CHAPTER 13

Classifying sequence for K-theory

Lecture 13: 1 November, 2005

Today I will discuss some of the consequences of the two homotopies I described last week.

Recall the second of these results. Let X be a compact manifold and consider

$$A : X \longrightarrow \mathcal{C}^\infty(\mathbb{S}; \mathrm{GL}(N, \mathbb{C})),$$

a family of smooth maps, so $A \in \mathcal{C}^\infty(X \times \mathbb{S}; \mathrm{GL}(N, \mathbb{C}))$, satisfying the normalization condition that $A(x, 1) = \mathrm{Id}$ for all $x \in X$. We are permitted to stabilize the family by embedding $\mathrm{GL}(N, \mathbb{C})$ in $\mathrm{GL}(M, \mathbb{C})$ for $M \geq N$. Then for M sufficiently large we can find a homotopy, which is to say a family $A_t \in \mathcal{C}^\infty(X \times [0, 1]_t \times \mathbb{S}; \mathrm{GL}(M, \mathbb{C}))$, such that $A_0 = A$ and

$$A_1(x) = \pi_-(x)e^{-i\theta} + \pi_0(x) + \pi_+(x)e^{i\theta} \tag{L13.1}$$

where π_-, π_0 and π_+ are three smooth families of projections which are mutually commuting and sum to the identity.

L13.1. Numerical index for the circle.

Corollary 5. *If $P \in \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ is an elliptic Toeplitz operator, so $\sigma_0(P) \in \mathcal{C}^\infty(\mathbb{S}; \mathrm{GL}(N, \mathbb{C}))$ then*

$$\mathrm{ind}(P) = \dim(\mathrm{null}(P)) - \dim(\mathrm{null}(P^*)) = \frac{i}{2\pi}\int_{\mathbb{S}} \mathrm{Tr}\left(\sigma_0(P)^{-1}\frac{d\sigma_0(P)}{d\theta}\right) d\theta. \tag{L13.2}$$

Proof. For a single symbol (i.e. $X = \{\mathrm{pt}\}$) of the form (L13.1) we can prove (L13.2) directly. Namely

$$P = \pi_- L + \pi_0 + \pi_+ U \tag{L13.3}$$

is a Toeplitz operator with this symbol, since L has symobl $e^{-i\theta}$ and U has symbol $e^{i\theta}$. The null space of P is

$$\mathrm{null}(P) = \pi_-(\mathbb{C}^N) \tag{L13.4}$$

and since the adjoint is $P^* = \pi_- U + \pi_0 + \pi_+ L$

$$\mathrm{null}(P^*) = \pi_+(\mathbb{C}^N) \Longrightarrow \mathrm{ind}(P) = \mathrm{rank}(\pi_-) - \mathrm{rank}(\pi_+). \tag{L13.5}$$

On the other hand, with $\sigma_0(P)$ given by A_1,

$$A_1^{-1}\frac{dA_1}{d\theta} = -i\pi_- + i\pi_+ \Longrightarrow \int_{\mathbb{S}} \mathrm{Tr}\left(A_1^{-1}\frac{dA_1}{d\theta}\right) d\theta = -2\pi i\,\mathrm{ind}(P) \tag{L13.6}$$

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which is (L13.2) in this special case.

The homotopy argument shows that every elliptic symbol $p \in \mathcal{C}^\infty(\mathbb{S}; \mathrm{GL}(N, \mathbb{C}))$ normalized by $p(1) = \mathrm{Id}$ is stably homotopic to one of the form (L13.1). Setting $Q = p(1) \in \mathrm{GL}(N, \mathbb{C})$ it follows that any elliptic operator may be written as a product $P = QP'$ where $\sigma(P')$ satisfies the normalization condition and $Q \in \mathrm{GL}(N, \mathbb{C})$. Since Q is an isomorphism, the index of P is equal to the index of P'. Moreover, since Q is independent of θ,

$$\text{(L13.7)} \quad \int_{\mathbb{S}} \mathrm{Tr}\left(\sigma_0(P)^{-1}\frac{d\sigma_0(P)}{d\theta}\right) d\theta = \int_{\mathbb{S}} \mathrm{Tr}\left(\sigma_0(P')^{-1}Q^{-1}\frac{dQ\sigma_0(P')}{d\theta}\right) d\theta$$
$$= \int_{\mathbb{S}} \mathrm{Tr}\left(\sigma_0(P')^{-1}\frac{d\sigma_0(P')}{d\theta}\right) d\theta.$$

Thus, it suffices to prove the index formula for P', i.e. to assume the normalization condition for P. Now, the index of a curve of elliptic operators is constant and we also know, from Proposition 22, that the right side of (L13.2) is homotopy invariant, i.e. is constant along a curve of elliptic symbols and holds at the end point. Thus (L13.2) must hold in general. $\square$

A similar argument works for elliptic pseudodifferential operators on the circle, with the resulting formula being 'the same' except there are now two circles forming the boundary of $\overline{T^*\mathbb{S}}$.

