18.156 LECTURE 3

Manifolds: Definitions

(1) A paracompact, Hausdorff topological space, M, with a covering by smoothly-related coordinate patches.

Coordinate patch in topological space: Open set with a homeomorphism to an open set of $\mathbb{R}^n$.

Smooth relation of coordinate patches, $F_i : U_i \longrightarrow U'_i$ $i = 1, 2$ is the condition that either $U_1 \cap U_2 = \emptyset$ or else $F_2 \circ F_1^{-1} : F_1(U_1 \cap U_2) \longrightarrow F_2(U_1 \cap U_2)$ is a diffeomorphism of open sets.

(2) A set M with a given algebra of functions (over the reals here) denoted $\mathcal{C}^\infty(M)$ such that

Points of M are separated by $\mathcal{C}^\infty(M)$.

For each point $p \in M$ there exist 'local coordinates at p' meaning functions $f_i \in \mathcal{C}^\infty(M)$, $i = 1, \dots, n$ (n fixed independently of the point) such that for some set $U_p \ni p$, $F_p = (f_1, \dots, f_n) : U_p \longrightarrow \mathbb{R}^n$ is a bijection to an open set U'_p and

$$
\begin{gathered}
(F_p^{-1})^* : \mathcal{C}^\infty(M)\big|_{U_p} \longrightarrow \mathcal{C}^\infty(U'_p) \\
F_p^* : \mathcal{C}_c^\infty(U'_p) \longrightarrow \mathcal{C}^\infty(M) \text{ (by extension as zero) is an injection} \\
\text{a countable collection of these sets } U_p \text{ cover } M \\
f \in \mathcal{C}^\infty(M) \text{ iff } (F_p^{-1})^* f\big|_{U_p} \in \mathcal{C}^\infty(U'_p)\ \forall\, p \in M.
\end{gathered}
\tag{1}
$$

(3) A set M with a $\mathcal{C}^\infty$ algebra of real-valued functions $\mathcal{C}^\infty(M)$ in which there exist N generating elements g_i such that the map $G = (g_1, \dots, g_N) : M \longrightarrow \mathbb{R}^N$ is a bijection onto $G(M) \subset \mathbb{R}^N$ where for each $p \in M$, $G(M)$ is defined in a neighbourhood of $G(p)$ by $N - n$ smooth functions with independent differentials. It follows that $\mathcal{C}^\infty(M) = G^* \mathcal{C}^\infty(\mathbb{R}^N)$.

These you might think of these as the (standard) covering definition, the algebraic definition and the extrinsic definition respectively. Note that a $\mathcal{C}^\infty$ algebra (of real-valued functions) is one such that $u(f_1, \dots, f_k)$ is also in the algebra for any elements f_k and any $u \in \mathcal{C}^\infty(\mathbb{R}^k)$.

I do not plan to go through the equivalence of these in class. If anyone wants to write it out (and thereby check that I haven't forgotten something) I am happy to help and give feedback.

So, I will simply assume all these properties and go on to discuss properties of manifolds which follow directly from these. The basic examples are open subsets of $\mathbb{R}^n$ and embedded submanifolds, as in the last definition (in particular it follows that one can consider embedded submanifolds of open subsets of $\mathbb{R}^n$).

Now, what follows easily. Well, we make various natural definitions consistent with the case of open sets.

(1) A smooth map $F : M \longrightarrow N$ is a map such that $F^* \mathcal{C}^\infty(N) \subset \mathcal{C}^\infty(M)$.

(2) The cotangent space at a point $p \in M$ is the quotient $T_p^* M = \mathcal{I}(p)/\mathcal{I}(p)^2$ of the ideal $\mathcal{I}(p) \subset \mathcal{C}^\infty(M)$ of functions which vanish at p by the ideal which is the (finite) span of products.
(3) The differential of a function $df(p) \in T_p^* M$ is the image of $f - f(p)$.
(4) The differential of a smooth map is naturally a linear map $F_p^* = dF(p) : T^*_{F(p)} N \longrightarrow T_p^* M$.
(5) The cotangent fibres at all points can be combined into a smooth manifold, the cotagent bundle $T^*M = \bigcup_{p \in M} T_p^* M$ as a set. The coordinate form of a $\mathcal{C}^\infty$ structure on T^*M is given by the maps from sets $T_U^* M = T^*U = \bigcup_{p \in U} T_p^* M$ to $U' \times \mathbb{R}^n$ given by the coordinate definition of the cotangent spaces.

