),$$

Before trying to check these results and more, we need to look at the properties of the relative compactification. In particular we want to show some linear invariance as a prelude to eventual coordinate invariance, as in Lemma 2. The definition itself corresponds to choosing a transversal subspace to V and writing W as a product

$$(L16.18) \quad W = V \times U, \quad V = \{\zeta = 0\}, \quad U = \{\eta = 0\}.$$

We want to show that, up to a diffeomorphism, ${}^V\overline{W}$ does not depend on the choice of U , otherwise the notation is defective (to say the least)!

Points of ${}^V\overline{W}$ fall into four classes, those in the interior, those at which $t = 0$ but $s > 0$, those at which $t > 0$ but $s = 0$ and those at which $t = 0, s = 0$. We can introduce local coordinates near each such point, although it is simpler just to introduce a local generating system (i.e. a set of functions which are smooth and which contain a coordinate system). We can safely ignore the interior, since this is just \mathbb{R}^n with global coordinates η, ζ . As in (1+.35), (1+.36) observe that

$$(L16.19) \quad \text{Near } t = 0, s > 0, \quad \frac{1}{|\eta|}, \quad \frac{\zeta}{|\eta|}, \quad \frac{\eta}{|\eta|} \text{ generate,}$$

$$(L16.20) \quad \text{near } t > 0, s = 0, \quad \frac{1}{|\zeta|}, \quad \frac{\zeta}{|\zeta|}, \quad \eta \text{ generate}$$

$$(L16.21) \quad \text{and near } t = 0, s = 0, \quad \frac{1}{|\eta|}, \quad \frac{|\eta|}{|\zeta|}, \quad \frac{\eta}{|\eta|}, \quad \frac{\zeta}{|\zeta|} \text{ generate}$$

(where ‘generate’ can be read as ‘are smooth and generate’).

In fact, to see the first of these, observe that $t = 0$ implies $|\eta| = \infty$, meaning that in a sufficiently small neighbourhood (in ${}^V\overline{W}$) of such a point $|\eta| > R$ for any preassigned R ($R > 10$ say below.) Since $s > 0$ at the point, $s \geq s_0 > 0$ in a neighbourhood for some $s_0 > 0$, so

$$(L16.22) \quad s = \left(1 + \frac{|\zeta|^2}{1 + |\eta|^2}\right)^{-\frac{1}{2}} > s_0 \implies |\zeta| < C|\eta|$$

where $C > 0$ depends on R and s_0 , especially the latter. Thus as we approach the first type of boundary point, $|\eta| \rightarrow \infty$, maybe $|\zeta| \rightarrow \infty$ but no faster than a multiple of $|\eta|$ and (L16.19) follows since we can replace t by $1/|\eta|$, s by $|\zeta|/|\eta|$ etc. Similarly at the second type of boundary point $t \geq t_0 > 0$ in some neighbourhood so $|\eta|$ is bounded above, i.e. η is finite. Hence $|\zeta| \rightarrow \infty$ as we approach the point, since $s \rightarrow 0$. Thus we can replace (t, η') by η itself, then s by $1/|\zeta|$ and ζ' by $\zeta/|\zeta|$, giving (L16.20). In the third case, of a point on the corner, along any sequence approaching such a point, $|\eta| \rightarrow \infty$, since $t \rightarrow 0$ and $|\zeta| \rightarrow \infty$ since $s \rightarrow 0$ (using (L16.22)). In fact $|\zeta|/|\eta| \rightarrow \infty$ for the same reason. From this (L16.21) follows.

Note that near any particular point on the boundary we just need to drop one, or in the last case two, of the ‘spherical’ variables to get a coordinate system.

As noted back in (1+.34) this allows us to see which of the constant and linear vector fields lift. Namely

$$(L16.23) \quad \partial_{\eta_i}, \quad \partial_{\zeta_k}, \quad \eta_i \partial_{\zeta_k}, \quad \zeta_l \partial_{\zeta_k}, \quad \eta_i \partial_{\eta_j}$$

all lift to be smooth on ${}^V\overline{W}$ (for the obvious range of the indices) but $\zeta_l \partial_{\eta_i}$ does not. To see this, we can use homogeneity in terms of the coordinates derived from the generating functions. For instance, look at the corner and denote the first two

functions in (L16.21) as r and R and an appropriate choice of coordinates from the spherical variables as ω . Then each of the vector fields lifts to be of the form

$$(L16.24) \quad a(r, R, \omega)\partial_r + b(r, R, \omega)\partial_R + V(r, R, \omega)$$

