18.156 – Spring 2008 – Graduate Analysis
Elliptic regularity and Scattering

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Contents

Preface

Initially at least, these are the lecture notes from last year – Spring 2007 – that I am using in Spring 2008. The first part of the course this year will be quite similar, but in the second half I will concentrate on spectral and scattering theory.

CHAPTER 1

Distributions

Summary of parts of 18.155 and a little beyond. With some corrections by Jacob Bernstein incorporated.

1. Fourier transform

The basic properties of the Fourier transform, tempered distributions and Sobolev spaces form the subject of the first half of this course. I will recall and slightly expand on such a standard treatment.

2. Schwartz space.

The space $\mathcal{S}(\mathbb{R}^n)$ of all complex-volumed functions with rapidly decreasing derivatives of all orders is a complete metric space with metric

$$\begin{aligned} d(u,v) &= \sum_{k=0}^{\infty} 2^{-k} \frac{\|u-v\|_{(k)}}{1+\|u-v\|_{(k)}} \text{ where} \\ \|u\|_{(k)} &= \sum_{|\alpha|+|\beta|\le k} \sup_{z\in\mathbb{R}^n} |z^\alpha D_z^\beta u(z)|. \end{aligned} \tag{1.1}$$

Here and below I will use the notation for derivatives

$$D_z^\alpha = D_{z_1}^{\alpha_1}\dots,D_{z_n}^{\alpha_n},\ D_{z_j} = \frac{1}{i}1\frac{\partial}{\partial z_j}.$$

These norms can be replaced by other equivalent ones, for instance by reordering the factors

$$\|u\|'_{(k)} = \sum_{|\alpha|+|\beta|\le k} \sup_{z\in\mathbb{R}^n} |D_z^\beta(z^\beta u)|.$$

In fact it is only the cumulative effect of the norms that matters, so one can use

$$\|u\|''_{(k)} = \sup_{z\in\mathbb{R}^n} |\langle z\rangle^{2k}(\Delta+1)^k u| \tag{1.2}$$

in (1.1) and the same topology results. Here

$$\langle z\rangle^2 = 1+|z|^2,\ \Delta = \sum_{j=1}^n D_j^2$$

(so the Laplacian is formally positive, the geometers' convention). It is not quite so trivial to see that inserting (1.2) in (1.1) gives an equivalent metric.

3. Tempered distributions.

The space of (metrically) continuous linear maps

$$f:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathbb{C} \tag{1.3}$$

is the space of tempered distribution, denoted $\mathcal{S}'(\mathbb{R}^n)$ since it is the dual of $\mathcal{S}(\mathbb{R}^n)$. The continuity in (1.3) is equivalent to the estimates

$$\exists\ k,\ C_k>0 \text{ s.t. } |f(\varphi)|\le C_k\|\varphi\|_{(k)}\ \forall\ \varphi\in\mathcal{S}(\mathbb{R}^n). \tag{1.4}$$

There are several topologies which can be considered on $\mathcal{S}'(\mathbb{R}^n)$. Unless otherwise noted we consider the *uniform topology* on $\mathcal{S}'(\mathbb{R}^n)$; a subset $U\subset\mathcal{S}'(\mathbb{R}^n)$ is open in the uniform topology if for every $u\in U$ and every k sufficiently large there exists $\delta_k>0$ (both k and δ_k depending on u) such that

$$v\in\mathcal{S}'(\mathbb{R}^n),\ |(u-u)(\varphi)\le\delta_k\|\varphi\|_{(k)}\Rightarrow v\in U.$$

For linear maps it is straightforward to work out continuity conditions. Namely

$$\begin{aligned}P&:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^m)\\ Q&:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}'(\mathbb{R}^m)\\ R&:\mathcal{S}'(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^m)\\ S&:\mathcal{S}'(\mathbb{R}^n)\longrightarrow\mathcal{S}'(\mathbb{R}^m)\end{aligned}$$

are, respectively, continuous for the metric and uniform topologies if

$$\begin{aligned}&\forall\ k\ \exists\ k',\ C \text{ s.t. } \|P\varphi\|_{(k)}\le C\|\varphi\|_{(k')}\ \forall\ \varphi\in\mathcal{S}(\mathbb{R}^n)\\ &\exists\ k,\ k',\ C \text{ s.t. } |Q\varphi(\psi)|\le C\|\varphi\|_{(k)}\|\psi\|_{(k')}\\ &\forall\ k,\ k'\ \exists\ C \text{ s.t. } |u(\varphi)|\le\|\varphi\|_{(k')}\ \forall\ \varphi\in\mathcal{S}(\mathbb{R}^n)\Rightarrow\|Ru\|_{(k)}\le C\\ &\forall\ k'\ \exists\ k,\ C,\ C' \text{ s.t. } \|u(\varphi)\|_{(k)}\le\|\varphi\|_{(k)}\forall\ \varphi\in\mathcal{S}(\mathbb{R}^n)\Rightarrow|Su(\psi)|\le C'\|\psi\|_{(k')}\ \forall\ \psi\in\mathcal{S}(\mathbb{R}^n).\end{aligned}$$

The particular case of R, for $m=0$, where at least formally $\mathcal{S}(\mathbb{R}^0)=\mathbb{C}$, corresponds to the reflexivity of $\mathcal{S}(\mathbb{R}^n)$, that

$$\begin{gathered}R:\mathcal{S}'(\mathbb{R}^n)\longrightarrow\mathbb{C} \text{ is cts. iff } \exists\ \varphi\in\mathcal{S}(\mathbb{R}^n) \text{ s.t.}\\ Ru=u(\varphi) \text{ i.e. } (\mathcal{S}'(\mathbb{R}^n))'=\mathcal{S}(\mathbb{R}^n).\end{gathered}$$

In fact another extension of the middle two of these results corresponds to the Schwartz kernel theorem:

$$\begin{aligned}Q:&\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}'(\mathbb{R}^m) \text{ is linear and continuous}\\ &\text{iff } \exists\ Q\in\mathcal{S}'(\mathbb{R}^m\times\mathbb{R}^n) \text{ s.t. } (Q(\varphi))(\psi)=Q(\psi\boxtimes\varphi)\ \forall\ \varphi\in\mathcal{S}(\mathbb{R}^m)\ \psi\in\mathcal{S}(\mathbb{R}^n).\\ R:&\mathcal{S}'(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n) \text{ is linear and continuous}\\ &\text{iff } \exists\ R\in\mathcal{S}(\mathbb{R}^m\times\mathbb{R}^n) \text{ s.t. } (Ru)(z)=u(R(z,\cdot)).\end{aligned}$$

Schwartz test functions are dense in tempered distributions

$$\mathcal{S}(\mathbb{R}^n)\hookrightarrow\mathcal{S}'(\mathbb{R}^n)$$

where the *standard inclusion* is via Lebesgue measure

$$\mathcal{S}(\mathbb{R}^n)\ni\varphi\mapsto u_\varphi\in\mathcal{S}'(\mathbb{R}^n),\ u_\varphi(\psi)=\int_{\mathbb{R}^n}\varphi(z)\psi(z)dz. \tag{1.5}$$

The basic operators of differentiation and multiplication are transferred to $\mathcal{S}'(\mathbb{R}^n)$ by duality so that they remain consistent with the (1.5):

$$D_z u(\varphi) = u(-D_z\varphi)$$
$$fu(\varphi) = u(f\varphi)\ \forall\ f \in \mathcal{S}(\mathbb{R}^n)).$$

In fact multiplication extends to the space of function of polynomial growth:

$$\forall\ \alpha \in \mathbb{N}_0^n\ \exists\ k \text{ s.t. } |D_z^\alpha f(z)| \leq C\langle z\rangle^k.$$

Thus such a function is a multiplier on $\mathcal{S}(\mathbb{R}^n)$ and hence by duality on $\mathcal{S}'(\mathbb{R}^n)$ as well.

4. Fourier transform

Many of the results just listed are best proved using the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$
$$\mathcal{F}\varphi(\zeta) = \hat{\varphi}(\zeta) = \int e^{-iz\zeta}\varphi(z)dz.$$

This map is an isomorphism that extends to an isomorphism of $\mathcal{S}'(\mathbb{R}^n)$

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$
$$\mathcal{F}\varphi(D_{z_j}u) = \zeta_j\mathcal{F}u,\ \mathcal{F}(z_j u) = -D_{\zeta_j}\mathcal{F}u$$

and also extends to an isomorphism of $L^2(\mathbb{R}^n)$ from the dense subset

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^2)\mathrm{d}ense,\ \|\mathcal{F}\varphi\|_{L^2}^2 = (2\pi)^n\|\varphi\|_{L^2}^2. \tag{1.6}$$

5. Sobolev spaces

Plancherel's theorem, (**??**), is the basis of the definition of the (standard, later there will be others) Sobolev spaces.

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n);\ (1+|\zeta|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)\}$$
$$\|u\|_s^2 = \int_{\mathbb{R}^n}(1+|\zeta|^2)^s|\hat{u}(\zeta)|d\zeta,$$

where we use the fact that $L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is a well-defined injection (regarded as an inclusion) by continuous extension from (1.5). Now,

$$D^\alpha : H^s(\mathbb{R}^n) \longrightarrow H^{s-|\alpha|}(\mathbb{R}^n)\ \forall\ s,\ \alpha. \tag{1.7}$$

The Sobolev spaces are Hilbert spaces, so their duals are (conjugate) isomorphic to themselves. However, in view of our inclusion $L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, we habitually identify

$$(H^s(\mathbb{R}^n))' = H^{-s}(\mathbb{R}^n),$$

with the 'extension of the L^2 paring'

$$(u, v) = \text{``}\int u(z)v(z)dz'' = (2\pi)^{-n}\int_{\mathbb{R}^n}\langle\zeta\rangle^s\hat{u}\cdot\langle\zeta\rangle^{-s}\hat{u}d\zeta.$$

Note that then (5) is a linear, not a conjugate-linear, isomorphism since (5) is a real pairing.

The Sobolev spaces decrease with increasing s,

$$H^s(\mathbb{R}^n) \subset H^{s'}(\mathbb{R}^n)\ \forall\ s \geq s'.$$

One essential property is the relationship between the 'L^2 derivatives' involved in the definition of Sobolev spaces and standard derivatives. Namely, the Sobolev embedding theorem:

$$\begin{aligned} s > \frac{n}{2} &\Longrightarrow H^s(\mathbb{R}^n) \subset \mathcal{C}^0_\infty(\mathbb{R}^n) \\ &\qquad = \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ its continuous and bounded}\}. \\ s > \frac{n}{2} + k,\ k \in \mathbb{N} &\Longrightarrow H^s(\mathbb{R}^n) \subset \mathcal{C}^k_\infty(\mathbb{R}^n) \\ &\qquad \overset{\text{def}}{=} \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ s.t. } D^\alpha u \in \mathcal{C}^0_\infty(\mathbb{R}^n)\ \forall\ |\alpha| \le k\}. \end{aligned}$$

For positive integral s the Sobolev norms are easily written in terms of the functions, without Fourier transform:

$$\begin{gathered} u \in H^k(\mathbb{R}^n) \Leftrightarrow D^\alpha u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| \le k \\ \|u\|_k^2 = \sum_{|\alpha| \le k} \int_{\mathbb{R}^n} |D^\alpha u|^2 dz. \end{gathered}$$

For negative integral orders there is a similar characterization by duality, namely

$$\begin{aligned} H^{-k}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \text{ s.t. } ,\ \exists\ u_\alpha \in L^2(\mathbb{R}^n),\ |\alpha| \ge k \\ u = \sum_{|\alpha| \le k} D^\alpha u_\alpha\}. \end{aligned}$$

In fact there are similar "Hölder" characterizations in general. For $0 < s < 1$, $u \in H^s(\mathbb{R}^n) \Longrightarrow u \in L^2(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^{2n}} \frac{|u(z) - u(z')|^2}{|z - z'|^{n+2s}} dz dz' < \infty. \tag{1.8}$$

Then for $k < s < k+1$, $k \in \mathbb{N}$ $u \in H^s(\mathbb{R}^2)$ is equivalent to $D^\alpha \in H^{s-k}(\mathbb{R}^n)$ for all $|\alpha| \in k$, with corresponding (Hilbert) norm. Similar realizations of the norms exist for $s < 0$.

