CHAPTER 11

Toeplitz operators

Lecture 11: 25 October, 2005

Today I want to start working towards the contractibility of the group which I will call $G^0_{\mathcal{T}}$ and which I have not yet defined. As mentioned last time it is made up out of the Toeplitz algebra, hence the subscript $\mathcal{T}$. For the moment I will prove some preliminary results about the Toeplitz algebra and make a start on the contractibility.

The most basic result I will not prove in full detail – it is a good excerise!

LEMMA 22. *The Szegő projector $S : \mathcal{C}^\infty(\mathbb{S}) \longrightarrow \mathcal{C}^\infty_+(\mathbb{S})$ given explicitly in terms of the Fourier series expansion by*

$$Su(\theta) = \sum_{k\ge 0} c_k e^{ik\theta} \text{ if } u = \sum_{k\in\mathbb{Z}} c_k e^{ik\theta} \tag{L11.1}$$

is an element of $\Psi^0(\mathbb{S})$; it is a self-adjoint projection ($S^2 = S^* = S$) and its amplitude, the local Fourier transform of its kernel with respect to a normal fibration, vanishes rapidly at infinity in one (the negative) direction.

HINT ONLY, CARRIED OUT BELOW. Think of $\mathbb{S}$ as the boundary of the unit disk $\mathbb{D}$ in the complex plane. The elements of $\mathcal{C}^\infty_+(\mathbb{S})$ are actually those which have extension to $\mathcal{C}^\infty(\mathbb{D})$ (smooth up to the boundary that means) which are holomorphic in the interior. Then S can be obtained as the boundary value of the map

$$\tilde{S}(f)(z) = \frac{1}{2\pi i}\int_{\mathbb{S}} \frac{f(e^{i\theta})}{z - e^{i\theta}} d\theta,\ |z| < 1, \tag{L11.2}$$

interpreted as a contour integral. Applied to $e^{ik\theta} = \tau^k$ $k \ge 0$ it gives z^k in the interior and applied to $e^{-ik\theta} = z^{-k}$, $k > 1$, it gives zero as can be checked using Cauchy's formula. From this the kernel of S can be recovered in terms of the limit as $|z| \uparrow 1$ of $(z - \tau)^{-1}$. Certainly then the kernel is smooth away from the diagonal and one can compute the Fourier transform transversal to the diagonal of the kernel (cut off near the diagonal) and show that it is an element of $\mathcal{C}^\infty(\overline{T^*\mathbb{S}})$. A little contour shoving will show that it vanishes rapidly in the negative direction and approaches 1 in the positive direction. □

Now, the Toeplitz algebra

$$\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N) = \left\{A \in \Psi^0(\mathbb{S};\mathbb{C}^N); A = SAS\right\}, \tag{L11.3}$$

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which is regarded as an algebra of operators on $\mathcal{C}^\infty_+(\mathbb{S};\mathbb{C}^N)$, is topologically simpler than the whole of the algebra, as we shall see. I will proceed to prove some results for this and the whole algebra, leaving the 'stabilization' to next time.

First is the simplest basic result leading to the definition of the (analytic) families index of a family of elliptic pseudodifferential operators. I will do this for the circle but the proof will later be shown to extend almost unchanged to a general manifold. The circle is much simpler than the general case, at the moment, because we have a sequence of smoothing projections

$$\pi_r : \mathcal{C}^\infty(\mathbb{S}) \ni u = \sum_k c_k e^{ik\theta} \longmapsto \pi_r u = \sum_{|k|\le r} c_k e^{ik\theta} \in \mathcal{C}^\infty(\mathbb{S}). \tag{L11.4}$$

We extend these to act componentwise on vector-valued functions. The crucial property that these projections have is that

$$A \in \Psi^{-\infty}(\mathbb{S};\mathbb{C}^N) \Longrightarrow A\pi_r \to A \text{ in } \Psi^{-\infty}(\mathbb{S};\mathbb{C}^N). \tag{L11.5}$$

