CHAPTER 12

Linearization of symbols

Lecture 12: 27 October, 2005

Today I will go through the second homotopy that I will use next time to construct the classifying sequence for K-theory. This construction is due to Atiyah ([**1**]). The question is the extent to which one can simplify, or bring to normal form, a family of loops in $\mathrm{GL}(N,\mathbb{C})$. Thus, for a given smooth compact manifold X suppose we have a smooth map $a : X \longrightarrow \mathcal{C}^\infty(\mathbb{S};\mathrm{GL}(N,\mathbb{C}))$ which is the same thing as an element of $\mathcal{C}^\infty(X\times\mathbb{S};\mathrm{GL}(N,\mathbb{C}))$. I will assume that it satisfies the normalization condition

$$a\big|_{1\in\mathbb{S}} \equiv \mathrm{Id}\,. \tag{L12.1}$$

We are allowed to make deformations, i.e. homotopies, of the family and we are also permitted to *stabilize* the family by embedding $\mathrm{GL}(N,\mathbb{C}) \hookrightarrow \mathrm{GL}(M,\mathbb{C})$ for any $M \geq N$, as the subgroup

$$\mathrm{GL}(N,\mathbb{C}) \ni a \longrightarrow \begin{pmatrix} a & 0 \\ 0 & \mathrm{Id}_{M-N}\end{pmatrix} \in \mathrm{GL}(M,\mathbb{C}). \tag{L12.2}$$

The result shown by Atiyah is that by such stabilization and deformation (always through invertibles of course) we may arrive at a family

$$\tilde{a}(x) = \pi_-(x)e^{-i\theta} + \pi_0(x) + \pi_+(x)e^{i\theta} \tag{L12.3}$$

where π_-, π_0 π_+ are three smooth maps into the projections on $\mathbb{C}^M$ which mutually commute for each x and sum to the identity

$$\pi_-(x) + \pi_0(x) + \pi_+(x) = \mathrm{Id} \ \ \forall\ x \in X. \tag{L12.4}$$

Notice that this is just the normalization condition (L12.1) for a family of the form (L12.3).

To construct such a (stable) homotopy, we first consider the Fourier expansion of a

$$a(x,\theta) = \sum_{j=-\infty}^{\infty} a_j(x)e^{ij\theta}. \tag{L12.5}$$

The coefficients here are smooth functions valued in $N\times N$ matrices, namely

$$a_j(x) = \frac{1}{2\pi}\int_{\mathbb{S}} e^{-ij\theta}a(x,\theta)d\theta \tag{L12.6}$$

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which vanish rapidly with j, so for any differential operator P on X and any $q \in \mathbb{N}$.

$$\sup_X |Pa_j(x)| \le C_q(1+|j|)^{-q}. \tag{L12.7}$$

Thus the series (L12.5) converges rapidly in any C^p norm and there exists q such that with

(L12.8)

$$a_{(q)}(x,\theta) = \sum_{|j|\le q} a_j(x)e^{ij\theta},\ a_t = (1-t)a + ta_{(q)} : [0,1]\times X\times\mathbb{S} \longrightarrow \mathrm{GL}(N,\mathbb{C}).$$

We can also maintain the normalization condition under the homotopy since $c_t(x) = a_t(x,1) : [0,1]\times X \longrightarrow \mathrm{GL}(N,\mathbb{C})$ is the identity at $t=0$ so $c_t^{-1}(x)a_t(x,\theta)$ is a new homotopy to a trigonometric polynomial satisfying the normalization condition. Thus a and $a_{(q)}$ are homotopic, so we can consider instead $a_{(q)}$ and just suppose that a is a trigonometric polynomial of some degree satisfying the normalization condition.

Thus $a(x,\theta) = b(x,z)\big|_{z=e^{-i\theta}}$ where

$$b(x,z) = z^{-q}b'(x,z),\ b' : X\times\mathbb{C} \longrightarrow M(N,\mathbb{C}) \text{ a polynomial of degree } 2q \tag{L12.9}$$

and of course b' is invertible on the circle and $b'(x,1) = \mathrm{Id}$. Now we will use a simple form of stabilization to separate off the z^{-q} factor. Add another $N\times N$ identity block and consider the 2×2 block rotation

$$R_\tau = \begin{pmatrix} \cos(\tau)\,\mathrm{Id}_N & \sin(\tau)\,\mathrm{Id}_N \\ -\sin(\tau)\,\mathrm{Id}_N & \cos(\tau)\,\mathrm{Id}_N \end{pmatrix}. \tag{L12.10}$$