Proposition 28. *If $P \in \Psi^0(\mathbb{S}; \mathbb{C}^N)$ is elliptic*

(L13.8)
$$\mathrm{ind}(P) = \dim(\mathrm{null}(P)) - \dim(\mathrm{null}(P^*)) = \frac{i}{2\pi}\int_{S^*\mathbb{S}} \mathrm{Tr}\left(\sigma_0(P)^{-1}\frac{d\sigma_0(P)}{d\theta}\right) d\theta.$$

As a consequence of this one can see that the index of any differential operator on the circle vanishes. Namely, the principal symbol of a differential operator is a homogeneous polynomial $p(\theta)\tau^k$ so the restrictions to $\pm\infty$ are $(-1)^k p(\theta)$ as sections of the trivial homogeneity bundle. The signs cancel in (L13.8) and the orientations are opposite, so the terms cancel each other.

L13.2. Contractibility of the Toeplitz group. The central consequence of the two homotopies discussed last week is the weak contractibility of the normalized and stabilized group of invertible Toeplitz operators. Let me recall the definition. We start with the Szegő projector, $S \in \Psi^0(\mathbb{S})$ which projects a smooth function on the circle to its non-negative-frequency part, $S : \mathcal{C}^\infty(\mathbb{S}) \longrightarrow \mathcal{C}^\infty_+(\mathbb{S})$. Then the Toeplitz algebra is the compression of the pseudodifferential algebra to the range of S :

$$\text{(L13.9)} \qquad \Psi^0_{\mathcal{T}}(\mathbb{S}) = S\Psi^0(\mathbb{S})S$$

which we think of as operators on $\mathcal{C}^\infty_+(\mathbb{S})$. There is no problem considering matrices of such operators, forming the algebras $\Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ but we want to consider the 'fully stabilized' algebra which is the Toeplitz algebra 'with values in the smoothing operators' on another compact manifold Y (and maybe acting on a bundle E.)

So, consider

$$\text{(L13.10)} \qquad \mathcal{C}^\infty(Y^2; \Psi^0(\mathbb{S}) \otimes E) = I^{-\dim Y/2}(Y^2 \times \mathbb{S}^2, \mathrm{Diag}_{\mathbb{S}} \otimes \mathrm{Hom}(E))$$

where for simplicity of notation I am leaving out the density bundles, since they are trivial anyway. From the results we have proved for conormal distributions, this is an algebra where the product can be interpreted in several equivalent ways. Perhaps the clearest is to do the composition in $\mathbb{S}$ first. Thus, if $A, B \in \mathcal{C}^\infty(Y^2; \Psi^0(\mathbb{S}) \otimes E)$ then

$$A(y, y') \circ_{\mathbb{S}} B(z, z') \in \mathcal{C}^\infty(Y^4; \Psi^0(\mathbb{S}) \otimes \operatorname{Hom}(E)_L \otimes \operatorname{Hom}(E)_R) \tag{L13.11}$$

where the two copies of $\operatorname{Hom}(E)$ are on the left two and the right two copies of Y. We can then restrict to $y' = z$, compose in the two copies of $\operatorname{Hom}(E)$ and integrate out the z variable giving the composite

$$(A \circ B)(y, y') = \int_Y A(y, z) \circ_{\mathbb{S}} B(z, y') \tag{L13.12}$$

where we really do need to be carrying the densities along to do the integration invariantly.