where V is a vector field in the ω 's. The vector field is certainly smooth in $r > 0$, $R > 0$ since that is the interior. Moreover the scaling $r \rightarrow \lambda r$, $\lambda > 0$ corresponds precisely to $\eta \rightarrow \eta/\lambda$ and $\zeta \rightarrow \zeta/\lambda$. Under this combined scaling, all of the vector fields in (L16.23) are homogeneous, or degrees 0 or 1 (as is $\zeta_i\partial_{\eta_i}$). On the other hand the scaling $R \rightarrow \lambda R$ (with other variables fixed) corresponds precisely to $\zeta \rightarrow \zeta/\lambda$. Under this scaling all the vector fields in (L16.23) are homogeneous of degrees 0 or 1 still (whereas $\zeta_i\partial_{\eta_i}$ is homogeneous of degree -1). This homogeneity translates to homogeneity of the individual terms in (L16.24) and shows that the coefficients are all homogeneous of positive degrees, hence the vector fields lift to be smooth (and if you look a little more carefully, $\zeta_i\partial_{\eta_i}$ definitely does not.) They are all tangent to both boundary hypersurfaces. I have just been talking about a neighbourhood of the corner but the other regions of the boundary are similar with the discussion simpler (basically one of these homogeneities persists at each).

This proves Lemma 2. From (L16.11) we see immediately that the definition only depends on V , U , and the choice of Euclidean metrics on these spaces. That is, the group $O(n-k) \times O(k)$, which acts on W once the decomposition (L16.18) and choice of Euclidean metrics is fixed, lifts to act smoothly on ${}^V\overline{W}$, namely (O_η, O_ζ) acts through $(t, s, \eta', \zeta') \mapsto (t, s, O_\eta\eta', O_\zeta\zeta')$. To show that the whole group

$$(L16.25) \quad \{A \in \text{GL}(W); AV = V\} \text{ lifts to act smoothly on } {}^V\overline{W}$$

lifts to act smoothly on ${}^V\overline{W}$ observe that in terms of a splitting $W = V \oplus U$, this group consists of the lower triangular block matrices

$$(L16.26) \quad \begin{pmatrix} A' & 0 \\ S & A'' \end{pmatrix}, \quad A' \in \text{GL}(U), \quad A'' \in \text{GL}(V), \quad S \in \text{hom}(U, V).$$

We have already seen the invariance under the block diagonal, orthogonal, matrices and modulo those (needed just to make sure that A' and A'' are both positively oriented) such a matrix can be connected to the identity in the group. Thus, it can be written as a product of exponentials of elements of the Lie algebra. However, the Lie algebra is spanned by the linear vector fields in (L16.23) so these exponentials are given by the integration of smooth vector fields on ${}^V\overline{W}$ and so all lift to diffeomorphisms.

Thus in fact the definition of ${}^V\overline{W}$ does not depend on the choices made in the explicit map (L16.11). This justifies the notation ${}^V\overline{W}$ for the compactification of a vector space W with respect to a subspace V . Note that there really is asymmetry in the definition, as there has to be if it is independent of the choice of U , the transversal, but not of V . One can also see this in terms of the important map back to the radial compactification.

LEMMA 24. *The identification of the interiors of ${}^V\overline{W}$ and \overline{W} with W extends to a smooth surjective map*

$$(L16.27) \quad \beta: {}^V\overline{W} \longrightarrow \overline{W}.$$

PROOF. We only need to compare the compactification map (L16.11) with that corresponding to the radial compactification expressed in terms of these variables

$$(L16.28) \quad R : W \ni w = (\eta, \zeta) \mapsto (\tau, \eta'', \zeta'') = \left(\frac{1}{(1 + |\eta|^2 + |\zeta|^2)^{\frac{1}{2}}}, \frac{\eta}{(1 + |\eta|^2 + |\zeta|^2)^{\frac{1}{2}}}, \frac{\zeta}{(1 + |\eta|^2 + |\zeta|^2)^{\frac{1}{2}}} \right) \in \mathbb{R} \times W.$$

Clearly

$$(L16.29) \quad \tau = st, \quad \eta'' = s\eta', \quad \zeta'' = \zeta'$$

which shows that the map (L16.27) exists and is smooth. \square

Notice from (L16.29) that β maps the boundary hypersurface $\{t = 0, s > 0\}$ in ${}^V\overline{W}$ onto the boundary of \overline{W} except for the part $\partial\overline{V}$ where we regard $\overline{V} \subset \overline{W}$. This is actually the alternative construction of ${}^V\overline{W}$ which I will record here even though I have not defined the notion of blow up. It means ‘introduce polar coordinates around the submanifold.’