Replacing a by

$$R_{-\tau}\begin{pmatrix} z^{-q}\,\mathrm{Id}_N & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix} R_\tau \begin{pmatrix} b'(x,z) & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix}$$

gives a homotopy for $\tau \in [0,\pi/2]$ which rotates the z^{-q} into the second block, finishing at

$$\begin{pmatrix} b'(x,z) & 0 \\ 0 & z^{-q}\,\mathrm{Id}_N \end{pmatrix}. \tag{L12.11}$$

We then proceed to discuss these two blocks separately, of course the first is a good deal more complicated than the second. We will stabilize the first block by another p blocks each $N\times N$, where $p = 2q$ is the degree of the polynomial b' (notice that a polynomial of degree p has $p+1$ terms.) Thus we replace the first block by

$$\begin{pmatrix} b'(x,z) & 0 & 0 & \dots & 0 \\ 0 & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{Id}_N \end{pmatrix} \tag{L12.12}$$

and what is crucial is that this is invertible on $X\times\mathbb{S}$ where the circle is $|z| = 1$ now. Since this matrix is block diagonal, we can keep invertibity while adding absolutely any terms above the diagonal. What I want to do is to choose polynomials valued in $N \times N$ matrices (no invertibility condition of course) and deform (L12.12) to

$$\begin{pmatrix} b'(x,z) & c_1(x,z) & c_2(x,z) & \dots & c_p(x,z) \\ 0 & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{Id}_N \end{pmatrix}. \tag{L12.13}$$

To do this, just put a $t \in [0,1]$ in front of the c_j's. We can imagine (L12.13) as postmuliplied by the identity, then deform the identity to

$$\begin{pmatrix} \mathrm{Id}_N & 0 & 0 & \dots & 0 \\ -z & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & -z & \mathrm{Id}_N & 0 \\ 0 & 0 & \dots & -z & \mathrm{Id}_N \end{pmatrix} \tag{L12.14}$$

which has $-z$ all along the 'subdiagonal'. This is a lower-triangular perturbation so is still invertible and homotopic to the identity. Thus, without having chosen the c_j, we have deformed the matrix to the product

(L12.15)

$$\begin{pmatrix} b'(x,z) & c_1(x,z) & c_2(x,z) & \dots & c_p(x,z) \\ 0 & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{Id}_N \end{pmatrix} \begin{pmatrix} \mathrm{Id}_N & 0 & 0 & \dots & 0 \\ -z & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & -z & \mathrm{Id}_N & 0 \\ 0 & 0 & \dots & -z & \mathrm{Id}_N \end{pmatrix}$$
$$= \begin{pmatrix} g_0(x) & g_1(x) & g_2(x) & \dots & g_p(x) \\ -z & \mathrm{Id}_N & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & -z & \mathrm{Id}_N \end{pmatrix} = p_+(x,z).$$

Here

$$g_0 = b' - zc_1,\ g_1 = c_1 - zc_2, \dots, g_{p-1} = c_{p-1} - zc_p,\ g_p = c_p. \tag{L12.16}$$

Observe that we can choose the c_j's successively to be polynomials of degree $p-j$ so that each of the g_j's is a constant matrix, i.e. does not depend on z at all. In fact the g_j's are just the coefficient matrices of b'.

At the end of this deformation the (enlarged) block corresponding to $b'(x,z)$ has been reduced to a linear, in z, matrix. We can proceed in the same way with the other, simpler block with entry $z^{-p}\,\mathrm{Id}_N$, but replacing z by $1/z$. This shows that there is a homotopy, after appropriate stabilization, to a matrix of the form

$$a(x,\theta) = a_-(x)e^{-i\theta} + a_0'(x) + a_0''(x) + a_+(x)e^{i\theta} \tag{L12.17}$$

through invertible matrices. Here the a_- and a_0' matrices form one block and the $a_0'' + a_+$ form another. As before we can enforce the normalization condition, that at the point $1 \in \mathbb{S}$ the matrix is the identity, simply by multiplying by the inverse of this matrix. Thus we can assume that *both* the blocks in the discussion above satify the normalization condition. Thus

$$a_-(x) + a_0'(x) = \mathrm{Id}_{p_-},\ a_0''(x) + a_+(x) = \mathrm{Id}_{p_+}. \tag{L12.18}$$

It follows that $a_-(x)$ and $a_0'(x)$ and $a_0''(x)$ and $a_+(x)$ commute for each x and these two block commute with each other. Thus, in the combined form (L12.17) it follows that $a_-(x)$, $a_0(x)$ and $a_+(x)$ are commuting matrices, for each x, summing to the identity.

