CHAPTER 14

Bott periodicity

Lecture 14: 3 November, 2005

Recall that I defined the standard K-theory of a compact manifold as the set of equivalence classes of pairs of complex vector bundles

$$K^0(X) = \{(E_+, E_-)\}/\sim \tag{L14.1}$$

where equivalence is the existence of a stable isomorphism. In particular $(E_+, E_-) \sim (E_+ \oplus H, E_- \oplus H)$ so these really are formal differences in the sense that we can 'cancel' an H from both terms.

Although the equivalence relation here is stable bundle isomorphism, it is important to realize that it implies the equivalence of homotopic bundles.

Lemma 23. *If E is a complex vector bundle over $[0,1] \times X$ then as bundles over X, $E_0 = E\big|_{\{0\}\times X}$ and $E_1 = E\big|_{\{1\}\times X}$ are isomorphic.*

L14.1. Proof of Theorem 7. We have also defined

$$K^{-2}(X) = [X \times \mathbb{S}, X \times \{1\}; G^{-\infty}(E;E), \mathrm{Id}] \tag{L14.2}$$

as the homotopy classes of pointed maps from $X \times \mathbb{S}$ into the 'smoothing group'. Theorem 7 asserts that these two abelian groups are isomorphic where the map between them is constructed by regularizing the null bundle of an elliptic family of Toeplitz operators as follows
(L14.3)

$$\begin{tikzcd}
{[\sigma_0(A)] \in K^{-2}(X)} \arrow[ddrr, dotted] & & \\
\mathcal{C}^\infty(X \times \mathbb{S}; \mathrm{GL}(N,\mathbb{C})) \arrow[u] & & \\
A \in \mathcal{C}^\infty(X; \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)) \arrow[u, "\sigma_0"] \arrow[rr] & & {[\mathrm{null}(A(\mathrm{Id}-\pi_k)), \mathrm{null}((\mathrm{Id}-\pi_k)A^*] \in K^0(X)}
\end{tikzcd}$$

So we have to show first that this is really does define a map

$$\mathrm{ind} : K^{-2}(X) \longrightarrow K^0(X). \tag{L14.4}$$

We first check that the element of $K^0(X)$ does not depend on k and it does not depend on the choice of A with fixed symbol $\sigma_0(A) = a \in \mathcal{C}^\infty(X \times \mathbb{S}; \mathrm{GL}(N,\mathbb{C}))$. This will give us a map

$$\mathcal{C}^\infty(X \times \mathbb{S}; \mathrm{GL}(N,\mathbb{C})) \longrightarrow K^0(X). \tag{L14.5}$$

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Two operators A_0 and A_1 with the same symbol are homotopic through the linear homotopy $A_t = (1-t)A_0 + tA_0$. Choosing k large it follows from our earlier arguments that $((\operatorname{Id} - \pi_k)A_t)$ is a smooth bundle over $X \times [0,1]$ and hence that the pairs of bundles

$$[\operatorname{null}(A_0(\operatorname{Id} - \pi_k)), \operatorname{null}((\operatorname{Id} - \pi_k)A_1^*)] \text{ and } [\operatorname{null}(A_1(\operatorname{Id} - \pi_k)), \operatorname{null}((\operatorname{Id} - \pi_k)A_1^*)]$$

(in which the null bundles are constant and trivial) define the same element in $K^0(X)$.

Thus it remains to consider the effect of taking different values of k. By assumption k is chosen large enough that $A(\operatorname{Id} - \pi_k)$ has null bundle equal to that of $\operatorname{Id} - \pi_k$. So it is enough to consider the effect of increasing k to $k+1$. The null bundle of $A(\operatorname{Id} - \pi_{k+1})$ is just increased by the trivial bundle $e^{i(k+1)\theta}\mathbb{C}^N$. Since none of these elements are annihilated by $A(x)$, the range of $A(\operatorname{Id} - \pi_k)$ is just the range of $A(x)(\operatorname{Id} - \pi_{k+1})$ plus $A(x)(e^{i(k+1)\theta}\mathbb{C}^N)$. Since A is a smooth isomorphism onto this space, it is a trivial bundle of rank N, with the trivialization given by $A(x)$ itself. Thus the null space of $(\operatorname{Id} - \pi_{k+1})A^*$, being the annihilator of the range with respect to the chosen innner product, must be equal to the null space of $(\operatorname{Id} - \pi_{k+1})A^*$ plus a trivial bundle of rank N. Thus increasing k by 1 does not change the element in $K^0(X)$.