One simple consequence of this is that

$$\mathcal{C}^\infty_\infty(\mathbb{R}^n) = \bigcap_k \mathcal{C}^k_\infty(\mathbb{R}^n) = \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ s.t. } |D^\alpha u| \text{ is bounded } \forall\ \alpha\}$$

is a multiplier on *all* Sobolev spaces

$$\mathcal{C}^\infty_\infty(\mathbb{R}^n) \cdot H^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)\ \forall\ s \in \mathbb{R}.$$

6. Weighted Sobolev spaces.

It follows from the Sobolev embedding theorem that

$$\bigcap_s H^s(\mathbb{R}^n) \subset \mathcal{C}^\infty_\infty(\mathbb{R}^n); \tag{1.9}$$

in fact the intersection here is quite a lot smaller, but nowhere near as small as $\mathcal{S}(\mathbb{R}^n)$. To discuss decay at infinity, as will definitely want to do, we may use weighted Sobolev spaces.

The ordinary Sobolev spaces do not effectively define decay (or growth) at infinity. We will therefore also set

$$\begin{aligned} H^{m,l}(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n);\ \langle z\rangle^{\ell} u \in H^m(\mathbb{R}^n)\},\ m, \ell \in \mathbb{R}, \\ &= \langle z\rangle^{-\ell} H^m(\mathbb{R}^n)\,, \end{aligned}$$

where the second notation is supported to indicate that $u \in H^{m,l}(\mathbb{R}^n)$ may be written as a product $\langle z\rangle^{-\ell} v$ with $v \in H^m(\mathbb{R}^n)$. Thus

$$H^{m,\ell}(\mathbb{R}^n) \subset H^{m',\ell'}(\mathbb{R}^n) \text{ if } m \geq m' \text{ and } \ell \geq \ell',$$

so the spaces are decreasing in each index. As consequences of the *Schwartz structure theorem*

$$\begin{aligned} \mathcal{S}'(\mathbb{R}^n) &= \bigcup_{m,\ell} H^{m,\ell}(\mathbb{R}^n) \\ \mathcal{S}(\mathbb{R}^n) &= \bigcap_{m,\ell} H^{m,\ell}(\mathbb{R}^n). \end{aligned} \tag{1.10}$$

This is also true 'topologically' meaning that the first is an 'inductive limit' and the second a 'projective limit'.

Similarly, using some commutation arguments

$$\begin{aligned} D_{z_j} &: H^{m,\ell}(\mathbb{R}^n) \longrightarrow H^{m-1,\ell}(\mathbb{R}^n),\ \forall\ m,\ elll \\ \times z_j &: H^{m,\ell}(\mathbb{R}^n) \longrightarrow H^{m,\ell-1}(\mathbb{R}^n). \end{aligned}$$

Moreover there is symmetry under the Fourier transform

$$\mathcal{F} : H^{m,\ell}(\mathbb{R}^n) \longrightarrow H^{\ell,m}(\mathbb{R}^n) \text{ is an isomorphism } \forall\ m,\ \ell.$$

As with the usual Sobolev spaces, $\mathcal{S}(\mathbb{R}^n)$ is dense in all the $H^{m,\ell}(\mathbb{R}^n)$ spaces and the continuous extension of the L^2 paring gives an identification

$$\begin{gathered} H^{m,\ell}(\mathbb{R}^n) \cong (H^{-m,-\ell}(\mathbb{R}^n))' \text{ fron} \\ H^{m,\ell}(\mathbb{R}^n) \times H^{-m,-\ell}(\mathbb{R}^n) \ni u, v \mapsto \\ (u, v) = \text{“}\int u(z)v(z)dz''. \end{gathered}$$

Let R_s be the operator defined by Fourier multiplication by $\langle\zeta\rangle^s$:

$$R_s : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\ \widehat{R_s f}(\zeta) = \langle\zeta\rangle^s \hat{f}(\zeta). \tag{1.11}$$

Lemma 1. *If $\psi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$M_s = [\psi, R_s *] : H^t(\mathbb{R}^n) \longrightarrow H^{t-s+1}(\mathbb{R}^n) \tag{1.12}$$

is bounded for each t.

Proof. Since the Sobolev spaces are defined in terms of the Fourier transform, first conjugate and observe that (1.12) is equivalent to the boundeness of the integral operator with kernel

$$K_{s,t}(\zeta, \zeta') = (1+|\zeta|^2)^{\frac{t-s+1}{2}} \hat{\psi}(\zeta-\zeta') \left((1 + |\zeta'|^2)^{\frac{s}{2}} - (1 + |\zeta|^2)^{\frac{s}{2}}\right) (1+|\zeta'|^2)^{-\frac{t}{2}} \tag{1.13}$$

on $L^2(\mathbb{R}^n)$. If we insert the characteristic function for the region near the diagonal

$$|\zeta - \zeta'| \leq \frac{1}{4}(|\zeta| + |\zeta'|) \Longrightarrow |\zeta| \leq 2|\zeta'|,\ |\zeta'| \leq 2|\zeta| \tag{1.14}$$

then $|\zeta|$ and $|\zeta'|$ are of comparable size. Using Taylor's formula

(1.15)
$$(1+|\zeta'|^2)^{\frac{s}{2}}-(1+|\zeta|^2)^{\frac{s}{2}}=s(\zeta-\zeta')\cdot\int_0^1(t\zeta+(1-t\zeta')\left(1+|t\zeta+(1-t)\zeta'|^2\right)^{\frac{s}{2}-1}dt$$
$$\Longrightarrow\left|(1+|\zeta'|^2)^{\frac{s}{2}}-(1+|\zeta|^2)^{\frac{s}{2}}\right|\le C_s|\zeta-\zeta'|(1+|\zeta|)^{s-1}.$$

It follows that in the region (1.14) the kernel in (1.13) is bounded by

$$C|\zeta-\zeta'||\hat{\psi}(\zeta-\zeta')|. \tag{1.16}$$

In the complement to (1.14) the kernel is rapidly decreasing in ζ and ζ' in view of the rapid decrease of $\hat{\psi}$. Both terms give bounded operators on L^2, in the first case using the same estimates that show convolution by an element of $\mathcal{S}$ to be bounded. □

LEMMA 2. *If $u\in H^s(\mathbb{R}^n)$ and $\psi\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ then*

$$\|\psi u\|_s\le\|\psi\|_{L^\infty}\|u\|_s+C\|u\|_{s-1} \tag{1.17}$$

where the constant depends on s and ψ but not u.

PROOF. This is really a standard estimate for Sobolev spaces. Recall that the Sobolev norm is related to the L^2 norm by

$$\|u\|_s=\|\langle D\rangle^s u\|_{L^2}. \tag{1.18}$$

Here $\langle D\rangle^s$ is the convolution operator with kernel defined by its Fourier transform

$$\langle D\rangle^s u=R_s*u,\ \widehat{R_s}(\zeta)=(1+|\zeta|^2)^{\frac{s}{2}}. \tag{1.19}$$

To get (1.17) use Lemma 1.

From (1.12), (writing 0 for the L^2 norm)

(1.20)
$$\|\psi u\|_s=\|R_s*(\psi u)\|_0\le\|\psi(R_s*u)\|_0+\|M_s u\|_0$$
$$\le\|\psi\|_{L^\infty}\|R_s u\|_0+C\|u\|_{s-1}\le\|\psi\|_{L^\infty}\|u\|_s+C\|u\|_{s-1}.$$

This completes the proof of (1.17) and so of Lemma 2. □

7. Multiplicativity

Of primary importance later in our treatment of non-linear problems is some version of the multiplicative property

$$A^s(\mathbb{R}^n)=\begin{cases}H^s(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n) & s\le\frac{n}{2}\\ H^s(\mathbb{R}^n) & s>\frac{n}{2}\end{cases}\text{ is a }\mathcal{C}^\infty\text{ algebra.} \tag{1.21}$$

Here, a $\mathcal{C}^\infty$ algebra is an algebra with an additional closure property. Namely if $F:\mathbb{R}^N\longrightarrow\mathbb{C}$ is a $\mathcal{C}^\infty$ function vanishing at the origin and $u_1,\ \dots,\ u_N\in A^s$ are *real-valued* then

$$F(u_1,\dots,u_n)\in A^s.$$

I will only consider the case of real interest here, where s is an integer and $s>\frac{n}{2}$. The obvious place to start is

LEMMA 3. *If $s>\frac{n}{2}$ then*

$$u,\ v\in H^s(\mathbb{R}^n)\Longrightarrow uv\in H^s(\mathbb{R}^n). \tag{1.22}$$

PROOF. We will prove this directly in terms of convolution. Thus, in terms of weighted Sobolev spaces $u \in H^s(\mathbb{R}^n) = H^{s,0}(\mathbb{R}^n)$ is equivalent to $\hat{u} \in H^{0,s}(\mathbb{R}^n)$. So (1.22) is equivalent to

$$u,\ v \in H^{0,s}(\mathbb{R}^n) \Longrightarrow u * v \in H^{0,s}(\mathbb{R}^n). \tag{1.23}$$

Using the density of $\mathcal{S}(\mathbb{R}^n)$ it suffices to prove the estimate

$$\|u * v\|_{H^{0,s}} \le C_s \|u\|_{H^{0,s}} \|v\|_{H^{0,s}} \text{ for } s > \frac{n}{2}. \tag{1.24}$$

Now, we can write $u(\zeta) = \langle\zeta\rangle^{-s}u'$ etc and convert (1.24) to an estimate on the L^2 norm of

$$\langle\zeta\rangle^{-s}\int \langle\xi\rangle^{-s}u'(\xi)\langle\zeta-\xi\rangle^{-s}v'(\zeta-\xi)d\xi \tag{1.25}$$

in terms of the L^2 norms of u' and $v' \in \mathcal{S}(\mathbb{R}^n)$.

Writing out the L^2 norm as in the proof of Lemma 1 above, we need to estimate the absolute value of

$$\int\int\int d\zeta d\xi d\eta \langle\zeta\rangle^{2s}\langle\xi\rangle^{-s}u_1(\xi)\langle\zeta-\xi\rangle^{-s}v_1(\zeta-\xi)\langle\eta\rangle^{-s}u_2(\eta)\langle\zeta-\eta\rangle^{-s}v_2(\zeta-\eta) \tag{1.26}$$

in terms of the L^2 norms of the u_i and v_i. To do so divide the integral into the four regions,

$$\begin{aligned}
&|\zeta-\xi| \le \frac{1}{4}(|\zeta|+|\xi|),\ |\zeta-\eta| \le \frac{1}{4}(|\zeta|+|\eta|)\\
&|\zeta-\xi| \le \frac{1}{4}(|\zeta|+|\xi|),\ |\zeta-\eta| \ge \frac{1}{4}(|\zeta|+|\eta|)\\
&|\zeta-\xi| \ge \frac{1}{4}(|\zeta|+|\xi|),\ |\zeta-\eta| \le \frac{1}{4}(|\zeta|+|\eta|)\\
&|\zeta-\xi| \ge \frac{1}{4}(|\zeta|+|\xi|),\ |\zeta-\eta| \ge \frac{1}{4}(|\zeta|+|\eta|).
\end{aligned} \tag{1.27}$$

Using (1.14) the integrand in (1.26) may be correspondingly bounded by

$$\begin{aligned}
&C\langle\zeta-\eta\rangle^{-s}|u_1(\xi)||v_1(\zeta-\xi)|\cdot\langle\zeta-\xi\rangle^{-s}|u_2(\eta)||v_2(\zeta-\eta)|\\
&C\langle\eta\rangle^{-s}|u_1(\xi)||v_1(\zeta-\xi)|\cdot\langle\zeta-\xi\rangle^{-s}|u_2(\eta)||v_2(\zeta-\eta)|\\
&C\langle\zeta-\eta\rangle^{-s}|u_1(\xi)||v_1(\zeta-\xi)|\cdot\langle\xi\rangle^{-s}|u_2(\eta)||v_2(\zeta-\eta)|\\
&C\langle\eta\rangle^{-s}|u_1(\xi)|v_1(\zeta-\xi)|\cdot\langle\xi\rangle^{-s}|u_2(\eta)||v_2(\zeta-\eta)|.
\end{aligned} \tag{1.28}$$

Now applying Cauchy-Schwarz inequality, with the factors as indicated, and changing variables appropriately gives the desired estimate. □

Next, we extend this argument to (many) more factors to get the following result which is close to the Gagliardo-Nirenberg estimates (since I am concentrating here on L^2 methods I will not actually discuss the latter).