Now, we can compress the operators onto the range of S as before, or equivalently consider directly the smooth maps into the Toeplitz algebra $\mathcal{C}^\infty(Y^2; \Psi^0_{\mathcal{T}}(\mathbb{S}) \otimes E)$. I will denote this space with the product (L13.12) as $\Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S}, Y; E)$. The symbol map on $\Psi^0_{\mathcal{T}}(\mathbb{S})$ extends to give a symbol map which is multiplicative and takes values in the loops in smoothing operators

$$\sigma_0 : \Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S}, Y; E) \longrightarrow \mathcal{C}^\infty(\mathbb{S}; \Psi^{-\infty}(Y; E)),\ \sigma_0(AB) = \sigma_0(A)\sigma_0(B). \tag{L13.13}$$

This algebra does effectively stabilize the matrix-valued Toeplitz operators since we can embed the $N \times N$ matrices as a subalgebra of $\Psi^{-\infty}(Y; E)$, just by choosing an N-dimensional subspace of $\mathcal{C}^\infty(Y; E)$, and then

$$\Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N) \hookrightarrow \Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S}, Y; E) \tag{L13.14}$$

as a subalgebra acting on the subspace. Of course such an inclusion is not natural, but any two choices are homotopic through such embeddings, simply by rotating one subspace of $\mathcal{C}^\infty(Y; E)$ into the other.

Finally then we come to the group which is

$$\begin{aligned} G^0_{\mathcal{T}}(\mathbb{S}, Y; E) = \big\{A \in \Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S}, Y; E);\ \operatorname{Id} + \sigma(A) \in G^{-\infty}_{(1)}(Y; E),& \\ (\operatorname{Id} + A)^{-1} - \operatorname{Id} \in \Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S}, Y; E) \text{ and } \sigma_0(A)\big|_{1 \in \mathbb{S}} = 0\big\}.& \end{aligned} \tag{L13.15}$$

The first condition is ellipticity (recall that $G^{-\infty}_{(1)}(Y; E)$ is the loop group for $G^{-\infty}(Y; E)$, corresponding to maps from the circle). The last condition is the normalization condition. Since the symbol, fixed at a point on the circle, takes values in $G^{-\infty}(Y; E)$ this effectively kills off a whole classifying space for odd K-theory. We need this to get the result we are after, namely

Theorem 5. *The topological group $G^0_{\mathcal{T}}(\mathbb{S}, Y; E)$ is weakly contractible, i.e. if $f : X \longrightarrow G^0_{\mathcal{T}}(\mathbb{S}, Y; E)$ is any smooth map from a compact manifold then there is a smooth homotopy $f_\cdot : X \times [0, 1] \longrightarrow G^0_{\mathcal{T}}(\mathbb{S}, Y; E)$ with $f_0 = f$ and $f_1 \equiv \operatorname{Id}$.*

It is easy to see that continuous maps are approximable by smooth maps – or indeed the proof below carries through in the continuous case with only a little extra care.

Conjecture 1. *The group $G^0_{\mathcal{T}}(\mathbb{S}, Y; E)$ is dominated by a CW complex and as a result is actually contractible.*

Proof. Basically this amounts to putting the two homotopies, discussed earlier, together. First however we need to discuss the topology, to check that we do indeed have a topological group – in the infinite dimensional case such as this one needs to be careful. The topology on the space of conormal distributions of any fixed order, for a fixed submanifold, is very like the $\mathcal{C}^\infty$ topology. Namely we know that a conormal distribution is the sum of a smooth term and the inverse Fourier transform of a symbol and we can write this as

$$I^m(X,Z;E) \ni u \Longrightarrow \phi u = F^*\mathcal{F}^{-1}(a),\ a \in \mathcal{C}^\infty(\overline{N^*Z}; E\big|_Z \otimes N_{-m'} \otimes \Omega_{\text{fib}}) \tag{L13.16}$$

where $\phi \in \mathcal{C}^\infty(X)$ cuts off close to Z in the collar neighbourhood fixed by F. With such choices (including the identification of E on the collar neighbourhood with $E\big|_Z$) made, a and $\phi u \in \mathcal{C}^\infty(X;E)$ are determined and we can impose the usual $\mathcal{C}^\infty$ topology on them. That is, the seminorms on $I^m(X,Z;E)$ are those giving uniform convergence of all derivatives for a and ϕu. This gives a metric topology on $I^m(X,Z;E)$ with respect to which it is complete. Of course it is necessary to check that different choices of cutoff, normal fibration and bundle identification lead to the same topology but this follows directly from the earlier proofs (and I should have mentioned it ...).