NB. Smooth maps do not in general induce smooth maps on the cotangent bundles – because they are only defined over the image of the map.
(6) The dual bundle, TM defined in the same way using the dual spaces $T_p M = (T_p^* M)'$ is also a smooth manifold and the maps defined as the dual of the F_p^* *do* combine to give a smooth map $F_* : TM \longrightarrow TN$, the crucial thing being only that it is defined everywhere.

So, in principle there is are a lot of basic constructions to do. Anyone unfamilar with these should probably use, at least initially, the third definition of a manifold above and refer everything back to the underlying Euclidean space (although this does not really simplify things!)

Now, I will proceed to define differential operators on manifolds and the spaces they map between. First let's consider distributions. As well as the space $\mathcal{C}^\infty(M)$ and its complexification – just pairs of real maps thought of as a map into $\mathbb{C}$ with complex multiplication therefore defined – we consider the subspace of compactly supported functions

$$\mathcal{C}_c^\infty(M) \subset \mathcal{C}^\infty(M). \tag{2}$$

To make various computations we will use

Proposition 1 (Partitions of unity). *Any open cover O_a, $a \in A$, of any manifold M there is a partition of unity subordinate to it, that is a countable collection $\phi_i \in \mathcal{C}_c^\infty(M)$ of real-valued functions of compact support with $0 \le \phi_i(0) \le 1$ for all $p \in M$ such that*

$$\begin{gathered} \forall\ i,\ \operatorname{supp}(\phi_i) \subset O_a \text{ for some } a \in A \\ \forall\ i,\ \operatorname{supp}(\phi_i) \cap \operatorname{supp}(\phi_j) \neq \emptyset \text{ for finitely many } j \\ \sum_i \phi_i(p) = 1\ \forall\ p \in M. \end{gathered} \tag{3}$$

Now, back to local elliptic regularity and the three little Lemmas from the end of Lecture 2.

Lemma 1. *For any $s \in \mathbb{R}$ each $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a multiplier on $H^s(\mathbb{R}^n)$ and*

$$\|\phi u\|_{H^s} \le \sup |\phi| \|u\|_{H^s} + C_{s,\phi} \|u\|_{H^{s-1}}\ \forall\ u \in H^s(\mathbb{R}^n). \tag{4}$$

Lemma 2. *For any $\phi \in \mathcal{S}(\mathbb{R}^n)$ and any t, $s \in \mathbb{R}$,*

$$[\phi, \langle D \rangle^t] : H^s(\mathbb{R}^n) \longrightarrow H^{s-t+1}(\mathbb{R}^n) \text{ is bounded.} \tag{5}$$

Lemma 3. *For any ψ, $\phi \in \mathcal{S}(\mathbb{R}^n)$, and $s \in \mathbb{R}$, convolution operator by $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ gives a uniformly bounded linear map*

$$[\psi, \phi_\epsilon *] : H^s(\mathbb{R}^n) \longrightarrow H^{s+1}(\mathbb{R}^n),\ 0 < \epsilon \leq 1. \tag{6}$$

Let's start with

Proof of Lemma 2. We know that $\langle D\rangle^s$ is an isomorphism of $H^s(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ so it suffices to show that the operator (on tempered distributions)

$$\langle D\rangle^{s-t+1}[\phi, \langle D\rangle^t]\langle D\rangle^{-s} \tag{7}$$

is bounded on $L^2(\mathbb{R}^n)$.