LEMMA 4. *If $s > \frac{n}{2}$, $N \ge 1$ and $\alpha_i \in \mathbb{N}_0^k$ for $i = 1, \dots, N$ are such that*

$$\sum_{i=1}^{N} |\alpha_i| = T \le s$$

then

$$u_i \in H^s(\mathbb{R}^n) \Longrightarrow U = \prod_{i=1}^{N} D^{\alpha_i}u_i \in H^{s-T}(\mathbb{R}^n),\ \|U\|_{H^{s-T}} \le C_N \prod_{i=1}^{N}\|u_i\|_{H^s}. \tag{1.29}$$

PROOF. We proceed as in the proof of Lemma 3 using the Fourier transform to replace the product by the convolution. Thus it suffices to show that

$$u_1 * u_2 * u_3 * \cdots * u_N \in H^{0,s-T} \text{ if } u_i \in H^{0,s-\alpha_i}. \tag{1.30}$$

Writing out the convolution symmetrically in all variables,

$$u_1 * u_2 * u_3 * \cdots * u_N(\zeta) = \int_{\zeta=\sum\limits_i \xi_i} u_1(\xi_1)\cdots u_N(\xi_N) \tag{1.31}$$

it follows that we need to estimate the L^2 norm in ζ of

$$\langle\zeta\rangle^{s-T} \int_{\zeta=\sum\limits_i \xi_i} \langle\xi_1\rangle^{-s+a_1} v_1(\xi_1)\cdots\langle\xi_N\rangle^{-s+a_N} v_N(\xi_N) \tag{1.32}$$

for N factors v_i which are in L^2 with the $a_i = |\alpha|_i$ non-negative integers summing to $T \le s$. Again writing the square as the product with the complex conjuage it is enough to estimate integrals of the type

$$\begin{aligned} \int_{\{(\xi,\eta)\in\mathbb{R}^{2N};\sum\limits_i \xi_i=\sum\limits_i \eta_i\}} &\langle\sum_i \xi\rangle^{2s-2T}\langle\xi_1\rangle^{-s+a_1} \\ &v_1(\xi_1)\cdots\langle\xi_N\rangle^{-s+a_N} v_N(\xi_N)\langle\eta_1\rangle^{-s+a_1}\bar{v}_1(\eta_1)\cdots\langle\eta_N\rangle^{-s+a_N}\bar{v}_N(\eta_N). \end{aligned} \tag{1.33}$$

This is really an integral over $\mathbb{R}^{2N-1}$ with respect to Lebesgue measure. Applying Cauchy-Schwarz inequality the absolute value is estimated by

$$\int_{\{(\xi,\eta)\in\mathbb{R}^{2N};\sum\limits_i \xi_i=\sum\limits_i \eta_i\}} \prod_{i=1}^N |v_i(\xi_i)|^2 \langle\sum_l \eta_l\rangle^{2s-2T}\prod_{i=1}^N\langle\eta_i\rangle^{-2s+2a_i} \tag{1.34}$$

The domain of integration, given by $\sum\limits_i \eta_i = \sum\limits_i \xi_i$, is covered by the finite number of subsets Γ_j on which in addition $|\eta_j| \ge |\eta_i|$, for all i. On this set we may take the variables of integration to be η_i for $i \ne j$ and the ξ_l. Then $|\eta_i| \ge |\sum\limits_l \eta_l|/N$ so the second part of the integrand in (1.34) is estimated by

$$\langle\eta_j\rangle^{-2s+2a_j}\langle\sum_l \eta_l\rangle^{2s-2T}\prod_{i\ne j}\langle\eta_i\rangle^{-2s+2a_i} \le C_N\langle\eta_j\rangle^{-2T+2a_j}\prod_{i\ne j}\langle\eta_i\rangle^{-2s+2a_i} \le C_N'\prod_{i\ne j}\langle\eta_i\rangle^{-2s} \tag{1.35}$$

Thus the integral in (1.34) is finite and the desired estimate follows. □

PROPOSITION 1. *If* $F \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R})$ *and* $u \in H^s(\mathbb{R}^n)$ *for* $s > \frac{n}{2}$ *an integer then*

$$F(z, u(z)) \in H^s_{\text{loc}}(\mathbb{R}^n). \tag{1.36}$$

PROOF. Since the result is local on $\mathbb{R}^n$ we may multiply by a compactly supported function of z. In fact since $u \in H^s(\mathbb{R}^n)$ is bounded we also multiply by a compactly supported function in $\mathbb{R}$ without changing the result. Thus it suffices to show that

$$F \in \mathcal{C}^\infty_c(\mathbb{R}^n \times \mathbb{R}) \Longrightarrow F(z, u(z)) \in H^s(\mathbb{R}^n). \tag{1.37}$$

Now, Lemma 4 can be applied to show that $F(z, u(z)) \in H^s(\mathbb{R}^n)$. Certainly $F(z, u(z)) \in L^2(\mathbb{R}^n)$ since it is continuous and has compact support. Moreover, differentiating s times and applying the chain rule gives

$$D^\alpha F(z, u(z)) = \sum F_{\alpha_1,\dots,\alpha_N}(z, u(z)) D^{\alpha_1} u \cdots D^{\alpha_N} u \tag{1.38}$$

where the sum is over all (finitely many) decomposition with $\sum_{i=1}^{N} \alpha_i \le \alpha$ and the $F_{\cdot}(z, u)$ are smooth with compact support, being various derivitives of $F(z, u)$. Thus it follows from Lemma 4 that all terms on the right are in $L^2(\mathbb{R}^n)$ for $|\alpha| \le s$. $\square$

Note that slightly more sophisticated versions of these arguments give the full result (1.21) but Proposition 1 suffices for our purposes below.

8. Some bounded operators

LEMMA 5. *If $J \in \mathcal{C}^k(\Omega^2)$ is properly supported then the operator with kernel J (also denoted J) is a map*

$$J : H^s_{\text{loc}}(\Omega) \longrightarrow H^k_{\text{loc}}(\Omega) \ \forall\ s \ge -k. \tag{1.39}$$

CHAPTER 2

Elliptic Regularity

$\boxed{\boxed{\text{0.6Q; Revised: 6-8-2007; Run: February 7, 2008}}}$

Includes some corrections noted by Tim Nguyen and corrections by, and some suggestions from, Jacob Bernstein.

1. Constant coefficient operators

A linear, constant coefficient differential operator can be thought of as a map

$$P(D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ of the form } P(D)u(z) = \sum_{|\alpha|\le m} c_\alpha D^\alpha u(z),$$
$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n},\ D_j = \frac{1}{i}\frac{\partial}{\partial z_j}, \tag{2.1}$$

but it also acts on various other spaces. So, really it is just a polynomial $P(\zeta)$ in n variables. This 'characteristic polynomial' has the property that

$$\mathcal{F}(P(D)u)(\zeta) = P(\zeta)\mathcal{F}u(\zeta), \tag{2.2}$$

which you may think of as a little square

$$\begin{CD} \mathcal{S}(\mathbb{R}^n) @>P(D)>> \mathcal{S}(\mathbb{R}^n) \\ @A\mathcal{F}AVV @AVA\mathcal{F}A \\ \mathcal{S}(\mathbb{R}^n) @>>P\times> \mathcal{S}(\mathbb{R}^n) \end{CD} \tag{2.3}$$

and this is why the Fourier tranform is especially useful. However, this still does not solve the important questions directly.

Question 1. $P(D)$ is always injective as a map (2.1) but is usually not surjective. When is it surjective? If $\Omega \subset \mathbb{R}^n$ is a non-empty open set then

$$P(D) : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega) \tag{2.4}$$

is never injective (unless $P(\zeta)$ is constnat), for which polynomials is it surjective?

The first three points are relatively easy. As a map (2.1) $P(D)$ is injective since if $P(D)u = 0$ then by (2.2), $P(\zeta)\mathcal{F}u(\zeta) = 0$ on $\mathbb{R}^n$. However, a zero set, in $\mathbb{R}^n$, of a non-trivial polynomial alwasys has empty interior, i.e. the set where it is non-zero is dense, so $\mathcal{F}u(\zeta) = 0$ on $\mathbb{R}^n$ (by continuity) and hence $u = 0$ by the invertibility of the Fourier transform. So (2.1) is injective (of course excepting the case that P is the zero polynomial). When is it surjective? That is, when can every $f \in \mathcal{S}(\mathbb{R}^n)$ be written as $P(D)u$ with $u \in \mathcal{S}(\mathbb{R}^n)$? Taking the Fourier transform again, this is the same as asking when every $g \in \mathcal{S}(\mathbb{R}^n)$ can be written in the form $P(\zeta)v(\zeta)$ with

$v \in \mathcal{S}(\mathbb{R}^n)$. If $P(\zeta)$ has a zero in $\mathbb{R}^n$ then this is not possible, since $P(\zeta)v(\zeta)$ always vanishes at such a point. It is a little trickier to see the converse, that $P(\zeta) \neq 0$ on $\mathbb{R}^n$ implies that $P(D)$ in (2.1) is surjective. Why is this not obvious? Because we need to show that $v(\zeta) = g(\zeta)/P(\zeta) \in \mathcal{S}(\mathbb{R}^n)$ whenever $g \in \mathcal{S}(\mathbb{R}^n)$. Certainly, $v \in \mathcal{C}^\infty(\mathbb{R}^n)$ but we need to show that the derivatives decay rapidly at infinity. To do this we need to get an estimate on the rate of decay of a non-vanishing polynomial

LEMMA 6. *If P is a polynomial such that $P(\zeta) \neq 0$ for all $\zeta \in \mathbb{R}^n$ then there exists $C > 0$ and $a \in \mathbb{R}$ such that*

$$|P(\zeta)| \geq C(1 + |\zeta|)^a. \tag{2.5}$$

PROOF. This is a form of the Tarski-Seidenberg Lemma. Stated loosely, a semi-algebraic function has power-law bounds. Thus

$$F(R) = \inf\{|P(\zeta)|; |\zeta| \leq R\} \tag{2.6}$$

is semi-algebraic and non-vanishing so must satisfy $F(R) \geq c(1 + R)^a$ for some $c > 0$ and a (possibly negative). This gives the desired bound.

Is there an elementary proof? □

Thirdly the non-injectivity in (2.4) is obvious for the opposite reason. Namely for any non-constant polynomial there exists $\zeta \in \mathbb{C}^n$ such that $P(\zeta) = 0$. Since

$$P(D)e^{i\zeta\cdot z} = P(\zeta)e^{i\zeta\cdot z} \tag{2.7}$$

such a zero gives rise to a non-trivial element of the null space of (2.4). You can find an extensive discussion of the density of these sort of 'exponential' solutions (with polynomial factors) in all solutions in Hörmander's book [**?**].

What about the surjectivity of (2.4)? It is not always surjective unless Ω is *convex* but there are decent answers, to find them you should look under P-convexity in [**?**]. If $P(\zeta)$ is elliptic then (2.4) is surjective for every open Ω; maybe I will prove this later although it is not a result of great utility.

2. Constant coefficient elliptic operators

To discuss elliptic regularity, let me recall that any constant coefficient differential operator of order m defines a continuous linear map

$$P(D) : H^{s+m}(\mathbb{R}^n) \longmapsto H^s(\mathbb{R}^n). \tag{2.8}$$

Provided P is not the zero polynomial this map is *always* injective. This follows as in the discussion above for $\mathcal{S}(\mathbb{R}^n)$. Namely, if $u \in H^{s+m}(\mathbb{R}^n)$ then, by definition, $\hat{u} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and if $Pu = 0$ then $P(\zeta)\hat{u}(\zeta) = 0$ off a set of measure zero. Since $P(\zeta) \neq 0$ on an open dense set it follows that $\hat{u} = 0$ off a set of measure zero and so $u = 0$ as a distribution.