Proposition 24. *Suppose that $A : X \longrightarrow \Psi^0(\mathbb{S};\mathbb{C}^N)$ is a smooth family of elliptic pseudodifferential operators, parameterized by a compact manifold X, then there exists a smooth family $B : X \longrightarrow \Psi^{-\infty}(\mathbb{S};\mathbb{C}^N)$ such that $(A(x)+B(x))^{-1} \in \Psi^0(\mathbb{S};\mathbb{C}^N)$ exists for each $x \in X$, if and only if for large enough r the*

$$F_r(x) = \operatorname{null}((\operatorname{Id}-\pi_r)A^*(x)) \in \mathcal{C}^\infty(\mathbb{S};\mathbb{C}^N) \tag{L11.6}$$

form a smooth vector bundle over X which is bundle-isomorphic to a trivial bundle of dimension $(2r+1)N$.

Proof. First we show that for r large enough, the $F_r(x)$ do indeed form a smooth vector bundle over X. Since $A(x)$ is an elliptic family, there is a smooth family $Q : X \longrightarrow \Psi^0(\mathbb{S};\mathbb{C}^N)$ of parametrices for the $A(x)$, so

$$Q(x)A(x) = \operatorname{Id} - R(x),\ R \in \mathcal{C}^\infty(X;\Psi^{-\infty}(\mathbb{S};\mathbb{C}^N)). \tag{L11.7}$$

Composing on the right with $\operatorname{Id}-\pi_r$ we get

$$Q(x)A(x)(\operatorname{Id}-\pi_r) = (\operatorname{Id}-R'_r(x))(\operatorname{Id}-\pi_r),\ R'_r(x) = R(x)(\operatorname{Id}-\pi_r), \tag{L11.8}$$

where the fact that $(\operatorname{Id}-\pi_r)(\operatorname{Id}-\pi_r) = (\operatorname{Id}-\pi_r)$ has been used. Since $R(x)\pi_r \to R(x)$ uniformly as a family of smoothing operators (i.e. in the $\mathcal{C}^\infty$ toplogy) we know that for large enough r the inverse

$$(\operatorname{Id}-R'_r(x))^{-1} = \operatorname{Id}-S_r(x),\ S_r \in \mathcal{C}^\infty(X;\Psi^{-\infty}(\mathbb{S};\mathbb{C}^N) \tag{L11.9}$$

exists. Composing on the left with the inverse and then with the operator $\operatorname{Id}-\pi_r$ and setting $Q'(x) = (\operatorname{Id}-\pi_r)(\operatorname{Id}-S_r(x))Q(x)$ we find that

$$Q'(x)A'_r(x) = \operatorname{Id}-\pi_r,\ A'_r(x) = A(x) - A(x)\pi_r. \tag{L11.10}$$

From this it follows that the null space of A'_r is precisely

$$\operatorname{null}(A(x)(\operatorname{Id}-\pi_r)) = \operatorname{null}(\operatorname{Id}-\pi_r) \tag{L11.11}$$

the span of the Fourier terms with wavenumber $|k| \le r$. This is a trivial vector bundle over X of dimension $(2r+1)N$. Certainly, the left in (L11.11) contains the right and if $(\operatorname{Id}-\pi_r)u = u$ then (L11.10) shows that $A'_r(x)u \ne 0$.

It also follows from (L11.10) that the $F_r(x)$ form a smooth vector bundle over X. To see this, recall that we know the (numerical) index of $A(x)$ to be a homotopy invariant. In particular it is a fixed integer for all x (well, in each component of X) for $A'_r(x)$. Since

$$\operatorname{ind}(A'_r(x)) = (2r+1)N - \dim(F_r(x)) \tag{L11.12}$$

is locally constant, the $F_r(x)$ have locally constant dimensions and this is enough to guarantee that they vary smoothly with $x \in X$. In fact $Q'(x)$ has range the same as $\operatorname{Id} - \pi_r$ and hence has null space which is of constant dimension and $A'_r(x)Q'(x) = \operatorname{Id} - G(x)$ has null space which is a smooth bundle isomorphic to F_r.