The spaces of pseudodifferential operators are just special cases of conormal distributions so they also have such topologies. Moreover the proof of the composition theorem shows the continuity of composition with respect to this topolgy, so we have the first condition needed for a topological group, that

$$G^0_{\mathcal{T}}(\mathbb{S},Y;E) \times G^0_{\mathcal{T}}(\mathbb{S},Y;E) \ni (A,B) \longmapsto AB \in G^0_{\mathcal{T}}(\mathbb{S},Y;E) \tag{L13.17}$$

is continuous with respect to the topology inherited from $\Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S},Y;E)$. We also need to check that the same is true for

$$G^0_{\mathcal{T}}(\mathbb{S},Y;E) \ni A \longmapsto A^{-1} \in G^0_{\mathcal{T}}(\mathbb{S},Y;E). \tag{L13.18}$$

This is the usual stumbling block. In fact, the way we constructed the inverse was to first use the ellipticity to construct a parametrix and then the parameterix was 'corrected ' to the inverse by adding a smoothing operator. The construction of the parametrix is locally uniform on compact sets – it involves summation of the Taylor series for the symbol. The construction of the compensating smoothing term is also locally uniform. The uniqueness of the inverse (given that it exists) gives continuity on compact sets. This is enough to give the continuity in (L13.18) since the topology is metrizable, so it is enough to prove sequential continuity. In fact the set of invertible elliptic elements is open (within the subspace fixed by the normalization condition).

Now we proceed in 5 steps.

1) Given such a smooth map $f : X \longleftarrow G^0_{\mathcal{T}}(\mathbb{S},Y;E)$ we first approximate closely, and uniformly on X, by elements of $G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ using (L13.14) and hence deform into this smaller algebra. This follows exactly as in the approximation of smoothing operators by finite rank operators discussed earlier, the only difference is that in (L13.10) our smoothing operators are valued in the Toeplitz operators. So, simply decompose Y^2 into small product sets $U_i \times U_j$ over which the bundle E is trivial and which are embedded in the torus. Using the product of a partition of unity from Y and Fourier expansion on the torus allows us to approximate f arbitrarily closely. Note that the fact that the smooth functions are valued in the linear space $\Psi^0_{\mathcal{T}}(\mathbb{S})$ makes very little differece, since this is essentially the same as $\mathcal{C}^\infty(Z)$ for a compact manifold Z (in fact we can reduce to that case for the symbol and the smoothing error). It follows that the approximation is uniform on X and when enough terms in the Fourier series are taken the resulting finite rank family

(on Y) lies in $G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ and is homotopic to f in $G^0_{\mathcal{T}}(\mathbb{S},Y;E)$. Notice that we can maintain the normalization condition by first ignoring it and then afterwards composing with the inverse of $\sigma_0(f.)\big|_{1\in\mathbb{S}}$ thought of as a map from $X\times[0,1]$ into $G^{-\infty}(Y;E)$.

2) Now we are reduced to a smooth map from X into $G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$. This was the setting in which the homotopy given by Atiyah was discussed above. By first approximating the symbol by its truncated Fourier expansion and then stabilizing (depending on the order of the truncated symbol as a trigonomentric polynomial) we get a homotopy for the symbol, stabilized to an element of $\mathcal{C}^\infty(\mathbb{S};\mathrm{GL}(M,\mathbb{C}))$, for M large, to a symbol of the form

$$\pi_-(x)e^{-i\theta}+\pi_0(x)+\pi_+(x)e^{i\theta}\in\mathcal{C}^\infty(\mathbb{S};\mathrm{GL}(M,\mathbb{C})). \tag{L13.19}$$

Here the smooth families of projections π_-, π_0 and π_+ are mututally commuting and sum to the identity.

A smooth family of operators with the symbol (L13.19) is

$$A(x)=\pi_-(x)L+\pi_0(x)+\pi_+(x)U\in\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^M). \tag{L13.20}$$

This is certainly elliptic and we know that we may stabilize the null spaces to a bundle by considering $A(x)(\mathrm{Id}-\pi_k)$ for large enough k, where π_k is projection onto the span of the $e^{ij\theta}$ for $0\le j\le k$. The null space is then equal to that of $\mathrm{Id}-\pi_k$ and we are interested in the null bundle of the adjoint

$$\begin{aligned}\mathrm{null}((\mathrm{Id}-\pi_k)A(x)^*)&=\mathrm{null}\left((\mathrm{Id}-\pi_k)(\pi_-(x)U+\pi_0(x)+\pi_+(x)L)\right)\\ =\mathrm{sp}\{e^{ij\theta}\mathbb{C}^M,\ 0\le j\le(k-1),\ &(\pi_0(x)+\pi_+(x))\mathbb{C}^Me^{ik\theta},\ \pi_+(x)\mathbb{C}^Me^{i(k+1)\theta}\}.\end{aligned} \tag{L13.21}$$