As a map (2.8), $P(D)$ is is seldom surjective. It is said to be elliptic (either as a polynomial or as a differential operator) if it is of order m and there is a constant $c > 0$ such that

$$|P(\zeta)| \geq c(1 + |\zeta|)^m \text{ in } \{\zeta \in \mathbb{R}^n; |\zeta| > 1/c\}. \tag{2.9}$$

PROPOSITION 2. *As a map* (2.8)*, for a given s, $P(D)$ is surjective if and only if P is elliptic and $P(\zeta) \neq 0$ on $\mathbb{R}^n$ and then it is a topological isomorphism for every s.*

Proof. Since the Sobolev spaces are defined as the Fourier transforms of the weighted L^2 spaces, that is

$$f \in H^t(\mathbb{R}^n) \Longleftrightarrow (1+|\zeta|^2)^{t/2}\hat{f} \in L^2(\mathbb{R}^n) \tag{2.10}$$

the sufficiency of these conditions is fairly clear. Namely the combination of ellipticity, as in (2.9), and the condition that $P(\zeta) \neq 0$ for $\zeta \in \mathbb{R}^n$ means that

$$|P(\zeta)| \geq c(1+|\zeta|^2)^{m/2},\ c>0,\ \zeta \in \mathbb{R}^n. \tag{2.11}$$

From this it follows that $P(\zeta)$ is bounded above and below by multiples of $(1+|\zeta|^2)^{m/2}$ and so maps the weighted L^2 spaces into each other

$$\times P(\zeta) : H^{0,s+m}(\mathbb{R}^n) \longrightarrow H^{0,s}(\mathbb{R}^n),\ H^{0,s} = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n); \langle\zeta\rangle^s u(\zeta) \in L^2(\mathbb{R}^n)\}, \tag{2.12}$$

giving an isomorphism (2.8) after Fourier transform.

The necessity follows either by direct construction or else by use of the closed graph theorem. If $P(D)$ is surjective then multiplication by $P(\zeta)$ must be an isomorphism between the corresponding weighted space $H^{0,s}(\mathbb{R}^n)$ and $H^{0,s+m}(\mathbb{R}^n)$. By the density of functions supported off the zero set of P the norm of the inverse can be seen to be the inverse of

$$\inf_{\zeta\in\mathbb{R}^n} |P(\zeta)|\langle\zeta\rangle^{-m} \tag{2.13}$$

which proves ellipticity. □

Ellipticity is reasonably common in appliactions, but the condition that the characteristic polynomial not vanish at all is frequently not satisfied. In fact one of the questions I want to get to in this course – even though we are interested in variable coefficient operators – is improving (2.8) (by changing the Sobolev spaces) to get an isomorphism at least for homogeneous elliptic operators (which do not satisfy the second condition in Proposition 2 because they vanish at the origin). One reason for this is that we want it for monopoles.

Note that ellipticity itself is a condition on the principal part of the polynomial.

Lemma 7. *A polynomial* $P(\zeta) = \sum\limits_{\alpha|\leq m} c_\alpha \zeta^\alpha$ *of degree* m *is elliptic if and only if its leading part*

$$P_m(\zeta) = \sum_{|\alpha|=m} c_\alpha\zeta^\alpha \neq 0 \ \text{on}\ \mathbb{R}^n\setminus\{0\}. \tag{2.14}$$

Proof. Since the principal part is homogeneous of degree m the requirement (2.14) is equivalent to

$$|P_m(\zeta)| \geq c|\zeta|^m,\ c = \inf_{|\zeta|=1}|P(\zeta)| > 0. \tag{2.15}$$

Thus, (2.9) follows from this, since

$$|P(\zeta)| \geq |P_m(\zeta)| - |P'(\zeta)| \geq c|\zeta|^m - C|\zeta|^{m-1} - C \geq \frac{c}{2}|\zeta|^m \text{ if } |\zeta| > C', \tag{2.16}$$

$P' = P - M_m$ being of degree at most $m-1$. Conversely, ellipticity in the sense of (2.9) implies that

$$|P_m(\zeta)| \geq |P(\zeta)| - |P'(\zeta)| \geq c|\zeta|^m - C|\zeta|^{m-1} - C > 0 \text{ in } |\zeta| > C' \tag{2.17}$$

and so $P_m(\zeta) \neq 0$ for $\zeta \in \mathbb{R}^n\setminus\{0\}$ by homogeneity. □

Let me next recall *elliptic regularity* for constant coefficient operators. Since this is a local issue, I first want to recall the local versions of the Sobolev spaces discussed in Chapter 1

Definition 1. If $\Omega \subset \mathbb{R}^n$ is an open set then

$$H^s_{\text{loc}}(\Omega) = \left\{u \in \mathcal{C}^{-\infty}(\Omega); \phi u \in H^s(\mathbb{R}^n)\ \forall\ \phi \in \mathcal{C}^\infty_c(\Omega)\right\}. \tag{2.18}$$

Again you need to know what $\mathcal{C}^{-\infty}(\Omega)$ is (it is the dual of $\mathcal{C}^\infty_c(\Omega)$) and that multiplication by $\phi \in \mathcal{C}^\infty_c(\Omega)$ defines a linear continuous map from $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ to $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and gives a bounded operator on $H^m(\mathbb{R}^n)$ for all m.

Proposition 3. *If $P(D)$ is elliptic, $u \in \mathcal{C}^{-\infty}(\Omega)$ is a distribution on an open set and $P(D)u \in H^s_{\text{loc}}(\Omega)$ then $u \in H^{s+m}_{\text{loc}}(\Omega)$. Furthermore if ϕ, $\psi \in \mathcal{C}^\infty_c(\Omega)$ with $\phi = 1$ in a neighbourhood of* $\operatorname{supp}(\psi)$ *then*

$$\|\psi u\|_{s+m} \le C\|\psi P(D)u\|_s + C'\|\phi u\|_{s+m-1} \tag{2.19}$$

for any $M \in \mathbb{R}$, with C' depending only on ψ, ϕ, M and $P(D)$ and C depending only on $P(D)$ (so neither depends on u).

Although I will no prove it here, and it is not of any use below, it is worth noting that (2.19) *characterizes* the ellipticity of a differential operator with smooth coefficients.

Proof. Let me discuss this in two slightly different ways. The first, older, approach is via direct regularity estimates. The second is through the use of a parametrix; they are not really very different!

First the regularity estimates. An easy case of Proposition 3 arises if $u \in \mathcal{C}^{-\infty}_c(\Omega)$ has compact support to start with. Then $P(D)u$ also has compact support so in this case

$$u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n) \text{ and } P(D)u \in H^s(\mathbb{R}^n). \tag{2.20}$$

Then of course the Fourier transform works like a charm. Namely $P(D)u \in H^s(\mathbb{R}^n)$ means that

$$\langle\zeta\rangle^s P(\zeta)\hat{u}(\zeta) \in L^2(\mathbb{R}^n) \Longrightarrow \langle\zeta\rangle^{s+m} F(\zeta)\hat{u}(\zeta) \in L^2(\mathbb{R}^n),\ F(\zeta) = \langle\zeta\rangle^{-m}P(\zeta). \tag{2.21}$$

Ellipticity of $P(\zeta)$ implies that $F(\zeta)$ is bounded above and below on $|\zeta| > 1/c$ and hence can be inverted there by a bounded function. It follows that, given any $M \in \mathbb{R}$ the norm of u in $H^{s+m}(\mathbb{R}^n)$ is bounded

$$\|u\|_{s+m} \le C\|u\|_s + C'_M\|u\|_M,\ u \in \mathcal{C}^{-\infty}(\Omega), \tag{2.22}$$

where the second term is used to bound the L^2 norm of the Fourier transform in $|\zeta| \le 1/c$.

To do the general case of an open set we need to use cutoffs more seriously. We want to show that $\psi u \in H^{s+m}(\mathbb{R}^n)$ where $\psi \in \mathcal{C}^\infty_c(\Omega)$ is some fixed but arbitrary element. We can always choose some function $\phi \in \mathcal{C}^\infty_c(\Omega)$ which is equal to one in a neighbourhood of the support of ψ. Then $\phi u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ so, by the Schwartz structure theorem, $\phi u \in H^{m+t-1}(\mathbb{R}^n)$ for some (unknown) $t \in \mathbb{R}$. We will show that $\psi u \in H^{m+T}(\mathbb{R}^n)$ where T is the smaller of s and t. To see this, compute

$$P(D)(\psi u) = \psi P(D)u + \sum_{|\beta| \le m-1, |\gamma| \ge 1} c_{\beta,\gamma} D^\gamma \psi D^\beta \phi u. \tag{2.23}$$

With the final ϕu replaced by u this is just the effect of expanding out the derivatives on the product. Namely, the $\psi P(D)u$ term is when no derivative hits ψ and the other terms come from at least one derivative hitting ψ. Since $\phi = 1$ on the support of ψ we may then insert ϕ without changing the result. Thus the first term on the right in (2.23) is in $H^s(\mathbb{R}^n)$ and all terms in the sum are in $H^t(\mathbb{R}^n)$ (since at most $m-1$ derivatives are involved and $\phi u \in H^{m+t-1}(\mathbb{R}^n)$ be definition of t). Applying the simple case discussed above it follows that $\psi u \in H^{m+r}(\mathbb{R}^n)$ with r the minimum of s and t. This would result in the estimate

$$\|\psi u\|_{s+m} \leq C\|\psi P(D)u\|_s + C'\|\phi u\|_{s+m-1} \tag{2.24}$$

provided we knew that $\phi u \in H^{s+m-1}$ (since then $t = s$). Thus, initially we only have this estimate with s replaced by T where $T = \min(s,t)$. However, the only obstruction to getting the correct estimate is knowing that $\psi u \in H^{s+m-1}(\mathbb{R}^n)$.

To see this we can use a bootstrap argument. Observe that ψ can be taken to be *any* smooth function with support in the interior of the set where $\phi = 1$. We can therefore insert a chain of functions, of any finite (integer) length $k \geq s-t$, between then, with each supported in the region where the previous one is equal to 1 :

$$\operatorname{supp}(\psi) \subset \{\phi_k = 1\}^\circ \subset \operatorname{supp}(\phi_k) \subset \cdots \subset \operatorname{supp}(\phi_1) \subset \{\phi = 1\}^\circ \subset \operatorname{supp}(\phi) \tag{2.25}$$

where ψ and ϕ were our initial choices above. Then we can apply the argument above with $\psi = \phi_1$, then $\psi = \phi_2$ with ϕ replaced by ϕ_1 and so on. The initial regularity of $\phi u \in H^{t+m-1}(\mathbb{R}^n)$ for some t therefore allows us to deduce that

$$\phi_j u \in H^{m+T_j}(\mathbb{R}^n),\ T_j = \min(s, t+j-1). \tag{2.26}$$

If k is large enough then $\min(s, t+k) = s$ so we conclude that $\psi u \in H^{s+m}(\mathbb{R}^n)$ for any such ψ and that (2.24) holds. $\square$

Although this is a perfectly adequate proof, I will now discuss the second method to get elliptic regularity; the main difference is that we think more in terms of operators and avoid the explicit iteration technique, by doing it all at once – but at the expense of a little more thought. Namely, going back to the easy case of a tempered distibution on $\mathbb{R}^n$ give the map a name:-

$$Q(D) : f \in \mathcal{S}'(\mathbb{R}^n) \longmapsto \mathcal{F}^{-1}\left(\hat{q}(\zeta)\hat{f}(\zeta)\right) \in \mathcal{S}'(\mathbb{R}^n),\ \hat{q}(\zeta) = \frac{1-\chi(\zeta)}{P(\zeta)}. \tag{2.27}$$

Here $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ is chosen to be equal to one on the set $|\zeta| \leq \frac{1}{c} + 1$ corresponding to the ellipticity estimate (2.9). Thus $\hat{q}(\zeta) \in \mathcal{C}^\infty(\mathbb{R}^n)$ is bounded and in fact

$$|D^\alpha_\zeta \hat{q}(\zeta)| \leq C_\alpha (1+|\zeta|)^{-m-|\alpha|}\ \forall\ \alpha. \tag{2.28}$$

This has a straightforward proof by induction. Namely, these estimates are trivial on any compact set, where the function is smooth, so we need only consider the region where $\chi(\zeta) = 0$. The inductive statement is that for some polynomials H_α,

$$D^\alpha_\zeta \frac{1}{P(\zeta)} = \frac{H_\alpha(\zeta)}{(P(\zeta))^{|\alpha|+1}},\ \deg(H_\alpha) \leq (m-1)|\alpha|. \tag{2.29}$$

From this (2.28) follows. Prove (2.29) itself by differentiating one more time and reorganizing the result.