Thus we have succeeded in 'stabilizing' the null spaces to a bundle and the complements to the range to a bundle by modifying $A(x)$ by a smoothing operator to $A'_r(x) = A(x) - A(x)\pi_r$. The 'families index of A' is the formal difference of the null bundle and complement to the range

$$[(A'_r(x) \ominus F_r] \in K^0(X) \tag{L11.13}$$

where for the moment I have not defined either the left or right sides of this inclusion.

Now, we can prove one direction of the Proposition. If for large r there is an isomorphism to a trivial bundle over dimension $(2r+1)N$ then we can interpret this as an isomorphism of F_r to $\operatorname{null}(A'_r(x))$ and in this sense it is given by a family of smoothing operators, which we can denote by $B'_r(x)$. Clearly $A'_r(x) + B'_r(x)$ is then a family of invertible operators, differing from the original family by smoothing operators as anticipated.

Conversely, suppose that such an invertible perturbation exists so $A(x) + B(x)$ is invertible for all $x \in X$. Since $B(x)\pi_r \to B(x)$ uniformly in the $\mathcal{C}^\infty$ topology, it follows that $A(x) + B(x)\pi_r$ is invertible for r large enough. Since this is equal to

$$A'_r(x) + B_r(x),\ B_r(x) = (A(x) + B(x))\pi_r \tag{L11.14}$$

where $A'_r(x) = A(x)(\operatorname{Id} - \pi_r)$ as before, it follows that $B_r(x)$ is an isomorphism from the null space, which is a trivial bundle of dimension $(2r+1)N$ to a complement to the range of $A'_r(x)$, and hence to F_r. □

In fact this result is not restricted to the circle but extends to an arbitrary compact manifold (and more generally for a fibration by compact manifolds) once we can find appropriate replacements for the projections π_r.

The proof passes over to the Toeplitz case essentially unchanged, if we interpret π_r as the projection onto the span of the Fourier terms with $0 \le k \le r$.

Corollary 3. *Suppose that $A : X \longrightarrow \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ is a smooth family of elliptic Toeplitz pseudodifferential operators, parameterized by a compact manifold X, then there exists a smooth family $B : X \longrightarrow \Psi^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ such that $(A(x) + B(x))^{-1} \in \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ exists for each $x \in X$, if and only if for large enough r the $F_r(x)$ defined by* (L11.6) *form a smooth vector bundle over X which is bundle-isomorphic to a trivial bundle of dimension $(r+1)N$.*

Corollary 4 (of proof). *If $A_t : X \longrightarrow \Psi^0_{\mathcal{T}}(X; \mathbb{C}^N)$ (or $\Psi^0(X; \mathbb{C}^N)$) is a curve of elliptic families, i.e. is a smooth map from $[0,1]_t \times X$ elliptic at each point, and is invertible for $t = 0$ then there exists a smooth family $B_t : X \longrightarrow \Psi^{-\infty}_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^N)$ with $B_0 = 0$ such that $A_t(x) + B_t(x)$ is invertible for all $t \in [0,1]$ and all X.*

Proof. As in the proof above, consider $A_t(x)(\operatorname{Id} - \pi_r)$. For r large enough this has null space equal to that of $\operatorname{Id} - \pi_r$ for all $t \in [0,1]$ and all $x \in X$ and there is then a smooth bundle over $[0,1]\times X$ complementary to the range. Applying Proposition 24 the restriction of this bundle to $t=0$ must be isomorphic to the null bundle, which is trivial and of dimension $(r+1)d$. Since $[0,1]\times X$ is contractible to $\{0\}\times X$ it follows that the bundle is trivial over the whole of $[0,1]\times X$ so applying the Proposition in the other direction there is a smoothing perturbation making the operator invertible. Following the last part of the proof, this perturbation can be chosen to vanish at $t=0$. $\square$

We will use this result later to lift homotopies of smooth elliptic families to homotopies of invertible families.

