PROBLEM SET 1, 18.156
DUE MIDNIGHT TUESDAY 6 MARCH, 2012

Problem 1.1

Go through the basic properties of the Clifford algebra $\mathrm{Cl}(V)$ and its complexification $\mathbb{C}\mathrm{l}(V)$.

(1) Recall the definition as the quotient of the tensor product by the ideal generated by the relations $v \otimes w + w \otimes v = 2(v, w)$ with respect to a (for us positive-definite) real bilinear form on a real vector space.
(2) Show that if e_i is an orthonormal basis then the images of the tensor products $e_{i_1} \otimes \dots e_{i_l}$, $i_1 < i_2 < \dots < i_l$ generate $\mathrm{Cl}(V)$ over the reals or $\mathbb{C}\mathrm{l}(V)$ over the complexes.
(3) Check that span of the generators with odd and even numbers of elements generate subspaces $\mathrm{Cl}_o(V)$ and $\mathrm{Cl}_e(V)$ where the latter is a subalgebra.
(4) Show that the subspaces spanned by the generators with at most k elements, $\mathrm{Cl}^{(k)}(V)$ form a filtration with $\mathrm{Cl}^{(k)}(V)/\mathrm{Cl}^{(k-1)}(V) = \Lambda^k V$ a natural linear isomorphism.
(5) Prove periodicity of $\mathbb{C}\mathrm{l}(V)$. If $n > 2$ take an orthonormal basis and consider the element $e = ie_{n-1}e_n$. Show that it is an involution and $\pi_\pm = \frac{1}{2}(\mathrm{Id} \pm e)$ are projections. Compute the elements $\pi_\pm e_i \pi_\pm$ for $i = n-1, n$ and show (by looking at the generators) that the $\pi_+ e_k \pi_+$, $1 \le k \le n-2$ generate an algebra isomorphic to $\mathbb{C}\mathrm{l}(\mathbb{R}^{n-2})$. Conclude that as an algebra

$$\mathbb{C}\mathrm{l}(V) \equiv \mathbb{C}\mathrm{l}(\mathbb{R}^{n-2}) \otimes \hom(\mathbb{C}^2) \tag{1.1}$$

where the latter is the algebra of 2×2 complex matrices.
(6) Check the cases $n = 1$ and $n = 2$, respectively $\mathbb{C} \oplus \mathbb{C}$ and $M(2)$ to conclude that

$$\mathbb{C}\mathrm{l}(V) \equiv \begin{cases} \hom(\mathbb{C}^{2^p}) & \dim V = 2p \\ \hom(\mathbb{C}^{2^p}) \oplus \hom(\mathbb{C}^{2^p}) & \dim V = 2p+1. \end{cases} \tag{1.2}$$

(7) Show that the subset of $\mathrm{Cl}(V)$ consisting of the products of (any finite number of) unit vectors forms a group, it is $\mathrm{Pin}(V)$. The subgroup with only products of even numbers of elements is $\mathrm{Spin}(V) \subset \mathrm{Pin}(V)$.
(8) Define (for $\dim V \ge 2$) a map $\mathrm{Pin}(V) \longrightarrow \mathrm{O}(V)$ (the group of orthogonal transformations of V) taking any unit vector $v \in V$

to the reflection in the plane orthogonal to v. Show that this is a surjective group homomorphism with kernel isomorphic to $\mathbb{Z}_2$ and that similarly $\mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)$ is a double cover.

(9) Show that the periodicity result above for $\mathbb{C}\mathrm{l}(V)$ gives a representation of $\mathrm{Spin}(V)$ in each dimension, that it is reducable in even dimensions as the sum of two (positive and negative) spin representations and irreducible in odd dimensions. These are the spin representations, show that they are not representations of $\mathrm{SO}(V)$.

Problem 1.2 – optional

If you have some an enthusiasm for algebra, go through the discussion of periodicity for the real Clifford algebra (if you want to do this, I very much suggest you look it up!) It is periodic of period 8.

Problem 1.3

(1) Recall the proof of 'Borel's Lemma', that given any sequence $f_j \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{n-1})$ with a fixed bound on their supports, there exists $f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with $\partial^k_n f\big|_{x_n=0} = f_k$ for all k.

(2) Using this, show that if $a_j \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ is a sequence of homogeneous functions of homogeneity $m_j \to -\infty$ as $j \to \infty$ then there exists a single function $a \in \mathcal{C}^\infty(\mathbb{R}^n)$ (their 'asymptotic sum') such that for all N and multiindices α

(1.3)
$$|\partial^\alpha_\xi(a(\xi) - \sum_{k=1}^{N} a_j(\xi))| \le C_{N,\alpha}|\xi|^{M(N)-|\alpha|} \text{ in } |\xi| > 1,\ M(N) = \sup_{j>N} m_j.$$

(3) Show that such an asymptotic sum is well defined up to an element of $\mathcal{S}(\mathbb{R}^n)$.