So, in view of the estimate with $\alpha = 0$ in (2.28),

$$Q(D) : H^s(\mathbb{R}^n) \longrightarrow H^{s+m}(\mathbb{R}^n) \tag{2.30}$$

is continuous for each s and it is also an essential inverse of $P(D)$ in the sense that as operators on $\mathcal{S}'(\mathbb{R}^n)$

$$Q(D)P(D) = P(D)Q(D) = \mathrm{Id} - E,\ E : H^s(\mathbb{R}^n) \longrightarrow H^\infty(\mathbb{R}^n)\ \forall\ s \in \mathbb{R}. \tag{2.31}$$

Namely, E is Fourier multiplication by a smooth function of compact support (namely $1 - \hat{q}(\zeta)P(\zeta)$. So, in the global case of $\mathbb{R}^n$, we get elliptic regularity by applying $Q(D)$ to both sides of the equation $P(D)u = f$ to find

$$f \in H^s(\mathbb{R}^n) \Longrightarrow u = Eu + Qf \in H^{s+m}(\mathbb{R}^n). \tag{2.32}$$

This also gives the esimate (2.22) where the second term comes from the continuity of E.

The idea then, is to do the same thing for $P(D)$ acting on functions on the open set Ω; this argument will subsequently be generalized to variable coefficient operators. The problem is that $Q(D)$ does not act on functions (or chapterdistributions) defined just on Ω, they need to be defined on the whole of $\mathbb{R}^n$ and to be tempered before the the Fourier transform can be applied and then multiplied by $\hat{q}(\zeta)$ to define $Q(D)f$.

Now, $Q(D)$ is a convolution operator. Namely, rewriting (2.27)

$$Q(D)f = Qf = q * f,\ q \in \mathcal{S}'(\mathbb{R}^n),\ \hat{q} = \frac{1 - \chi(\zeta)}{P(\zeta)}. \tag{2.33}$$

This in fact is exactly what (2.27) means, since $\mathcal{F}(q * f) = \hat{q}\hat{f}$. We can write out convolution by a smooth function (which q is not, but let's not quibble) as an integral

$$q * f(\zeta) = \int_{\mathbb{R}^n} q(z - z')f(z')dz'. \tag{2.34}$$

Restating the problem, (2.34) is an integral (really a distributional pairing) over the whole of $\mathbb{R}^n$ not the subset Ω. In essence the cutoff argument above inserts a cutoff ϕ in front of f (really of course in front of u but not to worry). So, let's think about inserting a cutoff into (2.34), replacing it by

$$Q_\psi f(\zeta) = \int_{\mathbb{R}^n} q(z - z')\chi(z, z')f(z')dz'. \tag{2.35}$$

Here we will take $\chi \in \mathcal{C}^\infty(\Omega^2)$. To get the integrand to have compact support in Ω for each $z \in \Omega$ we want to arrange that the projection onto the second variable, z'

$$\pi_L : \Omega \times \Omega \supset \operatorname{supp}(\chi) \longrightarrow \Omega \tag{2.36}$$

should be proper, meaning that the inverse image of a compact subset $K \subset \Omega$, namely $(\Omega \times K) \cap \operatorname{supp}(\chi)$, should be compact in Ω.

Let me strengthen the condition on the support of χ by making it more two-sided and demand that $\chi \in \mathcal{C}^\infty(\Omega^2)$ have proper support in the following sense:

$$\text{If } K \subset \Omega \text{ then } \pi_R\left((\Omega \times K) \cap \operatorname{supp}(\chi)\right) \cup \pi_L\left((L \times \Omega) \cap \operatorname{supp}(\chi)\right) \Subset \Omega. \tag{2.37}$$

Here π_L, $\pi_R : \Omega^2 \longrightarrow \Omega$ are the two projections, onto left and right factors. This condition means that if we multiply the integrand in (2.35) on the left by $\phi(z)$, $\phi \in \mathcal{C}_c^\infty(\Omega)$ then the integrand has compact support in z' as well – and so should exist at least as a distributional pairing. The second property we want of χ is that it should not change the properties of q as a convolution operator too much. This reduces to

$$\chi = 1 \text{ in a neighbourhood of } \operatorname{Diag} = \{(z,z); z \in \Omega\} \subset \Omega^2 \tag{2.38}$$

although we could get away with the weaker condition that

$$\chi \equiv 1 \text{ in Taylor series at } \operatorname{Diag}. \tag{2.39}$$

Before discussing why these conditions help us, let me just check that it is possible to find such a χ. This follows easily from the existence of a partition of unity in Ω as follows. It is possible to find functions $\phi_i \in \mathcal{C}^\infty_c(\Omega)$, $i \in \mathbb{N}$, which have locally finite supports (i.e. any compact subset of Ω only meets the supports of a finite number of the ϕ_i,) such that $\sum\limits_i \phi_i(z) = 1$ in Ω and also so there exist functions $\phi_i' \in \mathcal{C}^\infty_c(\Omega)$, also with locally finite supports in the same sense and such that $\phi_i' = 1$ on a neighborhood of the support of ϕ_i. The existence of such functions is a standard result, or if you prefer, an exercise.

Accepting that such functions exists, consider

$$\chi(z,z') = \sum_i \phi_i(z)\phi_i'(z'). \tag{2.40}$$

Any compact subset of Ω^2 is contained in a compact set of the form $K \times K$ and hence meets the supports of only a finite number of terms in (2.40). Thus the sum is locally finite and hence $\chi \in \mathcal{C}^\infty(\Omega^2)$. Moreover, its support has the property (2.37). Clearly, by the assumption that $\phi_i' = 1$ on the support of ϕ_i and that the latter form a partition of unity, $\chi(z,z) = 1$. In fact $\chi(z,z') = 1$ in a neighborhood of the diagonal since each z has a neighborhood N such that $z' \in N$, $\phi_i(z) \neq 0$ implies $\phi_i'(z') = 1$. Thus we have shown that such a cutoff function χ exists.

Now, why do we want (2.38)? This arises because of the following 'pseudolocal' property of the kernel q.

Lemma 8. *Any distribution q defined as the inverse Fourier transform of a function satisfying* (2.28) *has the property*

$$\operatorname{sing\,supp}(q) \subset \{0\} \tag{2.41}$$

Proof. This follows directly from (2.28) and the properties of the Fourier transform. Indeed these estimates show that

$$z^\alpha q(z) \in \mathcal{C}^N(\mathbb{R}^n) \text{ if } |\alpha| > n + N \tag{2.42}$$

since this is enough to show that the Fourier transform, $(i\partial_\zeta)^\alpha \hat{q}$, is L^1. At every point of $\mathbb{R}^n$, other than 0, one of the z_j is non-zero and so, taking $z^\alpha = z_j^k$, (2.42) shows that $q(z)$ is in C^N in $\mathbb{R}^n \setminus \{0\}$ for all N, i.e. (2.41) holds. □

Thus the distribution $q(z - z')$ is only singular at the diagonal. It follows that different choices of χ with the properties listed above lead to kernels in (2.35) which differ by smooth functions in Ω^2 with proper supports.

Lemma 9. *A properly supported smoothing operator, which is by defnninition given by an integral operator*

$$Ef(z) = \int_\Omega E(z,z')f(z')dz' \tag{2.43}$$

where $E \in \mathcal{C}^\infty(\Omega^2)$ has proper support (so both maps

$$\pi_L,\ \pi_R : \operatorname{supp}(E) \longrightarrow \Omega \tag{2.44}$$

are proper), defines continuous operators

$$E : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega),\ \mathcal{C}_c^{-\infty}(\Omega) \longrightarrow \mathcal{C}_c^\infty(\Omega) \tag{2.45}$$

and has an adjoint of the same type.

See the discussion in Chapter 1.

Proposition 4. *If $P(D)$ is an elliptic operator with constant coefficients then the kernel in* (2.35) *defines an operator $Q_\Omega : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega)$ which maps $H^s_{\mathrm{loc}}(\Omega)$ to $H^{s+m}_{\mathrm{loc}}(\Omega)$ for each $s \in \mathbb{R}$ and gives a 2-sided parametrix for $P(D)$ in Ω :*

$$P(D)Q_\Omega = \operatorname{Id} - R,\ Q_\Omega P(D) = \operatorname{Id} - R' \tag{2.46}$$

where R and R' are properly supported smoothing operators.

Proof. We have already seen that changing χ in (2.35) changes Q_Ω by a smoothing operator; such a change will just change R and R' in (2.46) to different properly supported smoothing operators. So, we can use the explicit choice for χ made in (2.40) in terms of a partition of unity. Thus, multiplying on the left by some $\mu \in \mathcal{C}_c^\infty(\Omega)$ the sum becomes finite and

$$\mu Q_\Omega f = \sum_j \mu \psi_j q * (\psi'_j f). \tag{2.47}$$

It follows that Q_Ω acts on $\mathcal{C}^{-\infty}(\Omega)$ and, from the properties of q it maps $H^s_{\mathrm{loc}}(\mathbb{R}^n)$ to $H^{s+m}_{\mathrm{loc}}(\mathbb{R}^n)$ for any s. To check (2.46) we may apply $P(D)$ to (2.47) and consider a region where $\mu = 1$. Since $P(D)q = \delta_0 - \tilde{R}$ where $\tilde{R} \in \mathcal{S}(\mathbb{R}^n)$, $P(D)Q_\Omega f = \operatorname{Id} - R$ where additional 'error terms' arise from any differentiation of ϕ_j. All such terms have smooth kernels (since $\phi'_j = 1$ on the support of ϕ_j and $q(z - z')$ is smooth outside the diagonal) and are properly supported. The second identity in (2.46) comes from the same computation for the adjoints of $P(D)$ and Q_Ω. □

3. Interior elliptic estimates

Next we proceed to prove the same type of regularity and estimates, (2.24), for elliptic differential operators with variable coefficients. Thus consider

$$P(z, D) = \sum_{|\alpha| \le m} p_\alpha(z) D^\alpha,\ p_\alpha \in \mathcal{C}^\infty(\Omega). \tag{2.48}$$

We now assume ellipticity, of fixed order m, for the polynomial $P(z, \zeta)$ for each $z \in \Omega$. This is the same thing as ellipticity for the principal part, i.e. the condition for each compact subset of Ω

$$|\sum_{|\alpha|=m} p_\alpha(z)\zeta^\alpha| \ge C(K)|\zeta|^m,\ z \in K \Subset \Omega, C(K) > 0. \tag{2.49}$$

Since the coefficients are smooth this and $\mathcal{C}^\infty(\Omega)$ is a multiplier on $H^s_{\mathrm{loc}}(\Omega)$ such a differential operator (elliptic or not) gives continuous linear maps

$$P(z, D) : H^{s+m}_{\mathrm{loc}}(\Omega) \longrightarrow H^s_{\mathrm{loc}}(\Omega),\ \forall\ s \in \mathbb{R},\ P(z, D) : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega). \tag{2.50}$$

Now, we arrived at the estimate (2.19) in the constant coefficient case by iteration from the case $M = s + m - 1$ (by nesting cutoff functions). Pick a point

$\bar z \in \Omega$. In a small ball around $\bar z$ the coefficients are 'almost constant'. In fact, by Taylor's theorem,

$$P(z,\zeta) = P(\bar z,\zeta) + Q(z,\zeta),\ Q(z,\zeta) = \sum_j (z-\bar z)_j P_j(z,\bar z,\zeta) \tag{2.51}$$

where the P_j are also polynomials of degree m in ζ and smooth in z in the ball (and in $\bar z$.) We can apply the estimate (2.19) for $P(\bar z, D)$ and $s = 0$ to find

$$\|\psi u\|_m \le C\|\psi\left(P(z,D)u - Q(z,D)\right)u\|_0 + C'\|\phi u\|_{m-1}. \tag{2.52}$$

Because the coefficients are small

$$\begin{aligned}\|\psi Q(z,D)u\|_0 \le \sum_{j,|\alpha|\le m} \|(z-\bar z)_j r_{j,\alpha} D^\alpha \psi u\|_0 + C'\|\phi u\|_{m-1} \\ \le \delta C\|\psi u\|_m + C'\|\phi u\|_{m-1}.\end{aligned} \tag{2.53}$$

What we would like to say next is that we can choose δ so small that $\delta C < \frac{1}{2}$ and so inserting (2.53) into (2.52) we would get

$$\begin{aligned}\|\psi u\|_m &\le C\|\psi P(z,D)u\|_0 + C\|\psi Q(z,D)u\|_0 + C'\|\phi u\|_{m-1} \\ &\le C\|\psi P(z,D)u\|_0 + \frac{1}{2}\|\psi u\|_m + C'\|\phi u\|_{m-1} \\ &\Longrightarrow \frac{1}{2}\|\psi u\|_m \le C\|\psi P(z,D)u\|_0 + C'\|\phi u\|_{m-1}.\end{aligned} \tag{2.54}$$

However, there is a problem here. Namely this is an *a priori* estimate – to move the norm term from right to left we need to know that it is *finite*. Really, that is what we are trying to prove! So more work is required. Nevertheless we will eventually get essentially the same estimate as in the constant coefficient case.