Now, let me turn to the first substantial homotopy computation of the two needed to construct the classifying sequence for K-theory. In this I will use two 'shifts' in the Toeplitz algebra. Namely

$$\begin{gathered}U:\mathcal{C}^\infty_+(\mathbb{S})\longrightarrow\mathcal{C}^\infty_+(\mathbb{S}),\ Uu=\sum_{k\ge0}c_ke^{i(k+1)\theta}\text{ if }u=\sum_{k\ge0}c_ke^{ik\theta}\text{ and}\\ L:\mathcal{C}^\infty_+(\mathbb{S})\longrightarrow\mathcal{C}^\infty_+(\mathbb{S}),\ Lu=\sum_{k\ge1}c_ke^{i(k-1)\theta}\text{ if }u=\sum_{k\ge0}c_ke^{ik\theta}.\end{gathered}\tag{L11.15}$$

Both are elliptic elements of $\Psi^0_{\mathcal{T}}(\mathbb{S})$ since they can be written

$$U=Se^{i\theta}S,\ L=Se^{-i\theta}S\tag{L11.16}$$

and they are essential inverses of each other

$$LU=\mathrm{Id},\ UL=\mathrm{Id}-\pi_0.\tag{L11.17}$$

In particular L has null space exactly the constants and the constants form a complement to the range of U. Thus

$$\mathrm{ind}(U)=-1,\ \mathrm{ind}(L)=1.\tag{L11.18}$$

Set

$$\tilde G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)=\left\{A\in\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N);A\text{ is elliptic and }A^{-1}\in\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)\right\}.\tag{L11.19}$$

The tilde here is to distinguish it from a smaller group I will discuss later. We can inject $\mathrm{GL}(N,\mathbb{C})\longrightarrow\tilde G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ as the operators (of the form $\mathrm{Id}+a,\ a\in\Psi^{-\infty}_{\mathcal{T}}(\mathbb{S},\mathbb{C}^N)$)

$$\mathrm{GL}(N,\mathbb{C})\ni g\longrightarrow\mathrm{Id}-\pi_0+\pi_0g\pi_0.\tag{L11.20}$$

Let us also consider, in the standard way of 'stabilization' that

$$\mathrm{GL}(N,\mathbb{C})\subset\mathrm{GL}(2N,\mathbb{C})$$

as the upper left corner in a 2×2 block decomposition

$$\mathrm{GL}(N,\mathbb{C})\ni g\longmapsto\begin{pmatrix}g&0\\0&\mathrm{Id}_N\end{pmatrix}\in\mathrm{GL}(2N,\mathbb{C}).\tag{L11.21}$$

Proposition 25. *If* $\mathrm{GL}(N,\mathbb{C})\longrightarrow\tilde G^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^{2N})$ *is embedded as a subgroup by combining* (L11.21) *and* (L11.20) *(for* $2N$ *in place of* N*) then the image subgroup is deformable to the identity.*

Proof. Dividing $\mathbb{C}^{2N}$ into $\mathbb{C}^N \oplus \mathbb{C}^N$ we can picture the operators as block 2×2 matrices with entries which are $N \times N$ matrices of Toeplitz operators. The subgroup $\mathrm{GL}(N,\mathbb{C})$ can then be identified with

$$M_0 = \begin{pmatrix} \mathrm{Id} - \pi_0 + \pi_0 g \pi_0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{L11.22}$$

This is the intial value of a curve of operators in $\Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^{2N})$

$$M_\theta = \begin{pmatrix} \cos\theta(\mathrm{Id}-\pi_0) + \pi_0 g\pi_0 & \sin\theta g U \\ -\sin\theta g^{-1} L & \cos\theta \,\mathrm{Id} \end{pmatrix},\ 0 \le \theta \le \frac{\pi}{2}. \tag{L11.23}$$