Thus we do have a map (L14.5). A homotopy of symbols can be lifted to a homotopy of operators and as we have already seen, this results in the same element in $K^0(X)$, so (L14.5) descends to the desired map (L14.4). So it remains to show that this is an isomorphism.

So, suppose $A \in \mathcal{C}^\infty(X; \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N))$ has symbol $a \in \mathcal{C}^\infty(X \times \mathbb{S}; \operatorname{GL}(N,\mathbb{C}))$ with $\operatorname{ind}(a) = 0 \in K^0(X)$. We can assume that $a(x,1) = \operatorname{Id}$, since $b(x) = a(x,1) \in \mathcal{C}^\infty(X; \operatorname{GL}(N,\mathbb{C})$ is a smooth family of matrices, hence trivially an element of $\mathcal{C}^\infty(X; \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N))$, which is invertible. Thus $A(x)$ and $b^{-1}(x)A(x)$ have the same index. Now, if $\operatorname{ind}(a) = 0$ then we know that there is a family $A \in \mathcal{C}^\infty(X; G^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N))$ with symbol a. Thus from the argument of last time we know that there is then an homotopy from a suitably stabilized a to the identity. Stabilizing a corresponds to stabilizing the operator by the identity on a bundle and so does not change the index. This if $\operatorname{ind}(a) = 0$ then a can be deformed to the identity and hence $[a] = 0 \in K^{-2}(X)$, so the map (L14.4) is injective.

Surjectivity of the index map also follows easily. First recall that any smooth complex bundle E over X can be complemented to a trivial bundle, i.e. can be embedded as a subbundle of a trivial bundle $\mathbb{C}^N$ (and hence for any larger N). Taking a pair of vector bundles, (E_+, E_-), let π_+ be the projection onto E_+ as a subbundle of $\mathbb{C}^N$ and similarly let π_- be projection onto E_- as a subbundle of $\mathbb{C}^M$. Then the symbol

$$a(x,\theta) = \pi_+(x)e^{-i\theta} + (\operatorname{Id}_N - \pi_+) + (\operatorname{Id}_M - \pi_-) + \pi_-(x)e^{i\theta} \tag{L14.6}$$

has index $[(E_+, E_-)]$. Indeed, it is the symbol of the elliptic family

$$\pi_+(x)L + (\operatorname{Id}_N - \pi_+) + (\operatorname{Id}_M - \pi_-) + \pi_-(x)U \in \mathcal{C}^\infty(X; \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^{N+M})) \tag{L14.7}$$

which has null space $\pi_+(x)\mathbb{C}^N$ (constant on the circle) and the null space of its adjoint

$$\pi_+(x)U + (\operatorname{Id}_N - \pi_+) + (\operatorname{Id}_M - \pi_-) + \pi_-(x)L \tag{L14.8}$$

is similalry $\pi_-(x)\mathbb{C}^M$ so indeed $\operatorname{ind}(a) = [(E_+, E_-)]$ shows the surjectivity.

An elliptic element, such as L, with index $1 = [(\mathbb{C}, 0)] \in L(\{\mathrm{pt}\})$, in the Toeplitz algebra is sometimes called a Bott element and the inverse $K^0(X) \longrightarrow K^{-2}(X)$ just constructed is the Bott map.

More generally, if X is a possibly non-compact manifold we still want definitions of $K_c^{-1}(X)$, $K_c^{-2}(X)$ and $K_c^0(X)$ reducing to $K^{-1}(X)$, $K^{-2}(X)$ and $K^0(X)$ in the compact case. The natural choice for the first two is to take maps into the same spaces as before but which now reduce to the identity outside a compact set (depending on the map). Homotopies are also required to be constant (and hence equal to the identity) outside some compact subset of X in

(L14.9)
$$K_c^{-1}(X) = \left\{ f \in \mathcal{C}^\infty(X; G^{-\infty}(Y; E)); f\big|_{X\setminus K} = \mathrm{Id},\ K \Subset X \right\} /\text{homotopy}$$
$$K_c^{-2}(X) =$$
$$\left\{ f \in \mathcal{C}^\infty(X \times \mathbb{S}; G^{-\infty}(Y; E)); f\big|_{(X\setminus K)\times\mathbb{S}} = \mathrm{Id} = f\big|_{X\times\{1\}},\ K \Subset X \right\} /\text{homotopy}.$$

Of course, this is consistent with out defintion for compact spaces.