Taking the Fourier transform this is equivalent to the boundedness of the operator

$$\langle \eta\rangle^{s-t+1}[\hat{\phi}*, \langle \eta\rangle^t]\langle \eta\rangle^{-s} \tag{8}$$

on $L^2(\mathbb{R}^n)$. Now, we can write out the kernel as an integral operator and it is just

$$\begin{gathered} K(\eta,\xi) = F(\eta,\xi)\hat{\phi}(\eta-\xi), \\ F(\eta,\xi) = \langle \eta\rangle^{s-t+1}\langle \xi\rangle^{-s+t} - \langle \eta\rangle^{s+1}\langle \xi\rangle^{-s}. \end{gathered} \tag{9}$$

Now we can check that for any N

$$|K(\eta,\xi)| \leq C_N(1+|\eta-\xi|)^{-N} \tag{10}$$

This is clear in any region $|\eta-\xi| \geq \delta(|\eta|+|\xi|)$, $\delta > 0$ since the polynomial growth of F in η or ξ is offset by the rapid decay of $\hat{\phi}(\eta-\xi)$. In the complementary region (take δ small) $|\eta|$ and $|\xi|$ are bounded by multiples of each other and $|\eta-\xi|$ by a multiple of either. Using Taylor's formula with remainder for F around the diagonal it follows that

$$|F(\eta,\xi)| \leq C|\eta-\xi| \tag{11}$$

and the same estimate follows.

Now an application of Cauchy-Schwartz, or Schur's Lemma gives boundedness on $L^2(\mathbb{R}^n)$. □

Proof of Lemma 1. A norm bound on $\phi \in \mathcal{S}(\mathbb{R}^n)$ as a multiplication operator on $H^s(\mathbb{R}^n)$ is equivalent to a norm bound on $\langle D\rangle^{-s}\phi\langle D\rangle^s$ as an operator on L^2. Writing this as $\phi + [\langle D\rangle^{-s}, \phi]\langle D\rangle^s$ the norm bound (4) follows from Lemma 2 applied to the second term. □

Proof of Lemma 3. Again identifying $H^s(\mathbb{R}^n)$ (with norm) as the image of $\langle D\rangle^{-s}$ applied to $L^2(\mathbb{R}^n)$ we see that the result is the same as the uniform boundedness of

$$\langle D\rangle^{s+1}[\psi, \phi_\epsilon *]\langle D\rangle^{-s} \tag{12}$$

on $L^2(\mathbb{R}^n)$. Conjugating by the Fourier transform as before this reduces to the uniform boundedness of

$$\langle \eta\rangle^{s+1}[\hat{\psi}*, \hat{\phi}(\epsilon\eta)]\langle \eta\rangle^{-s} \tag{13}$$

on $L^2(\mathbb{R}^n)$. Ignoring the 'hats' since these are just Schwartz functions the kernel of this as an integral operator is

$$K(\eta,\xi) = G(\eta,\xi)\psi(\eta-\xi)G(\eta,\xi) = \langle \eta\rangle^{s+1}\langle \xi\rangle^{-s}\left((\phi(\epsilon\xi) - \phi(\epsilon\eta)\right). \tag{14}$$

Now, reconsider the proof of Lemma 2 above and look for the same bound (10). In the first region the same argument applies since only the polynomial boundedness of the coefficient function $G(\eta,\xi)$ was used. In the second region we can again apply Taylor's formula with remainder, or just the mean value theorem, to bound the difference

$$|\phi(\epsilon\eta)-\phi(\epsilon\xi)|\le C\epsilon|\eta-\xi|(1+\epsilon|\xi|)^{-1},\ |\eta-\xi|\le\delta(|\eta|+|\xi|),\ \delta>0\text{ small.} \tag{15}$$

The coefficient of ϵ is important of course, since it means that we get (7) again, since the extra factor of $|\eta|$ can be absorbed by $\epsilon|\eta|\le C(1+\epsilon|\eta|)$. Thus the uniform boundedness follows. $\square$

Clearly there is a very close relationship between Lemmas 2 and 3 and in fact all three Lemmas can be absorbed into the 'L^2 boundedness of pseudodifferential operators'.

So, if I have time I will again review the proof of local elliptic regularity, but what I showed in Lecture 2 is that it follows from elliptic regularity for constant coefficient operators and these three results.