This is an elliptic family of Toeplitz operators (so the Id's can be read as S's) since it symbol is the invertible matrix

$$\begin{pmatrix} \cos\theta & \sin\theta g e^{i\theta} \\ -\sin\theta g^{-1} e^{-i\theta} & \cos\theta \end{pmatrix} \tag{L11.24}$$

(which has determinant 1). Now, M_θ has the property that for all $k > 0$,

$$M_\theta \begin{pmatrix} u_k e^{ik\theta} \\ v_k e^{i(k-1)\theta} \end{pmatrix} = \begin{pmatrix} f_k e^{ik\theta} \\ g_k e^{i(k-1)\theta} \end{pmatrix},\ \begin{pmatrix} f_k \\ g_k \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta g \\ -\sin\theta g^{-1} & \cos\theta \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} \tag{L11.25}$$

and maps $(u_0, 0)$ to $(gu_0, 0)$. From the invertibility of these matrices it follows that M_θ is a curve in $\tilde{G}^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^{2N})$.

At the end of this first deformation we have arrived at the initial point of the curve

$$M'_\theta = \begin{pmatrix} \cos\theta g(\mathrm{Id}-\pi_0) + \pi_0 g \pi_0 & \sin\theta U \\ -\sin\theta L & \cos\theta g^{-1}\,\mathrm{Id} \end{pmatrix}. \tag{L11.26}$$

(where now θ runs from $\pi/2$ back to 0). This has essentially the same properties as M_θ. Namely it is elliptic since the symbol matrix is

$$\begin{pmatrix} \cos\theta g & \sin\theta e^{i\theta} \\ -\sin\theta e^{-i\theta} & \cos\theta g^{-1} \end{pmatrix} \tag{L11.27}$$

which again has determinant 1 and satisfies the analogue of (L11.25) with the matrix replaced by

$$\begin{pmatrix} \cos\theta g & \sin\theta \\ -\sin\theta & \cos\theta g^{-1} \end{pmatrix} \tag{L11.28}$$

which is again invertible (and the same on the zero mode).

At the end of this second homotopy (all uniform on $\mathrm{GL}(N,\mathbb{C})$ of course) we have arrived at the 'Toeplitz operator' which is purely a matrix

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}. \tag{L11.29}$$

It is straightforward to see that this is homotopic to the identity in $\mathrm{GL}(2N,\mathbb{C})$ using a similar rotation but purely in matrices, namely

$$\begin{pmatrix} \cos\theta g & \sin\theta \\ -\sin\theta & \cos\theta g^{-1} \end{pmatrix},\ \theta \in [0,\pi/2] \tag{L11.30}$$

followed by

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},\ \theta \in [\pi/2, 0] \tag{L11.31}$$

finishing at the identity. $\square$

This result will allow us to show that the '$G^{-\infty}_{\mathcal{T}}$' part of the final group can be (weakly) deformed away. Next time I will start with Atiyah's proof of Bott periodicity modified to show how the invertible elliptic operators can be deformed into this smoothing subgroup. The combination of the two discussions will give the weak contractibility we are after.

11+. Addenda to Lecture 11

11+.1. Proof of (L11.5)**.** This is really just the convergence of Fourier series. Thus, for $f \in \mathcal{C}^\infty(\mathbb{S})$ the truncated Fourier series $\pi_r f \longrightarrow f$ in $\mathcal{C}^\infty(\mathbb{S})$ as $r \to \infty$. An element $A \in \Psi^{-\infty}(\mathbb{S})$ is represented by a smooth kernel, $A \in \mathcal{C}^\infty(\mathbb{S} \times \mathbb{S})$,

$$Af(\theta) = \int_{\mathbb{S}} A(\theta, \theta') f(\theta') d\theta'. \tag{11+.32}$$

Since π_r is self-adjoint and real,

$$A(\pi_r f)(\theta) = \int_{\mathbb{S}} A(\theta, \theta')(\pi_r f)(\theta') d\theta' = \int_{\mathbb{S}} A_r(\theta, \theta') f(\theta') d\theta' \tag{11+.33}$$

where A_r is obtained from A by the action of π_r in the second variable. For a smooth family of smooth functions, the Fourier series converges uniformly with all its derivatives. Thus

$$A\pi_r \longrightarrow A \in \mathcal{C}^\infty(\mathbb{S} \times \mathbb{S}) \tag{11+.34}$$

which is the topology on $\Psi^{-\infty}(\mathbb{S})$, as claimed in (L11.5).