Consider the matrices obtained by integration round the circle

$$\pi = \frac{1}{2\pi i}\int_{|z|=1} p_+(x,z)^{-1}\frac{dp_+(x,z)}{dz}\frac{dz}{z}. \tag{L12.19}$$

Since p_+ is invertible on the circle, this is a smooth matrix in x. Suppose for a moment that $a_+(x)$ is invertible. Then

$$p_+(x,z)^{-1} = (a_+(x))^{-1}(a_0''(x)+z)^{-1} \tag{L12.20}$$

and the contour integral (L12.19) may be evaluated by residues. In fact $\pi(x)$ is then the projection onto the span of those eigenvectors of $-a_+(x)^{-1}a_0''(x)$ with eigenvalues $|z|<1$ (and vanishing on the span of the eigenvectors with eigenvalues in $|z|>1$). We may always perturb $a_+(x)$ to $a_+(x)+s\,\mathrm{Id}$ for small s to make it invertible. So in the general case, without assuming that $a_+(x)$ is invertible, it follows that $\pi(x)$ is a projection (as the limit of a sequence of projections) and that it commutes with both $a_0''(x)$ and $a_+(x)$ (since these commute with the argument of the integral).

Decomposing $p_+(x,z)$ with respect to $\pi(x)$, with which it commutes, the term $p_+(x,z)\pi = a_1(x)+b_1(x)z$ has no zeros outside the unit circle so the matrix $(1-t)a_1(x)+b_1(x)z$ is invertible on the unit circle for all $t\in[0,1]$. Similarly $p_+(x,z)(\mathrm{Id}-\pi) = a_2(x)+b_2(x)z$ has no singular values inside the unit circle so $a_2(x)+(1-t)b_2(x)z$ remains invertible on the unit circle for all $t\in[0,1]$. Combining these two homotopies and premultiplying by the value at $z=1$ gives a homotopy of $p_+(x,z)$ to $\pi(x)+(\mathrm{Id}-\pi(x))z$ – indeed the end point is $a_2'(x)+b_1'(x)z$ where $a_2'(x)$ acts on the range of π and $b_1'(x)$ on the range of $\mathrm{Id}-\pi$ and the normalization condition holds.

Carrying out a similar analysis for $p_-(x,z)$ we obtain a homotopy, always keeping invertibility for $|z|=1$ from the initial map $a: X\times\mathbb{S}\longrightarrow \mathrm{GL}(N,\mathbb{C})$, after stabilization, to a family of the form (L12.3).

We will apply this homotopy to the symbols of a family of elliptic Toeplitz operators, $P: X\longrightarrow \Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$, allowing stabilization.

Proposition 26. *If P is a smooth family of invertible elliptic Toeplitz operators parameterized by the compact manifold X with symbols satisfying the normalization condition*

$$\sigma_0(P)\big|_{1\in\mathbb{S}} = \mathrm{Id} \tag{L12.21}$$

then, after stabilization, it may be smoothly deformed through invertible Toeplitz operators to $\tilde{P}: X\longrightarrow G^{-\infty}_{\mathcal{T}}(\mathbb{S};\mathbb{C}^M)$.

Proof. We can certainly apply the previous result to the symbol family, deforming it to the form (L12.3). We may then choose an elliptic family with these symbols which reduces to P at $t=0$. As shown above such a homotopy of families of elliptic operators which is invertible at $t=0$ may perturbed by a smoothing family, which vanishes at the intial point, to make the whole family invertible. Thus, we may suppose that we have an operator with symbol of the form (L12.4) and which is invertible. We can easily find an explicit family of operators with this symbol, namely

$$Q(x) = \pi_-(x)L + \pi_0(x) + \pi_+(x)U \tag{L12.22}$$

where L and U are the shift (down and up respectively) operators. Thus we can in fact suppose that $Q(x)$ is invertible after the addition of a smoothing family.