Theorem 1. *If $P(z,D)$ is an elliptic differential operator of order m with smooth coefficients in $\Omega \subset \mathbb{R}^n$ and $u \in \mathcal{C}^{-\infty}(\Omega)$ is such that $P(z,D)u \in H^s_{\text{loc}}(\Omega)$ for some $s \in \mathbb{R}$ then $u \in H^{s+m}_{\text{loc}}(\Omega)$ and for any ϕ, $\psi \in \mathcal{C}^\infty_c(\Omega)$ with $\phi = 1$ in a neighbourhood of* $\operatorname{supp}(\psi)$ *and $M \in \mathbb{R}$, there exist constants C (depending only on P and ψ) and C' (independent of u) such that*

$$\|\psi u\|_{m+s} \le C\|\psi P(z,D)u\|_s + C'\|\phi u\|_M. \tag{2.55}$$

There are three main things to do. First we need to get the *a priori* estimate first for general s, rather than $s = 0$, and then for general ψ (since up to this point it is only for ψ with sufficiently small support). One problem here is that in the estimates in (2.53) the L^2 norm of a product is estimated by the L^∞ norm of one factor and the L^2 norm of the other. For general Sobolev norms such an estimate does not hold, but something similar does; see Lemma 2. The proof of this theorem occupies the rest of this Chapter.

Proposition 5. *Under the hypotheses of Theorem 1 if in addition $u \in \mathcal{C}^\infty(\Omega)$ then* (2.55) *follows.*

Proof of Proposition 5. First we can generalize (2.52), now using Lemma 2. Thus, if ψ has support near the point $\bar z$

$$\begin{aligned}\|\psi u\|_{s+m} &\le C\|\psi P(\bar z,D)u\|_s + \|\phi Q(z,D)\psi u\|_s + C'\|\phi u\|_{s+m-1} \\ &\le C\|\psi P(\bar z,D)u\|_s + \delta C\|\psi u\|_{s+m} + C'\|\phi u\|_{s+m-1}.\end{aligned} \tag{2.56}$$

This gives the extension of (2.54) to general s (where we are assuming that u is indeed smooth):

$$\|\psi u\|_{s+m} \le C_s \|\psi P(z,D)u\|_s + C'\|\phi u\|_{s+m-1}. \tag{2.57}$$

Now, given a general element $\psi \in \mathcal{C}^\infty_c(\Omega)$ and $\phi \in \mathcal{C}^\infty_c(\Omega)$ with $\phi = 1$ in a neighbourhood of $\operatorname{supp}(\psi)$ we may choose a partition of unity ψ_j with respect to $\operatorname{supp}(\psi)$ for each element of which (2.57) holds for some $\phi_j \in \mathcal{C}^\infty_c(\Omega)$ where in addition $\phi = 1$ in a neighbourhood of $\operatorname{supp}(\phi_j)$. Then, with various constants

$$\begin{aligned} \|\psi u\|_{s+m} \le \sum_j \|\psi_j u\|_{s+m} \le C_s \sum_j \|\psi_j \phi P(z,D)u\|_s + C' \sum_j \|\phi_j \phi u\|_{s+m-1} \\ \le C_s(K)\|\phi P(z,D)u\|_s + C''\|\phi u\|_{s+m-1}, \end{aligned} \tag{2.58}$$

where K is the support of ψ and Lemma 2 has been used again. This removes the restriction on supports.

Now, to get the full (a priori) estimate (2.55), where the error term on the right has been replaced by one with arbitrarily negative Sobolev order, it is only necessary to iterate (2.58) on a nested sequence of cutoff functions as we did earlier in the constant coefficient case.

This completes the proof of Proposition 5. □

So, this proves *a priori* estimates for solutions of the elliptic operator in terms of Sobolev norms. To use these we need to show the regularity of solutions and I will do this by constructing parametrices in a manner very similar to the constant coefficient case.

Theorem 2. *If $P(z,D)$ is an elliptic differential operator of order m with smooth coefficients in $\Omega \subset \mathbb{R}^n$ then there is a continuous linear operator*

$$Q : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega) \tag{2.59}$$

such that

$$P(z,D)Q = \operatorname{Id} - R_R,\ QP(z,D) = \operatorname{Id} - R_L \tag{2.60}$$

where R_R, R_L are properly-supported smoothing operators.

That is, both R_R and R_L have kernels in $\mathcal{C}^\infty(\Omega^2)$ with proper supports. We will in fact conclude that

$$Q : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^{s+m}_{\mathrm{loc}}(\Omega),\ \forall\ s \in \mathbb{R} \tag{2.61}$$

using the *a priori* estimates.

To construct at least a first approximation to Q essentially the same formula as in the constant coefficient case suffices. Thus consider

$$Q_0 f(z) = \int_\Omega q(z, z-z')\chi(z,z')f(z')dz'. \tag{2.62}$$

Here q is defined as last time, except it now depends on both variables, rather than just the difference, and is defined by inverse Fourier transform

$$q_0(z,Z) = \mathcal{F}^{-1}_{\zeta \longmapsto Z}\hat{q}_0(z,\zeta),\ \hat{q}_0 = \frac{1-\chi(z,\zeta)}{P(z,\zeta)} \tag{2.63}$$

where $\chi \in \mathcal{C}^\infty(\Omega \times \mathbb{R})$ is chosen to have compact support in the second variable, so $\operatorname{supp}(\chi) \cap (K \times \mathbb{R}^n)$ is compact for each $K \Subset \Omega$, and to be equal to 1 on such a large set that $P(z,\zeta) \neq 0$ on the support of $1-\chi(z,\zeta)$. Thus the right side makes sense and the inverse Fourier transform exists.

Next we extend the esimates, (2.28), on the ζ derivatives of such a quotient, using the ellipticity of P. The same argument works for derivatives with respect to z, except no decay occurs. That is, for any compact set $K \Subset \Omega$

$$|D_z^\beta D_\zeta^\alpha \hat{q}_0(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{-m-|\alpha|},\ z \in K. \tag{2.64}$$

Now the argument, in Lemma 8, concerning the singularities of q_0 works with z derivatives as well. It shows that

$$(z_j - z'_j)^{N+k} q_0(z, z-z') \in \mathcal{C}^N(\Omega \times \mathbb{R}^n) \text{ if } k+m > n/2. \tag{2.65}$$

Thus,

$$\operatorname{sing\,supp} q_0 \subset \operatorname{Diag} = \{(z,z) \in \Omega^2\}. \tag{2.66}$$

The 'pseudolocality' statement (2.66), shows that as in the earlier case, changing the cutoff function in (2.62) changes Q_0 by a properly supported smoothing operator and this will not affect the validity of (2.60) one way or the other! For the moment not worrying too much about how to make sense of (2.62) consider (formally)

$$P(z,D)Q_0 f = \int_\Omega \left(P(z,D_Z)q_0(z,Z)\right)_{Z=z-z'} \chi(z,z')f(z')dz' + E_1 f + R_1 f. \tag{2.67}$$

To apply $P(z,D)$ we just need to apply D^α to $Q_0 f$, multiply the result by $p_\alpha(z)$ and add. Applying D_z^α (formally) under the integral sign in (2.62) each derivative may fall on either the 'parameter' z in $q_0(z,z-z')$, the variable $Z = z-z'$ or else on the cutoff $\chi(z,z')$. Now, if χ is ever differentiated the result vanishes near the diagonal and as a consequence of (2.66) this gives a smooth kernel. So any such term is included in R_1 in (2.67) which is a smoothing operator and we only have to consider derivatives falling on the first or second variables of q_0. The first term in (2.67) corresponds to *all* derivatives falling on the second variable. Thus

$$E_1 f = \int_\Omega e_1(z,z-z')\chi(z,z')f(z')dz' \tag{2.68}$$

is the sum of the terms which arise from at least one derivative in the 'parameter variable' z in q_0 (which is to say ultimately the coefficients of $P(z,\zeta)$). We need to examine this in detail. First however notice that we may rewrite (2.67) as

$$P(z,D)Q_0 f = \operatorname{Id} + E_1 + R'_1 \tag{2.69}$$

where E_1 is unchanged and R'_1 is a new properly supported smoothing operator which comes from the fact that

$$\begin{aligned} P(z,\zeta)q_0(z,\zeta) = 1-\rho(z,\zeta) \Longrightarrow \\ P(z,D_Z)q_0(z,Z) = \delta(Z) + r(z,Z),\ r \in \mathcal{C}^\infty(\Omega\times\mathbb{R}^n) \end{aligned} \tag{2.70}$$

from the choice of q_0. This part is just as in the constant coefficient case.

So, it is the new error term, E_1 which we must examine more carefully. This arises, as already noted, directly from the fact that the coefficients of $P(z,D)$ are not assumed to be constant, hence $q_0(z,Z)$ depends parameterically on z and this is differentiated in (2.67). So, using Leibniz' formula to get an explicit representation of e_1 in (2.68) we see that

$$e_1(z,Z) = \sum_{|\alpha|\le m,\ |\gamma|<m} p_\alpha(z)\binom{\alpha}{\gamma} D_z^{\alpha-\gamma} D_Z^\gamma q_0(z,Z). \tag{2.71}$$

The precise form of this expansion is not really significant. What *is* important is that at most $m-1$ derivatives are acting on the second variable of $q_0(z,Z)$ since all the terms where all m act here have already been treated. Taking the Fourier transform in the second variable, as before, we find that

$$\hat{e}_1(z,\zeta) = \sum_{|\alpha|\le m,\ |\gamma|<m} p_\alpha(z)\binom{\alpha}{\gamma} D_z^{\alpha-\gamma}\zeta^\gamma \hat{q}_0(z,\zeta) \in \mathcal{C}^\infty(\Omega\times\mathbb{R}^n). \tag{2.72}$$

Thus $\hat{e}_1$ is the sum of products of z derivatives of $q_0(z,\zeta)$ and polynomials in ζ of degree at most $m-1$ with smooth dependence on z. We may therefore transfer the estimates (2.64) to e_1 and conclude that

$$|D_z^\beta D_\zeta^\alpha \hat{e}_1(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{-1-|\alpha|}. \tag{2.73}$$

Let us denote by $S^m(\Omega\times\mathbb{R}^n)\subset\mathcal{C}^\infty(\Omega\times\mathbb{R}^n)$ the linear space of functions satisfying (2.64) when $-m$ is replaced by m, i.e.

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{m-|\alpha|} \Longleftrightarrow a\in S^m(\Omega\times\mathbb{R}^n). \tag{2.74}$$

This allows (2.73) to be written succinctly as $\hat{e}_1\in S^{-1}(\Omega\times\mathbb{R}^n)$.

To summarize so far, we have chosen $\hat{q}_0\in S^{-m}(\Omega\times\mathbb{R}^n)$ such that with Q_0 given by (2.62),

$$P(z,D)Q_0 = \operatorname{Id} + E_1 + R_1' \tag{2.75}$$

where E_1 is given by the same formula (2.62), as (2.68), where now $\hat{e}_1\in S^{-1}(\Omega\times\mathbb{R}^n)$. In fact we can easily generalize this discussion, to do so let me use the notation

$$\operatorname{Op}(a)f(z) = \int_\Omega A(z,z-z')\chi(z,z')f(z')dz', \quad \text{if } \hat{A}(z,\zeta)=a(z,\zeta)\in S^m(\Omega\times\mathbb{R}^n). \tag{2.76}$$

Proposition 6. *If $a\in S^{m'}(\Omega\times\mathbb{R}^n)$ then*

$$P(z,D)\operatorname{Op}(a) = \operatorname{Op}(pa) + \operatorname{Op}(b) + R \tag{2.77}$$

where R is a (properly supported) smoothing operator and $b\in S^{m'+m-1}(\Omega\times\mathbb{R}^n)$.