11+.2. Proof of Lemma 22. Following the 'hint' of the lecture, we first observe that restriction to the boundary gives an isomorphism

$$\mathcal{C}^\infty_{\mathrm{hol}}(\mathbb{D}) = \{u \in \mathcal{C}^\infty(\mathbb{D}); (\partial_x + i\partial_y)u = 0\} \longrightarrow \mathcal{C}^\infty_+(\mathbb{S}). \tag{11+.35}$$

Surjectivity follows easily, as indicated in the lecture, since if $a \in \mathcal{C}^\infty_+(\mathbb{S})$ then its Fourier series converges uniformly with all derivatives on the circle and since $e^{ik\theta} = z^k$ restricted to the circle and $|z^k| \le 1$ in the disc

$$u_a(z) = \sum_{k \ge 0} a_k z^k$$

converges uniformly on $\mathbb{D}$, with all derivatives, to a holomomorphic function (since the terms are holomorphic) restricting to a on the boundary. Moreover, all elements of $\mathcal{C}^\infty_{\mathrm{hol}}(\mathbb{D})$ arise this way, since the Fourier coefficients of the boundary value of $u \in \mathcal{C}^\infty_{\mathrm{hol}}(\mathbb{D})$ can be written, for $k < 0$, as

$$a_k = \frac{1}{2\pi} \int_{\mathbb{S}} e^{-ik\theta} u(z) d\theta = \lim_{r \uparrow 1} \int_{|z|=r} z^{-k} u(z) \frac{dz}{z} = 0 \tag{11+.36}$$

by Cauchy's integral formula. Thus the boundary value is in $\mathcal{C}^\infty_+(\mathbb{S})$ and if it vanishes then extending u as 0 outside the disc gives a continuous function on $\mathbb{R}^2$ which satisfies $(\partial_x + i\partial_y)\tilde{u} = 0$, in the sense of distributions, everywhere. Thus (by elliptic regularity) it is in fact an entire function of compact support, which must vanish identically. Thus the map is also injective.

Consider the integral in (L11.2). For $|z| < 1$ this certainly converges for any $f \in \mathcal{C}^\infty(\mathbb{S})$ (since $z - e^{i\theta} \ne 0$ with all its derivatives and by differentiation under the integral sign it is holomorphic in $|z| < 1$. Using the rapid convergence of the Fourier series we may interchange series and integral and conclude that for any $f \in \mathcal{C}^\infty(\mathbb{S})$,

(11+.37)
$$\tilde{S}(f)(z) = \sum_{k\in\mathbb{Z}} a_k u_k(z),\ u_k(z) = \frac{1}{2\pi i}\int_{\mathbb{S}} \frac{e^{ik\theta}}{e^{i\theta} - z} d\theta = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{\tau^k}{\tau - z}\frac{d\tau}{i\tau}.$$

For $k < 0$ there are no poles outside the unit disk, including at $\tau = \infty$, so by Cauchy's integral formula the $u_k(z) \equiv 0$, $k < 0$. For $z \geq 0$ there is a pole at ∞ and applying the residue formula, it evaluates to z^k. Thus in fact $\tilde{S}(f)(z)$ is the holomorphic extension, into $|z| < 1$, of $S(f)$. It is therefore smooth up to the boundary, by the discussion above, so indeed

$$S(f)(e^{i\theta}) = \lim_{r\uparrow 1} \tilde{S}(f)(z) \tag{11+.38}$$

as claimed.