LEMMA 25. *The relative compactification ${}^V\overline{W}$ is canonically identified with the manifold obtained by blowing up the boundary $\partial\overline{V}$ in \overline{W} (denoted by me $[\overline{W}, \partial\overline{V}]$).*

Now we know that the space of (model) product-type conormal distributions defined by (L16.14) is also invariant under linear transformations which preserve \mathbb{R}^k (as a subspace of \mathbb{R}^n) because the Fourier transform converts this to the action of the transpose, which preserves the annihilator in the dual and we may use Lemma 2 which implies in particular that on the ‘symbolic side’

$$(L16.30) \quad A^* \left(t^{-M} s^{-M'} \mathcal{C}^\infty({}^V\overline{W}) \right) = t^{-M} s^{-M'} \mathcal{C}^\infty({}^V\overline{W}) \quad \forall M, M' \in \mathbb{R} \text{ (or indeed } \mathbb{C}).$$

Recall that the whole thrust of this definition is towards (L16.15) – (L16.17). So, consider (L16.16) first. This is a consequence of (L16.27) and (L16.29). Namely we are to show that

$$(L16.31) \quad u \in I_S^m(\mathbb{R}^n, \{0\}) \implies u \in I_S^{m, m^+}(\mathbb{R}^n, \mathbb{R}^k)$$

where of course this must be true for any choice of k . By definition

$$(L16.32) \quad \hat{u} = R^* a, \quad a \in \rho^{-M} \mathcal{C}^\infty(\overline{W}), \quad W = \mathbb{R}^n$$

and then from (L16.27) and (L16.29) (which shows that $\beta^* \rho = st$)

$$(L16.33) \quad \beta^* a \in t^{-M} s^{-M} \mathcal{C}^\infty({}^V\overline{W}), \quad V = \mathbb{R}^{n-k}, \quad W = \mathbb{R}^n.$$

However, β is just the canonical extension of the identification of the interiors so of course, $\beta R_V = R$ since they are equal on the interiors. Thus (L16.33) and (L16.32) mean that

$$(L16.34) \quad \hat{u} = R_V^* b, \quad b \in t^{-M} s^{-M} \mathcal{C}^\infty({}^V\overline{W})$$

which is (L16.31) and hence (L16.16) (for the moment all the order-normalizations are messed up or omitted here).

Next consider (L16.17). We want to do much the same as (L16.33) but we do not quite have the right map (with a little more blow-up technology we could get it). So, let me proceed more by hand. We already observed in the run up to (L16.9) that

$$(L16.35) \quad u \in \mathcal{S}(\mathbb{R}_y^k; I_S^m(\mathbb{R}^{n-k}, \{0\})) \implies \hat{u} \in \mathcal{S}(\mathbb{R}_\eta^k, \rho_\zeta^{-M} \mathcal{C}^\infty(\overline{\mathbb{R}_\zeta^{n-k}})).$$

Ignoring the factor of ρ_ζ^{-M} for the moment we want to show that (L16.35) implies that \hat{u} extends from the interior (i.e. \mathbb{R}^n) to be smooth on ${}^V\overline{W}$. To do this we can consider the three regions of the boundary in (L16.19). Near the boundary $s = 0$, away from the corner in (L16.20), \hat{u} is smooth, since it is a smooth function of η and of the generating functions $1/|\zeta|$ and $\zeta/|\zeta|$ of $\overline{\mathbb{R}_\zeta^{n-k}}$. Near the remainder of the boundary, covered by (L16.19) and (L16.21), $|\eta| \rightarrow \infty$. from (L16.35) we know that \hat{u} is uniformly rapidly decreasing as $|\eta| \rightarrow \infty$, i.e.

$$(L16.36) \quad |\hat{u}(\eta, \zeta)| \leq C_N |\eta|^{-N} \text{ in } |\eta| > 1.$$

Since the entries of the Jacobian of the singular changes of variables from $\zeta/|\eta|$ to $\zeta/(1+|\zeta|^2)^{\frac{1}{2}}$ and $1/(1+|\zeta|^2)^{\frac{1}{2}}$ are bounded by powers of $|\eta|$ it follows that $\hat{u}(\eta, \zeta)$ is in fact smooth down to the boundary $t = 0$ at which it vanishes to infinite order. That is,

$$(L16.37) \quad \hat{u} \in t^\infty s^{-M} \mathcal{C}^\infty({}^V\overline{W}) \longrightarrow u \in I^{-\infty, m}(\mathbb{R}^n, \mathbb{R}^k, \{0\}).$$

Additional factors of ρ_ζ present no extra problems.

In fact we will later make use of the fact that

LEMMA 26. *Under the identification as functions on the interior*

$$(L16.38) \quad \mathcal{S}(\mathbb{R}_\eta^k; \rho_\zeta^{-M} \mathcal{C}^\infty(\overline{\mathbb{R}_\zeta^{n-k}})) \equiv t^\infty s^{-M} \mathcal{C}^\infty({}^V\overline{W}), \quad W = \mathbb{R}^n, \quad V = \mathbb{R}^{n-k}.$$

I will prove the partial converse of this, (L16.15) next time and go through the extension to vector bundles and submanifolds, much as before, leading to the definition of the pseudodifferential operators through (L16.3).