3) Now, the original family was invertible and we know that along a curve of elliptics, which is initially invertible, we may perturb by a smoothing family (initially zero) to maintain invertibility. Thus the family we arrive at, of the form (L13.20) can be perturbed to be invertible by a smoothing operator. As shown earlier this means that the null bundle and null bundle of the adjoint are bundle isomorphic once they are sufficiently stabilized. In this case this just means that the bundle (L13.21) is trivial, i.e. isomorphic to a trivial bundle of the same rank, for large enough k. Writing out (L13.21) this means

$$\mathrm{null}((\mathrm{Id}-\pi_k)A^*(\cdot))=\mathbb{C}^{kM}\oplus\left(\mathbb{C}^M\setminus\mathrm{Ran}(\pi-(\cdot)\right)\oplus\mathrm{Ran}(\pi_+(\cdot))\simeq\mathbb{C}^{(k+1)M}. \tag{L13.22}$$

This in turn means that there exists

$$F:\mathrm{Ran}(\pi_-(\cdot))\oplus\mathbb{C}^{(k+1)M}\longleftrightarrow\mathrm{Ran}(\pi_+(\cdot))\oplus\mathbb{C}^{(k+1)M}, \tag{L13.23}$$

i.e. that the ranges of these two projections are stably isomorphic.

Now, for any bundle, with projector π it is straightforward to see that the symbol

$$\begin{pmatrix}\pi(x)e^{-i\theta}+(\mathrm{Id}-\pi(x)) & 0\\ 0 & (\mathrm{Id}-\pi(x))+\pi(x)e^{i\theta}\end{pmatrix} \tag{L13.24}$$

is homotopic to the identity through invertible symbols. Indeed one such homotopy is

$$\begin{pmatrix}\cos(\theta)\pi(x)e^{-i\theta}+(\mathrm{Id}-\pi(x)) & \sin(\theta)\pi(x)\\ -\sin(\theta)\pi(x) & (\mathrm{Id}-\pi(x))+\cos(\theta)\pi(x)e^{i\theta}\end{pmatrix} \tag{L13.25}$$

rotating to $\pi/2$ and then back again without the exponentials. It follows that by using such a homotopy from the identity (in some other matrix block) the symbol in (L13.20) can be connected to one in which π_- and π_+ are increased by the same trivial projection corresponding to $\mathbb{C}^{kM}$. Then the isomorphism in (L13.23) can be used to deform this symbol to the identity. Namely, simplifying the notation by identifying $\pi_\pm$ with the stabilized projections, we may identify F as an isomorphism from the range of π_- to the range of π_+. Splitting the space into three, the ranges of π_-, π_0 and π_+ we may consider the homotopy (where the $\pi_\pm$'s are now redundant) from $\tau = 0$ to $\pi/2$

$$\begin{pmatrix} \cos(\tau)\pi_- e^{-i\theta} & 0 & \sin(\tau)F^{-1} \\ 0 & \pi_0 & 0 \\ -\sin(\tau)F & 0 & \cos(\tau)\pi_+ e^{i\theta} \end{pmatrix} \tag{L13.26}$$

and then back again without the exponentials, finishing at the identity.

This we have deformed the family of symbols to the identity after sufficient stabilization. As already noted this can be lifted to a deformation of invertibles, i.e. in $G^0_{\mathcal{T}}(\mathbb{S}, Y; E)$ which finishes at an element of $G^{-\infty}_{\mathcal{T}}(\mathbb{S} \times Y; E)$ (which is of finite rank in Y.)

4) At this stage in the deformation the symbol has been trivialized and we are reduced to a family $A \in \mathcal{C}^\infty(X; G^{-\infty}_{\mathcal{T}}(\mathbb{S} \times Y; E))$ which can be taken to be of finite rank in Y, i.e. to have image in a subgroup $G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ for large N. Even if it were not the case we can achieve this result directly by finite rank approximation in Y as before. Now, we further make a finite rank approximation in $\mathbb{S}$ by replacing the family by $(\mathrm{Id} - \pi_k)A(x)(\mathrm{Id} - \pi_k)$ which converges uniformly to $A(x)$ as $k \to \infty$. Taking k sufficiently large, the family may now be assumed to act on the finite dimensional subspace of $\mathcal{C}^\infty_+(\mathbb{S} \times Y; E)$ spanned by