For $K^0(X)$ we need to take a similar definition in which the two bundles (E_+, E_-) are isomorphic outside a compact set, where the isomorphism needs to be included in the data defining the element. Thus we consider triples (E_+, E_-, a) where $a \in \mathcal{C}^\infty(X \setminus K; \hom(E_+, E_-))$ is invertible for some compact set $K \Subset X$. Thus

$$K_c^0(X) = \{(E_+, E_-, a)\} / \sim \tag{L14.10}$$

for such data, where the equivalence relation is that

(L14.11)
$$\begin{gathered}(E_+, E_-, a_0) \sim (E_+, E_-, a_1)\\ \text{if } \exists \text{ a homotopy of isomorphisms } a_t : E_+ \longrightarrow E_- \text{ over } [0, 1]_t \times (X \setminus K)\\ \text{and } (E_+, E_-, a) \sim (F_+, F_-, b)\\ \text{if } \exists\ H \text{ and } F : E_+ \oplus F_- \oplus H \longleftrightarrow E_+ \oplus F_- \oplus H\\ \text{s.t. } F = a \oplus b \oplus \mathrm{Id}_H \text{ on } X \setminus K,\ K \Subset X.\end{gathered}$$

Note that a triple (E_+, E_-, a) defines the zero element in $K_c(X)$ if and only if there is a bundle H and an isomorphism $b : E_+ \oplus H \longrightarrow E_- \oplus H$ over X restricting to $a \oplus \mathrm{Id}_H$ outside some compact subset.

Exercise 18. Show that the index isomorphism (L14.4) carries over to the case of non-compact manifolds.

An important consequence of the existence of this index isomorphism is

Proposition 29. *[Bott periodicity, usual form] For any manifold X there is a natural isomorphism*

$$K_c^0(X) \longrightarrow K_c^0(\mathbb{R}^2 \times X). \tag{L14.12}$$

Proof. In our original definition of $K^{-2}(X)$ we can perturb any representative slightly so that the normalization condition $f\big|_{X\times\{1\}} = \mathrm{Id}$ can be arranged to hold on $X \times I$ for some neighbourhood I of $1 \in \mathbb{S}$ and similarly for homotopies. Identifying $\mathbb{S} \setminus \{1\}$ with $\mathbb{R}$ this shows that in terms of the non-compact notion

$$K_c^{-2}(X) = K_c^{-1}(\mathbb{R} \times X). \tag{L14.13}$$

Next consider $K_c^0(\mathbb{R} \times X)$, say for X compact; we shall show that

$$K_c^0(\mathbb{R} \times X) = K^{-1}(X). \tag{L14.14}$$

A bundle over $\mathbb{R} \times X$ is necessarily isomorphic to the lift of a bundle from X so any element is represented by two bundles E_+, E_- over X and isomorphisms between then over $(-\infty, -N) \times X$ and $(N, \infty) \times X$ for some N. By homotopy invariance, these isomorphisms can also be taken to be constant in the real variable. Then the isomorphism at $-N$ may be used to identify the bundles and the isomorphism at N becomes an isomorphism of a fixed bundle to itself. Stabilizing such an isomorphism by the identity on a complementary bundle gives an element of $K_c^{-1}(X)$ and it is straightforward to check that this element is well defined and leads to the isomorphism (L14.14).

Combining these two identifications we see that

$$K_c^0(\mathbb{R}^2 \times X) = K_c^{-1}(\mathbb{R} \times X) = K_c^{-2}(X) = K_c^0(X) \tag{L14.15}$$

where the last identification is using the index map. □

From this we can deduce that (for $k \geq 1$)

$$K^{-1}(\mathbb{S}^k) = \begin{cases} \mathbb{Z} & k \text{ odd} \\ \{0\} & k \text{ even.} \end{cases} \tag{L14.16}$$