On the other hand we may easily compute the (stabilized) null bundles of $Q(x)$ and its adjoint. Namely (for any $k \ge 0$ it is not really necessary to stabilize here)

$$\begin{aligned} \operatorname{null}(Q(x)(\operatorname{Id} - \pi_k)) &= \operatorname{sp}\{e^{ij\theta}\mathbb{C}^M,\ 0 \le j \le k\} \\ \operatorname{null}((\operatorname{Id} - \pi_k)Q^*(x)) &= \operatorname{sp}\{e^{ij\theta}\mathbb{C}^M,\ 0 \le j \le k-1, \\ &\qquad e^{ik\theta}(\pi_0 + \pi_+(x))\mathbb{C}^M,\ e^{i(k+1)\theta}\pi_+(x)\mathbb{C}^M\}. \end{aligned} \tag{L12.23}$$

Now we know that the assumption that $Q(x)$ has an invertible perturbation means that these two bundles must be isomorphic for large k. The first of these is just the trivial bundle of rank $(k+1)M$ whilst the second is the trivial bundle of rank kM plus the range of $\pi_0 + \pi_+$ plus another copy of the range of π_+. Since π_- complements $\pi_0 + \pi_+$ to a trivial bundle of range M, adding the range of π_- to both sides (with the identity isomorphism) this means there must be a vector bundle isomorphism

$$\mathbb{C}^L + \operatorname{Ran}(\pi_-) \simeq \mathbb{C}^L + \operatorname{Ran}(\pi_+). \tag{L12.24}$$

Now, observe that the $2L \times 2L$ block matrix

$$\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \tag{L12.25}$$

is homotopic to the identity using a simple rotation

$$\begin{pmatrix} \cos(\tau)e^{-i\theta} & \sin(\tau) \\ -\sin(\tau) & \cos(\tau)e^{i\theta} \end{pmatrix} \tag{L12.26}$$

to $\tau = \pi/2$, followed by the rotation back without the exponentials. Thus we can, by stabilizing, add such a matrix to the symbol of $Q(x)$. This replaces π_+ and π_- by trivially stabilized projections so that they have ranges which are bundle isomorphic. Finally then this allows us to perform a similar rotation to the identity. Namely, identifying the range of π_- with that of π_+ using a bundle isomorphism, F, we may consider the homotopy

$$\begin{pmatrix} \cos(\theta)e^{-i\theta}\pi_- & \sin(\tau)F & 0 \\ -\sin(\tau)F^{-1} & \cos(\tau)e^{i\theta} & 0 \\ 0 & 0 & \pi_0 \end{pmatrix} \tag{L12.27}$$

and then back again without the exponentials.

Thus the symbol can be deformed to the identity (after stabilization of course) which means that the operator can be deformed, through invertibles, to a family in $G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^L)$. $\square$

Next time I will show that this construction, together with the construction from last time, gives the weak contractibility of the stabilized, normalized Toeplitz group which I will now proceed to define.

Choose any compact manifold Y (as usual with positive dimension). Then we can consider the space

(L12.28)
$$\mathcal{C}^\infty(Y^2;\Psi^m(\mathbb{S})\otimes\Omega_R Y) = I^{m'}(\mathbb{S}^2\times Y^2,\mathrm{Diag}_{\mathbb{S}};\Omega_R(Y\times\mathbb{S}),\ m'=m-\frac{1}{2}\dim Y.$$

Apart from the (trivial) density factors this is just the space of smooth functions with values in the pseudodifferential operators on $\mathbb{S}$. However, we may also interpret it as the space of pseudodifferential operators on $\mathbb{S}$ 'with values in the smoothing operators on Y.' That is, there is a full operator composition on this space.

To see this, consider the Toeplitz action of $A\in\mathcal{C}^\infty(Y^2;\Psi^m(\mathbb{S})\otimes\Omega_R)$ on $u\in\mathcal{C}^\infty(\mathbb{S}\times Y)$

$$A(y,y')u(\theta,y'')\in\mathcal{C}^\infty(Y^3\times\mathbb{S};\Omega'_Y). \tag{L12.29}$$