$$e^{ij\theta} e_l,\ 0 \le j \le k,\ 0 \le l \le N. \tag{L13.27}$$

Now, again stabilize the group by expanding N to $(k+1)N$ by choosing k other independent subspaces of $\mathcal{C}^\infty(Y; E)$ of the same dimension. Then the basis in (L13.27) is expanded to

$$e^{ij\theta} e_{l,p},\ 0 \le j \le k,\ 0 \le l \le N,\ 0 \le p \le k \tag{L13.28}$$

where $e_{l,0} = e_l$ and of course the operator is the identity on the terms with $p > 0$. Then consider the rotation of basis elements in 2 dimensional spaces for $1 \le j \le k$, $1 \le l \le N$

$$\cos(\theta)e^{ij\theta}e_l + \sin(\theta)e_{l,j},\ -\sin(\theta)e^{ij\theta}e_l + \cos(\theta)e_{l,j},\ \theta \in [0, \pi/2] \tag{L13.29}$$

with all other elements held fixed. This has the effect of rotating all the non-trivial parts of the matrix into the 0 Fourier term with everything outside the constants on the circle being the identity.

5) The final step is then to follow the first homotopy of last week which allows such a matrix in $\mathrm{GL}(N, \mathbb{C}) \subset G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$, identified as the zero Fourier terms, to be deformed to the identity in $G^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^{2N})$. This completes the deformation to the identity. □

L13.3. Classifying sequence for K-theory. One reason this weakly contractible group is of interest here is that it gives a smooth classifying sequence for K-theory.

THEOREM 6. *There is a short exact sequence of topological groups*

$$G^{-\infty}_{\mathcal{T}}(\mathbb{S}\times Y;E) \longrightarrow G^{0,-\infty}_{\mathcal{T}}(\mathbb{S},Y;E) \longrightarrow G^{-\infty}_{(1),0}(Y;E)[[\rho]] \tag{L13.30}$$

in which the first group is classifying for odd K-theory, the second is weakly contractible and the third is (therefore) a reduced classifying group for even K-theory (i.e. the identity component of such a classifying group). The quotient group is a formal countable sum (i.e. the elements are sequences, written as power series in the indeterminant ρ) with leading term an element of $G^{-\infty}_{(1),0}(Y;E)$, the subgroup of the loop group $\mathcal{C}^\infty(\mathbb{S};G^{-\infty}(Y;E))$ consisting of the pointed loops (taking 1 to the identity) of index zero and with lower order terms which are arbitrary elements of $\mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}(Y;E))$.

Since (L13.30) is a short exact sequence of groups, there is a product induced on the quotient. This will show up a bit later.

PROOF. There is actually not too much to prove here since we have shown the weak contractibility. The leading term of the projection map is just the principal symbol σ. Thus, if $A \in G^{0,-\infty}_{\mathcal{T}}(\mathbb{S},Y;E)$ then we know that $\sigma_0(A) \in \mathcal{C}^\infty(\mathbb{S};G^{-\infty}(Y;E))$ has index zero (this follows from our first result today) and $\sigma_0(A)(1) = 0$ is the normalization condition on the symbol. This is precisely the definition of $G^{-\infty}_{(1),0}(Y;E)$ and the map is surjective since any such symbol of index zero is the symbol of an invertible operator.

To get the second map in (L13.30) we just consider a normal fibration around the diagonal in $\mathbb{S}$. Then the corresponding 'full symbol map' takes a conormal distribution in (L13.10) and maps it to the Taylor series at the circle at infinity of the transverse Fourier transform of the kernel (cut off near the diagonal of $\mathbb{S}$). This gives a short exact sequence of linear maps

$$\Psi^{-\infty}_{\mathcal{T}}(\mathbb{S}\times Y;E) \longrightarrow \mathcal{C}^\infty(Y^2;\Psi^0_{\mathcal{T}}(\mathbb{S})\otimes E) \longrightarrow \sum_{j=0}^{\infty}\rho^j\mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}(Y;E)). \tag{L13.31}$$

The only constraint on an elliptic operator to be perturbable is the already-noted requirement that the index vanish. Thus, for invertible perturbations of the identity we arrive at (L13.30). ☐