Proof. Follow through the discussion above with $\hat{q}_0$ replaced by a. □

So, we wish to get rid of the error term E_1 in (2.68) to as great an extent as possible. To do so we add to Q_0 a second term $Q_1=\operatorname{Op}(a_1)$ where

$$a_1 = -\frac{1-\chi}{P(z,\zeta)}\hat{e}_1(z,\zeta)\in S^{-m-1}(\Omega\times\mathbb{R}^n). \tag{2.78}$$

Indeed

$$S^{m'}(\Omega\times\mathbb{R}^n)S^{m''}(\Omega\times\mathbb{R}^n)\subset S^{m'+m''}(\Omega\times\mathbb{R}^n) \tag{2.79}$$

(pretty much as though we are multiplying polynomials) as follows from Leibniz' formula and the defining estimates (2.74). With this choice of Q_1 the identity (2.77) becomes

$$P(z,D)Q_1 = -E_1 + \mathrm{Op}(b_2) + R_2,\ b_2 \in S^{-2}(\Omega \times \mathbb{R}^n) \tag{2.80}$$

since $p(z,\zeta)a_1 = -\hat{e}_1 + r'(z,\zeta)$ where $\mathrm{supp}(r')$ is compact in the second variable and so contributes a smoothing operator and by definition $E_1 = \mathrm{Op}(\hat{e}_1)$.

Now we can proceed by induction, let me formalize it a little.

Lemma 10. *If $P(z,D)$ is elliptic with smooth coefficients on Ω then we may choose a sequence of elements $a_i \in S^{-m-i}(\Omega \times \mathbb{R}^n)$ $i = 0, 1, \dots,$ such that if $Q_i = \mathrm{Op}(a_i)$ then*

$$P(z,D)(Q_0 + Q_1 + \cdots + Q_j) = \mathrm{Id} + E_{j+1} + R'_j,\ E_{j+1} = \mathrm{Op}(b_{j+1}) \tag{2.81}$$

with R_j a smoothing operator and $b_j \in S^{-j}(\Omega \times \mathbb{R}^n)$, $j = 1, 2, \dots$.

Proof. We have already taken the first two steps! Namely with $a_0 = \hat{q}_0$, given by (2.63), (2.75) is just (2.81) for $j = 0$. Then, with a_1 given by (2.78), adding (2.80) to (2.78) gives (2.81) for $j = 1$. Proceeding by induction we may assume that we have obtained (2.81) for some j. Then we simply set

$$a_{j+1} = -\frac{1-\chi(z,\zeta)}{P(z,\zeta)} b_{j+1}(z,\zeta) \in S^{-j-1-m}(\Omega \times \mathbb{R}^n)$$

where we have used (2.79). Setting $Q_{j+1} = \mathrm{Op}(a_{j+1})$ the identity (2.77) becomes

$$P(z,D)Q_{j+1} = -E_{j+1} + E_{j+2} + R''_{j+1},\ E_{j+2} = \mathrm{Op}(b_{j+2}) \tag{2.82}$$

for some $b_{j+2} \in S^{-j-2}(\Omega \times \mathbb{R}^n)$. Adding (2.82) to (2.81) gives the next step in the inductive argument. □

Consider the error term in (2.81) for large j. From the estimates on an element $a \in S^{-j}(\Omega \times \mathbb{R}^n)$

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{-j-|\alpha|} \tag{2.83}$$

it follows that if $j > n + k$ then $\zeta^\gamma a$ is integrable in ζ with all its z derivatives for $|\zeta| \le k$. Thus the inverse Fourier transform has continuous derivatives in all variables up to order k. Applied to the error term in (2.81) we conclude that

$$E_j = \mathrm{Op}(b_j) \text{ has kernel in } \mathcal{C}^{j-n-1}(\Omega^2) \text{ for large } j. \tag{2.84}$$

Thus as j increases the error terms in (2.81) have increasingly smooth kernels.

Now, standard properties of operators and kernels, see Lemma 5, show that operator

$$Q_{(k)} = \sum_{j=0}^{k} Q_j \tag{2.85}$$

comes increasingly close to satisfying the first identity in (2.60), except that the error term is only finitely (but arbitrarily) smoothing. Since this is enough for what we want here I will banish the actual solution of (2.60) to the addenda to this Chapter.

LEMMA 11. *For k sufficiently large, the left parametrix $Q_{(k)}$ is a continuous operator on $\mathcal{C}^\infty(\Omega)$ and*

$$Q_{(k)} : H^s_{\text{loc}}(\Omega) \longrightarrow H^{s+m}_{\text{loc}}(\Omega) \ \forall \ s \in \mathbb{R}. \tag{2.86}$$

PROOF. So far I have been rather cavalier in treating $\operatorname{Op}(a)$ for $a \in S^m(\Omega \times \mathbb{R}^n)$ as an operator without showing that this is really the case, however this is a rather easy exercise in distribution theory. Namely, from the basic properties of the Fourier transform and Sobolev spaces

$$A(z, z - z') \in \mathcal{C}^k(\Omega; H^{-n-1+m-k}_{\text{loc}}(\Omega)) \ \forall \ k \in \mathbb{N}. \tag{2.87}$$

It follows that $\operatorname{Op}(a) : H^{n+1-m+k}_{\text{c}}(\Omega)$ into $\mathcal{C}^k(\Omega)$ and in fact into $\mathcal{C}^k_{\text{c}}(\Omega)$ by the properness of the support. In particular it does define an operator on $\mathcal{C}^\infty(\Omega)$ as we have been pretending and the steps above are easily justified.

A similar argument, which I will not give here since it is better to do it by duality (see the addenda), shows that for any fixed s

$$A : H^s_{\text{loc}}(\Omega) \longrightarrow H^S_{\text{loc}}(\Omega) \tag{2.88}$$

for some S. Of course we want something a bit more precise than this.

If $f \in H^s_{\text{loc}}(\Omega)$ then it may be approximated by a sequence $f_j \in \mathcal{C}^\infty(\Omega)$ in the topology of $H^s_{\text{loc}}(\Omega)$, so $\mu f_j \to \mu f$ in $H^s(\mathbb{R}^n)$ for each $\mu \in \mathcal{C}^\infty_c(\Omega)$. Set $u_j = Q_{(k)} f_j \in \mathcal{C}^\infty(\Omega)$ as we have just seen, where k is fixed but will be chosen to be large. Then from our identity $P(z, D)Q_{(k)} = \operatorname{Id} + R_{(k)}$ it follows that

$$P(z, D)u_j = f_j + g_j, \ g_j = R_{(k)} f_j \to R_{(k)} f \text{ in } H^N_{\text{loc}}(\Omega) \tag{2.89}$$

for k large enough depending on s and N. Thus, for k large, the right side converges in $H^s_{\text{loc}}(\Omega)$ and by (2.88), $u_j \to u$ in some $H^S_{\text{loc}}(\Omega)$. But now we can use the *a priori* estimates (2.55) on $u_j \in \mathcal{C}^\infty(\Omega)$ to conclude that

$$\|\psi u_j\|_{s+m} \le C\|\psi(f_j + g_j)\|_s + C''\|\phi u_j\|_S \tag{2.90}$$

to see that ψu_j is bounded in $H^{s+m}(\mathbb{R}^n)$ for any $\psi \in \mathcal{C}^\infty_c(\Omega)$. In fact, applied to the difference $u_j - u_l$ it shows the sequence to be Cauchy. Hence in fact $u \in H^{s+m}_{\text{loc}}(\Omega)$ and the estimates (2.55) hold for this u. That is, $Q_{(k)}$ has the mapping property (2.86) for large k. □

In fact the continuity property (2.86) holds for all $\operatorname{Op}(a)$ where $a \in S^m(\Omega \times \mathbb{R}^n)$, not just those which are parametrices for elliptic differential operators. I will comment on this below – it is one of the basic results on pseudodifferential operators.

There is also the question of the second identity in (2.60), at least in the same finite-order-error sense. To solve this we may use the transpose identity. Thus taking formal transposes this second identity should be equivalent to

$$P^t Q^t = \operatorname{Id} - R^t_L. \tag{2.91}$$

The transpose of $P(z, D)$ is the differential operotor

$$P^t(z, D) = \sum_{|\alpha| \le m} (-D)^\alpha_z p_\alpha(z). \tag{2.92}$$

This is again of order m and after a lot of differenttiation to move the coefficients back to the left we see that its leading part is just $P_m(z, -D)$ where $P_m(z, D)$ is the leading part of $P(z, D)$, so it is elliptic in Ω exactly when P is elliptic. To construct a solution to (2.92), up to finite order errors, we need just apply Lemma 10 to the transpose differential operator. This gives $Q'_{(N)} = \mathrm{Op}(a'_{(N)}$ with the property

$$P^t(z,D)Q'_{(N)} = \mathrm{Id} - R'_{(N)} \tag{2.93}$$

where the kernel of $R'_{(N)}$ is in $C^N(\Omega^2)$. Since this property is preserved under transpose we have indeed solved the second identity in (2.60) up to an arbitrarily smooth error.

Of course the claim in Theorem 2 is that the one operator satisfies both identities, whereas we have constructed two operators which each satisfy one of them, up to finite smoothing error terms

$$P(z,D)Q_R = \mathrm{Id} - R_R,\ \ Q_L P(z,D) = \mathrm{Id} - R_L. \tag{2.94}$$

However these operators must themselves be equal up to finite smoothing error terms since composing the first identity on the left with Q_L and the second on the right with Q_R shows that

$$Q_L - Q_L R_R = Q_L P(z,D) Q_R = Q_R - R_L Q_R \tag{2.95}$$

where the associativity of operator composition has been used. We have already checked the mapping property(2.86) for both Q_L and Q_R, assuming the error terms are sufficiently smoothing. It follows that the composite error terms here map $H^{-p}_{\mathrm{loc}}(\Omega)$ into $H^p_{\mathrm{loc}}(\Omega)$ where $p \to \infty$ with k with the same also true of the transposes of these operators. Such an operator has kernel in $C^{p'}(\Omega^2)$ where again $p' \to \infty$ with k. Thus the difference of Q_L and Q_R itself becomes arbitrarily smoothing as $k \to \infty$.

Finally then we have proved most of Theorem 2 except with arbitrarily finitely smoothing errors. In fact we have not quite proved the regularity statement that $P(z,D)u \in H^s_{\mathrm{loc}}(\Omega)$ implies $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$ although we came very close in the proof of Lemma 11. Now that we know that $Q_{(k)}$ is also a right parametrix, i.e. satisfies the second identity in (2.55) up to arbitrarily smoothing errors, this too follows. Namely from the discussion above $Q_{(k)}$ is an operator on $\mathcal{C}^{-\infty}(\Omega)$ and

$$Q_{(k)}P(z,D)u = u + v_k,\ \ \psi v_k \in H^{s+m}(\Omega)$$

for large enough k so (2.86) implies $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$ and the *a priori* estimates magically become real estimates on all solutions.

Addenda to Chapter 2

Asymptotic completeness to show that we really can get smoothing errors.
Some discussion of pseudodifferential operators – adjoints, composition and boundedness, but only to make clear what is going on.
Some more reassurance as regards operators, kernels and mapping properties – since I have treated these fairly shabbily!

CHAPTER 3

Coordinate invariance and manifolds

$\boxed{\text{0.6Q; Revised: 6-8-2007; Run: February 7, 2008}}$

For the geometric applications we wish to make later (and of course many others) it is important to understand how the objects discussed above behave under coordinate transformations, so that they can be transferred to manifolds (and vector bundles). The basic principle is that the results above are independent of the choice of coordinates, which is to say diffeomorphisms of open sets.

1. Local diffeomorphisms

Let $\Omega_i \subset \mathbb{R}^n$ be open and $f : \Omega_1 \longrightarrow \Omega_2$ be a diffeomorphism, so it is a $\mathcal{C}^\infty$ map, which is equivalent to the condition

$$f^*u \in \mathcal{C}^\infty(\Omega_1)\ \forall\ u \in \mathcal{C}^\infty(\Omega_2),\ f^*u = u \circ f,\ f^*u(z) = u(f(z)), \tag{3.1}$$

and has a $\mathcal{C}^\infty$ inverse $f^{-1} : \Omega_2 \longrightarrow \Omega_1$. Such a map induces an isomorphisms $f^* : \mathcal{C}^\infty_c(\Omega_2) \longrightarrow \mathcal{C}^\infty_c(\Omega_1)$ and $f^* : \mathcal{C}^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_1)$ with inverse $(f^{-1})^* = (f^*)^{-1}$.

Recall also that, as a homeomorphism, f^* identifies the (Borel) measurable functions on Ω_2 with those on Ω_1. Since it is continuously differentiable it also identifies $L^1_{\mathrm{loc}}(\Omega_2)$ with $L^1_{\mathrm{loc}}(\Omega_1)$ and

$$u \in L^1_{\mathrm{c}}(\Omega_2) \Longrightarrow \int_{\Omega_1} f^*u(z)|J_f(z)|dz = \int_{\Omega_1} u(z')dz',\ J_f(z) = \det \frac{\partial f_i(z)}{\partial z_j}. \tag{3.2}$$

The absolute value appears because the definition of the Lebesgue integral is through the Lebesgue measure.