Restricting to the diagonal and integrating gives

$$Au=\int_Y A(y,y')u(\theta,y')\in\mathcal{C}^\infty(\mathbb{S}\times Y) \tag{L12.30}$$

and this is a continuous linear operator. Operator composition therefore works in the obvious way, if $A\in\mathcal{C}^\infty(Y^2;\Psi^{m_1}(\mathbb{S})\otimes\Omega_R Y)$, $B\in\mathcal{C}^\infty(Y^2;\Psi^{m_2}(\mathbb{S})\otimes\Omega_R Y)$ then

$$AB(y,y')=\int_Y A(y,z)\circ_{\mathbb{S}} B(z,y')\in\mathcal{C}^\infty(Y^2;\Psi^{m_1+m_2}(\mathbb{S})\otimes\Omega_R Y) \tag{L12.31}$$

and with this product we will denote the space as

$$\mathcal{C}^\infty(Y^2;\Psi^m(\mathbb{S})\otimes\Omega_R Y)=\Psi^{m,-\infty}(\mathbb{S},Y). \tag{L12.32}$$

It is straightforward to do the same thing for operators between sections of any two vector bundles over Y or $Y\times\mathbb{S}$. We can also look at the elements which are valued in the Toeplitz operators and consider the algebra

$$\Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S},Y)=S\Psi^{0,-\infty}(\mathbb{S},Y)S. \tag{L12.33}$$

We 'really' view this algebra as a stabilization of all the $\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ each of which can be embedded in it as a subalgebra by taking a corresponding finite dimensional subspace of $\mathcal{C}^\infty(Y)$ and considering only operators acting on it. These can also be thought of in the form (L12.33) in that if $\pi_N\in\Psi^{-\infty}(Y)$ is a projection onto a subspace of dimension N then

$$\pi_N\Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S},Y)\pi_N\simeq\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N). \tag{L12.34}$$

The symbol maps in all these cases are surjective maps onto the corresponding spaces of smooth functions

$$\begin{gathered}
\Psi^0(\mathbb{S})\xrightarrow{\sigma_0}\mathcal{C}^\infty(\mathbb{S}_+\sqcup\mathbb{S}_-)\\
\Psi^0_{\mathcal{T}}(\mathbb{S})\xrightarrow{\sigma_0}\mathcal{C}^\infty(\mathbb{S}),\ \mathbb{S}=\mathbb{S}_+\\
\Psi^0(\mathbb{S};\mathbb{C}^N)\xrightarrow{\sigma_0}\mathcal{C}^\infty(\mathbb{S};M(N,\mathbb{C}))\\
\Psi^{0,-\infty}_{\mathcal{T}}(\mathbb{S})\xrightarrow{\sigma_0}\mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}(Y))
\end{gathered} \tag{L12.35}$$

and these are homomorphism of the corresponding algebras.

Now, we want to consider the group of invertible perturbations of the identity of this type. Notice that the fact that these operators are valued in smoothing operators means that they cannot be invertible, say acting on $\mathcal{C}^\infty(\mathbb{S}\times Y)$, on their own. We add a normalization condition for the same reason as it was added to the homotopy result above and consider

(L12.36)
$$\begin{aligned} G^{0,-\infty}_{\mathcal{I}}(\mathbb{S},Y) = \big\{A \in \Psi^{0,-\infty}_{\mathcal{I}}(\mathbb{S}); \operatorname{Id}+A \text{ is elliptic (i.e. } \sigma_0(A) \in \mathcal{C}^\infty(\mathbb{S};G^{-\infty}(Y)) \\ (\operatorname{Id}+A)^{-1} - \operatorname{Id} \in \Psi^{0,-\infty}_{\mathcal{I}}(\mathbb{S}),\ \sigma(A)\big|_{1\in\mathbb{S}} = 0\big\}. \end{aligned}$$

Sometimes I will get carried away and just denote this as $G^0_{\mathcal{I}}(\mathbb{S})$ even though it does depend on Y.

So, next time I will prove

Proposition 27. *The topological group $G^{0,-\infty}_{\mathcal{I}}(\mathbb{S},Y)$ is weakly contractible.*

The proof, as I said before, is obtained by combining the two homotopies that I have talked about today and last time. We will get some other things out of these as well. Why should we care about this? For one thing it means that the short exact sequence of groups

(L12.37)
$$G^{-\infty}_{\mathcal{I}}(\mathbb{S}\times Y) \longrightarrow G^{0,-\infty}_{\mathcal{I}}(\mathbb{S},Y) \longrightarrow Q$$

where Q is the quotient, is a classifying sequence for K-theory.

12+. Addenda to Lecture 12