There are two other closely related theorems that I will prove next time, again as consequences of the homotopies discussed earlier. To state them I need to define the 'usual' K-group $K^0(X)$ for a compact manifold X. Traditionally this is the starting point for topological K-theory, but I have instead approach the subject through

$$\begin{aligned} &K^{-1}(X) = [X;G^{-\infty}(Y;E)] \text{ and} \\ &\qquad K^{-2}(X) = [X;G^{-\infty}_{(1)}(Y;E)] = [X\times\mathbb{S}, X\times\{1\};G^{-\infty}(Y;E),\mathrm{Id}]. \end{aligned} \tag{L13.32}$$

We define $K^0(X)$ as the Grothendieck group associated to stable vector bundles (under direct sum). Thus if $E \longrightarrow X$ and $F \longrightarrow X$ are two vector bundles over X they are isomorphic if there is a diffeomorphism between the total spaces $E \longleftrightarrow F$ which maps the fibre E_x linearly to the fibre F_x; denote this relationship $E \equiv F$. To define $K^0(X)$ consider pairs of vector bundles (E_+,E_-) (also thought of as $\mathbb{Z}_2$-graded vector bundles) and the equivalence relation of stable isomorphism. That is

$$(E_+,E_-)\sim(F_+,F_-)\Longleftrightarrow\exists\ H \text{ s.t. } E_+\oplus F_-\oplus H\equiv E_-\oplus F_+\oplus H. \tag{L13.33}$$

It is straightforward to check that this is an equivalence relation and the set it defines, $K^0(X)$, is an abelian group under direct sum

$$[(E_+,E_-)]+[(F_+,F_-)]=[(E_+\oplus F_+,E_-\oplus F_-)]. \tag{L13.34}$$

THEOREM 7. *[Families index for the Toeplitz algebra] Given* $[a]\in K^{-2}(X)$, *represented by* $a\in\mathcal{C}^\infty(X\times\mathbb{S};\mathrm{GL}(N,\mathbb{C}))$ *with* $a(1)=\mathrm{Id}$, *choosing any smooth family of operators* $A\in\mathcal{C}^\infty(X;\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N))$ *with* $\sigma_0(A)=a$, *the stabilized 'families index'*

$$[(\mathrm{null}(A(x)(\mathrm{Id}-\pi_k),\mathrm{null}((\mathrm{Id}-\pi_k)A^*(x))]\in K^0(X) \tag{L13.35}$$

is well-defined for large k, *independent of the choice of* A, *and defines an isomorphism of abelain groups*

$$K^{-2}(X)\longrightarrow K^0(X). \tag{L13.36}$$

THEOREM 8. *[Bott periodicity] For any representative* $[(E_+,E_-)]\in K^0(X)$ *one can choose smooth families of commuting projections* $\pi_-(x)$, $\pi_0(x)$, $\pi_+(x)$ *on* $\mathbb{C}^N$ *for large* N *such that* $E_\pm$ *are isomorphic to the ranges of* $\pi_\pm$ *and* $\pi_-(x)+\pi_0(x)+\pi_+(x)=\mathrm{Id}$ *and then the element*

$$\pi_-(x)e^{-i\theta}+\pi_0(x)+\pi_+(x)e^{i\theta}\in\mathcal{C}^\infty(X\times\mathbb{S};\mathrm{GL}(N,\mathbb{C})) \tag{L13.37}$$

projects to a well-defined map

$$K^0(X)\longrightarrow K^{-2}(X) \tag{L13.38}$$

which is an isomorphism.

The maps in these two theorems are just inverses of each other (assuming that I have not messed up the signs).

13+. Addenda to Lecture 13

13+.1. Proof of Proposition 28. First choose an element $P_+\in\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ with $\sigma(P_+)=\sigma(P)\big|_{S^*_+\mathbb{S}}$. Then the operator $P_++(\mathrm{Id}-S)\in\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ has the same index as P_+ (the latter acting on $\mathcal{C}^\infty_+(\mathbb{S};\mathbb{C}^N)$) so the formula (L13.2) applies. We can also choose a 'negative' Toeplitz operator, $P_-\in\Psi^0_{-\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$, the Toepltiz algebra for the opposite orientation, with $\sigma(P_-)=\sigma(P)\big|_{S^*_-\mathbb{S}}$. Extending it as the identity on the positive side, P_+P_- is an elliptic operator with the same index as P and this index is $\mathrm{ind}(P_+)+\mathrm{ind}(P_-)$. This proves (L13.8).