16+. Addenda to Lecture 16

16+.1. More on the relative compactification. The relative compactification ${}^V\overline{W}$ is given by the map and image in (L16.11) for the vector spaces V and W in (L16.10). Observe that as well as the map (L16.27) there is a natural map

$$(16+.39) \quad {}^V\overline{W} \ni (t, s, \eta', \zeta') \longmapsto (t, \eta') \in \overline{W/V}.$$

Certainly this map is smooth and surjective in the model setting. Furthermore it follows from the form of the general element of $\text{GL}(W, V)$, i.e. an element of $A \in \text{GL}(W)$ such that $A(V) \subset V$,

$$(16+.40) \quad \text{GL}(W, V) \ni A = \begin{pmatrix} A' & 0 \\ Q & A'' \end{pmatrix}, \quad A(\eta, \zeta) = (A'\eta, Q\eta + A''\zeta)$$

that the map (16+.39) and the actions of $\text{GL}(W, V)$ and $\text{GL}(V)$ give a commutative diagram

$$(16+.41) \quad \begin{array}{ccc} \text{GL}(W, V) & \longrightarrow & \text{GL}(V) \\ \Big\| & & \Big\| \\ {}^V\overline{W} & \longrightarrow & \overline{W/V}. \end{array}$$

Thus the map (16+.39) is natural.

LEMMA 27. *The map (16+.39) is a fibration with fibre diffeomorphic to \bar{V} . Although not naturally a product the fibration is trivial and induces a fibration of the boundary face $H_W = \{t = 0\}$ of ${}^V\bar{W}$,*

$$(16+.42) \quad H_W \longrightarrow S(W/V) = \partial\overline{W/V} \text{ with fibre diffeomorphic to } \bar{V}.$$

The other boundary hypersurface, $H_V = \{s = 0\}$ naturally decomposes as a product

$$(16+.43) \quad H_V = SV \times \overline{W/V}, \quad (t, 0, \eta', \zeta') \longmapsto ((0, \zeta'), (t, \eta')).$$

PROOF. The map (16+.43) corresponds to the quotient of the group $GL(W, V)$ by the normal subgroup $\begin{pmatrix} \text{Id} & 0 \\ * & \text{Id} \end{pmatrix}$ in (16+.40). Namely it gives a commutative diagram

$$(16+.44) \quad \begin{array}{ccc} GL(W, V) & \longrightarrow & GL(V) \times GL(W/V) \\ \downarrow & & \downarrow \\ B_V & \longleftrightarrow & SV \times \overline{W/V} \end{array}$$

which shows that the product decomposition is natural.

On the other hand in (16+.42), on restriction to H_W the off-diagonal part of $GL(W, V)$ still acts non-trivially, so the map is only naturally a trivial fibration (that it is a fibration follows from the explicit form of (16+.42) which presents it as a product). \square

The invariance of these maps shows that they extend directly to the corresponding bundle settings. In the geometric case discussed below, where

$$(16+.45) \quad Z \subset Y \subset X$$

are submanifolds, the vector space W and subspace V are replaced by the bundle and subbundle

$$(16+.46) \quad N^*Y_Z \subset N^*Z$$

with the conormal bundles being relative to X . Note that the quotient N^*Z/N_Z^*Y may be naturally identified with the conormal bundle of Z as a submanifold of Y . Perhaps I will finally admit a relative notation for normal/conormal bundles and write

$$(16+.47) \quad {}^XN^*Z/{}^XN_Z^*Y = {}^YN^*Z.$$

Then the relative compactification becomes the manifold with corners

$$(16+.48) \quad N^*Y_Z \overline{N^*Z}$$

which has the two boundary hypersurfaces we can now associate to Z (corresponding to W above) and Y (corresponding to V) which are respectively fibred and have a product-bundle structure:-

$$(16+.49) \quad \begin{array}{ccc} \overline{{}^XN_Z^*Y} & \longrightarrow & H_Z \\ & & \downarrow \\ & & {}^YSN^*Z, \\ H_Y \equiv \overline{{}^YN^*Z} \times_Z & & {}^XSN_Z^*Y. \end{array}$$

We shall see later that there is a natural identification with the blow up

$$(16+.50) \quad H_Z \equiv [{}^XSN^*Z, {}^XS_Z^*Y],$$

of ${}^XS_Z^*Y$ as a submanifold of ${}^XSN^*Z$.