In fact we shall show that $K^{-1}(\mathbb{S}^k) = K_c(\mathbb{R}^k)$ then from (L14.14) $K_c(\mathbb{R}^k) = \mathrm{K}_c^{-1}(\mathbb{R}^{k+1})$ and (L14.16) follows. There is a map

$$K_c^{-1}(\mathbb{R}^k) \longrightarrow K^{-1}(\mathbb{S}^k) \tag{L14.17}$$

defined by identifying a point on the sphere as the point at infinity on $\mathbb{R}^k$. Then a map from $\mathbb{R}^k$ to $G^{-\infty}(Y; E)$ required to be the identity near infinity defines an element of $K^{-1}(\mathbb{S}^k)$. Homotopy with the value fixed near infinity as the identity implies homotopy on the sphere so this gives (L14.17). Moreover, using the connectedness of $G^{-\infty}(Y; E)$ every element of $K^{-1}(\mathbb{S}^k)$ must arise this way, since the value at the chosen point can be deformed to Id. Thus (L14.17) is surjective. An element can only go to zero if it is homotopic to the identiy through families which are constant near infinity. But then multiplying everywhere by the inverse of the value at infinity gives a homotopy which is the identity near infinity, so (L14.17) is an isomorphism and (L14.16) follows.

Corollary 6. *The homotopy groups of $G^{-\infty}$ are*

$$\pi_k(G^{-\infty}(Y; E)) = \begin{cases} \mathbb{Z} & k \textit{ odd} \\ \{0\} & k \textit{ even.} \end{cases} \tag{L14.18}$$

This is one justification for the statement that $G^{-\infty}$ is a classifying group for K-theory.

Next I want to give at least a preliminary statement of the Atiyah-Singer index theorem. I will discuss both the 'numerical' index and the families index. The formula for the former and the formula for the Chern character for the latter are of particular interest.

Given a compact manifold, Z, and two complex vector bundles E_+, and E_- over Z any elliptic operator $P \in \Psi^0(Z; E_+, E_-)$ (if one exists) has finite dimensional null space and its range has finite dimensional complement. The difference between these two integers is the (numerical) index of P

$$\operatorname{ind}(P) = \dim(\operatorname{null}(P)) - \dim(\mathcal{C}^\infty(Z; E_-)/P\mathcal{C}^\infty(Z; E_+)). \tag{L14.19}$$

We already know that this function is homotopy invariant, so it can only depend on the geometric data (Z, E_+, E_-) and the symbol $\sigma_0(P) \in \mathcal{C}^\infty(S^*Z; \hom(E_+, E_-))$.

Proposition 30. *The index defines a map*

$$\operatorname{ind}_{\mathrm{a}} : K_{\mathrm{c}}(T^*Z) \longrightarrow \mathbb{Z},\ \operatorname{ind}(P) = \operatorname{ind}_{\mathrm{a}}([(\pi^*E_+, \pi^*E_-, \sigma_0(P))]). \tag{L14.20}$$

Proof. Since $K_{\mathrm{c}}(T^*Z)$ is defined as the set of equivalence classes of triples (E_+, E_-, a), with a an isomorphism outside a compact set, we need to show first that, for T^*Z, every such classes arises from the symbol of an elliptic operator. Notice that the fibres of the cotangent bundle are contractible, being vector spaces. So it is a standard fact (and easy enough to check) that every vector bundle over T^*Z is bundle isomorphic to π^*E for some bundle over Z. Using the invariance under bundle isomorphisms in the definition of $K_{\mathrm{c}}(T^*Z)$ it follows that every element is represented by a triple corresponding to an elliptic operator – note that by the homotopy invariance in the definition of the equivalence relation we may assume that a is homogeneous of degree 0 (or any other degree you might choose). So it only remains to show that the index is constant on equivalence classes. As for the bundles themselves, bundle isomorphism over T^*Z are homotopic to their values at the zero section, i.e. to bundle isomorphisms over Z. Such a bundle isomorphism is invertible and hence has zero index as a (rather trivial) pseudodifferential operator. This, with the homotopy invariance, shows that the index map does project to a well-defined map (L14.20). $\square$

Not only is this map, the 'analytic index map' well defined but it is clearly a homomorphism, since we know that $\operatorname{ind}(AB) = \operatorname{ind}(A) + \operatorname{ind}(B)$.

Gelfand around 1960 asked what amounts to the question of identifying this map in topological terms and in particular to find a formula for it.

An answer to this, given by Atiyah and Singer, is to define another map, the topological index, and show that the two are equal. This second map is defined by 'trivializing the topology' of the space. Namely by embedding Z as a submanifold of a simple manifold, either Euclidean space or a sphere according to your taste. Then, and this is where most of the work is, an operator on the larger space is constructed which has the same index and with symbol which is 'derived' from that of the original operator. For appropriate operators on the sphere (trival near the point at infinity) the index can again be seen to be an isomophism and this allows the topological index to be defined, or the analytic index to be computed depending on how you look at it.

14+. Addenda to Lecture 14