It follows that $f^* : L^2_{\mathrm{loc}}(\Omega_2) \longrightarrow L^2_{\mathrm{loc}}(\Omega_1)$ is also an isomorphism. If $u \in L^2(\Omega_2)$ has support in some compact subset $K \Subset \Omega_2$ then f^*u has support in the compact subset $f^{-1}(K) \Subset \Omega_1$ and

$$\|f^*u\|^2_{L^2} = \int_{\Omega_1} |f^*u|^2 dz \le C(K) \int_{\Omega_1} |f^*u|^2 |J_f(z)| dz = C(K)\|u\|^2_{L^2}. \tag{3.3}$$

Distributions are defined by duality, as the continuous linear functionals:-

$$u \in \mathcal{C}^{-\infty}(\Omega) \Longrightarrow u : \mathcal{C}^\infty_c(\Omega) \longrightarrow \mathbb{C}. \tag{3.4}$$

We always embed the smooth functions in the distributions using integration. This presents a small problem here, namely it is not consistent under pull-back. Indeed if $u \in \mathcal{C}^\infty(\Omega_2)$ and $\mu \in \mathcal{C}^\infty_c(\Omega_1)$ then

$$\begin{gathered}\int_{\Omega_1} f^*u(z)\mu(z)|J_f(z)|dz = \int_{\Omega_2} u(z')(f^{-1})^*\mu(z')dz' \text{ or}\\ \int_{\Omega_1} f^*u(z)\mu(z)dz = \int_{\Omega_2} u(z')(f^{-1})^*\mu(z')|J_{f^{-1}}(z')|dz',\\ \text{since } f^*J_{f^{-1}} = (J_f)^{-1}.\end{gathered} \tag{3.5}$$

So, if we want distributions to be 'generalized functions', so that the identification of $u \in \mathcal{C}^\infty(\Omega_2)$ as an element of $\mathcal{C}^{-\infty}(\Omega_2)$ is consistent with the identification of $f^*u \in \mathcal{C}^\infty(\Omega_1)$ as an element of $\mathcal{C}^{-\infty}(\Omega_1)$ we need to use (3.5). Thus we *define*

$$f^* : \mathcal{C}^{-\infty}(\Omega_2) \longrightarrow \mathcal{C}^{-\infty}(\Omega_1) \text{ by } f^*u(\mu) = u((f^{-1})^*\mu|J_{f^{-1}}|). \tag{3.6}$$

There are better ways to think about this, namely in terms of densities, but let me not stop to do this at the moment. Of course one should check that f^* is a map as indicated and that it behaves correctly under composition, so $(f\circ g)^* = g^*\circ f^*$.

As already remarked, smooth functions pull back under a diffeomorphism (or any smooth map) to be smooth. Dually, vector fields push-forward. A vector field, in local coordinates, is just a first order differential operator without constant term

$$V = \sum_{j=1}^n v_j(z)D_{z_j},\ D_{z_j} = D_j = \frac{1}{i}\frac{\partial}{\partial z_j}. \tag{3.7}$$

For a diffeomorphism, the push-forward may be defined by

$$f^*(f_*(V)u) = Vf^*u \ \forall\ u \in \mathcal{C}^\infty(\Omega_2) \tag{3.8}$$

where we use the fact that f^* in (3.1) is an isomorphism of $\mathcal{C}^\infty(\Omega_2)$ onto $\mathcal{C}^\infty(\Omega_1)$. The chain rule is the computation of f_*V, namely

$$f_*V(f(z)) = \sum_{j,k=1}^n v_j(z)\frac{\partial f_j(z)}{\partial z_k}D_k. \tag{3.9}$$

As always this operation is natural under composition of diffeomorphism, and in particular $(f^{-1})_*(f_*)V = V$. Thus, under a diffeomorphism, vector fields push forward to vector fields and so, more generally, differential operators push-forward to differential operators.

Now, with these definitions we have

Theorem 3. *For every $s \in \mathbb{R}$, any diffeomorphism $f : \Omega_1 \longrightarrow \Omega_2$ induces an isomorphism*

$$f^* : H^s_{\mathrm{loc}}(\Omega_2) \longrightarrow H^s_{\mathrm{loc}}(\Omega_1). \tag{3.10}$$

Proof. We know this already for $s = 0$. To prove it for $0 < s < 1$ we use the norm on $H^s(\mathbb{R}^n)$ equivalent to the standard Fourier transform norm:-

$$\|u\|_s^2 = \|u\|^2_{L^2} + \int_{\mathbb{R}^{2n}} \frac{|u(z)-u(\zeta)|^2}{|z-\zeta|^{2s+n}}dzd\zeta. \tag{3.11}$$

See Sect 7.9 of [?]. Then if $u \in H^s_{\mathrm{c}}(\Omega_2)$ has support in $K \Subset \Omega_2$ with $0 < s < 1$, certainly $u \in L^2$ so $f^*u \in L^2$ and we can bound the second part of the norm in

(3.11) on f^*u :

$$\begin{aligned}(3.12)\quad \int_{\mathbb{R}^{2n}} \frac{|u(f(z)) - u(f(\zeta))|^2}{|z-\zeta|^{2s+n}} dz d\zeta \\ = \int_{\mathbb{R}^{2n}} \frac{|u(z') - u(\zeta')|^2}{|g(z') - g(\zeta')|^{2s+n}} |J_g(z')||J_g(\zeta')| dz' d\zeta' \\ \le C \int_{\mathbb{R}^{2n}} \frac{|u(z) - u(\zeta)|^2}{|z-\zeta|^{2s+n}} dz d\zeta\end{aligned}$$

since $C|g(z') - g(\zeta')| \ge |z' - \zeta'|$ where $g = f^{-1}$.

For the spaces of order $m+s$, $0 \le s < 1$ and $m \in \mathbb{N}$ we know that

$$(3.13)\qquad u \in H^{m+s}_{\text{loc}}(\Omega_2) \Longleftrightarrow Pu \in H^s_{\text{loc}}(\Omega_2)\ \forall\ P \in \text{Diff}^m(\Omega_2)$$

where $\text{Diff}^m(\Omega)$ is the space of differential operators of order at most m with smooth coefficients in Ω. As noted above, differential operators map to differential operators under a diffeomorphism, so from (3.13) it follows that $H^{m+s}_{\text{loc}}(\Omega_2)$ is mapped into $H^{m+s}_{\text{loc}}(\Omega_1)$ by f^*.

For negative orders we may proceed in the same way. That is if $m \in \mathbb{N}$ and $0 \le s < 1$ then

$$(3.14)\qquad u \in H^{s-m}_{\text{loc}}(\Omega_2) \Longleftrightarrow u = \sum_J P_J u_J,\ P_J \in \text{Diff}^m(\Omega_2),\ u_J \in H^s(\Omega_2)$$

where the sum over J is finite. A similar argument then applies to prove (3.10) for all real orders. □

Consider the issue of differential operators more carefully. If $P : \mathcal{C}^\infty(\Omega_1) \longrightarrow \mathcal{C}^\infty(\Omega_1)$ is a differential operator of order m with smooth coefficients then, as already noted, so is

$$(3.15)\qquad P_f : \mathcal{C}^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_2),\ P_f v = (f^{-1})^*(P f^* v).$$

However, the formula for the coefficients, i.e. the explicit formula for P_f, is rather complicated:-

$$(3.16)\qquad P = \sum_{|\alpha| \le m} \Longrightarrow P_f = \sum_{|\alpha| \le m} p_\alpha(g(z'))(J_f(z') D_{z'})^\alpha$$

since we have to do some serious differentiation to move all the Jacobian terms to the left.

Even though the formula (3.16) is complicated, the leading part of it is rather simple. Observe that we can compute the leading part of a differential operator by 'oscillatory testing'. Thus, on an open set Ω consider

$$(3.17)\qquad P(z,D)(e^{it\psi}u) = e^{it\psi} \sum_{k=0}^m t^k P_k(z,D)u,\ u \in \mathcal{C}^\infty(\Omega),\ \psi \in \mathcal{C}^\infty(\Omega),\ t \in \mathbb{R}.$$

Here the $P_k(z,D)$ are differential operators of order $m-k$ acting on u (they involve derivatives of ψ of course). Note that the only way a factor of t can occur is from a derivative acting on $e^{it\psi}$ through

$$(3.18)\qquad D_{z_j} e^{it\psi} = e^{it\psi} t \frac{\partial \psi}{\partial z_j}.$$

Thus, the coefficient of t^m involves no differentiation of u at all and is therefore multiplication by a smooth function which takes the simple form

$$\sigma_m(P)(\psi, z) = \sum_{|\alpha|=m} p_\alpha(z)(D\psi)^\alpha \in \mathcal{C}^\infty(\Omega). \tag{3.19}$$

In particular, the value of this function at any point $z \in \Omega$ is determined once we know $d\psi$, the differential of ψ at that point. Using this observation, we can easily compute the leading part of P_f given that of P in (3.15). Namely if $\psi \in \mathcal{C}^\infty(\Omega_2)$ and $(P_f)(z')$ is the leading part of P_f for

$$\begin{aligned}\sigma_m(P_f)(\psi', z')v &= \lim_{t\to\infty} t^{-m} e^{-it\psi} P_f(z', D_{z'})(e^{it\psi'}v) \\ &= \lim_{t\to\infty} t^{-m} e^{-it\psi} g^*(P(z, D_z)(e^{itf^*\psi'} f^*v) \\ &= g^*(\lim_{t\to\infty} t^{-m} e^{-itf^*\psi'} g^*(P(z, D_z)(e^{itf^*\psi'} f^*v) = g^* P_m(f^*\psi, z) f^*v.\end{aligned} \tag{3.20}$$

Thus

$$\sigma_m(P_f)(\psi', \zeta')) = g^*\sigma_m(P)(f^*\psi', z). \tag{3.21}$$

This allows us to 'geometrize' the transformation law for the leading part (called the principal symbol) of the differential operator P. To do so we think of $T^*\Omega$, for Ω and open subset of $\mathbb{R}^n$, as the union of the $T^*Z\Omega$, $z \in \Omega$, where $T^*_z\Omega$ is the linear space

$$\begin{aligned}&T^*_z\Omega = \mathcal{C}^\infty(\Omega)/\sim_z,\ \psi \sim_z \psi' \Longleftrightarrow \\ &\qquad \psi(Z) - \psi'(Z) - \psi(z) + \psi'(z) \text{ vanishes to second order at } Z = z.\end{aligned} \tag{3.22}$$

Essentially by definition of the derivative, for any $\psi \in \mathcal{C}^\infty(\Omega)$,

$$\psi \sim_z \sum_{j=1}^n \frac{\partial \psi}{\partial z}(z)(Z_j - z_j). \tag{3.23}$$

This shows that there is an isomorphism, given by the use of coordinates

$$T^*\Omega \equiv \Omega \times \mathbb{R}^n,\ [z, \psi] \longmapsto (z, d\psi(z)). \tag{3.24}$$

The point of the complicated-looking definition (3.22) is that it shows easily (and I recommend you do it explicitly) that any smooth map $h : \Omega_1 \longrightarrow \Omega_2$ induces a smooth map

$$h^*T^*\Omega_2 \longrightarrow T^*\Omega_1,\ h([h(z), \psi]) = [z, h^*\psi] \tag{3.25}$$

which for a diffeomorphism is an isomorphism.

Lemma 12. *The transformation law* (3.21) *shows that for any element* $P \in \mathrm{Diff}^m(\Omega)$ *the principal symbol is well-defined as an element*

$$\sigma(P) \in \mathcal{C}^\infty(T^*\Omega) \tag{3.26}$$

which furthermore transforms as a function under the pull-back map (3.25) *induced by any diffeomorphism of open sets.*

Proof. The formula (3.19) is consistent with (3.23) and hence with (3.21) in showing that $\sigma_m(P)$ is a well-defined function on $T^*\Omega$. □

2. Manifolds

I will only give a rather cursory discussion of manifolds here. The main cases we are interested in are practical ones, the spheres $\mathbb{S}^n$ and the balls $\mathbb{B}^n$. Still, it is obviously worth thinking about the general case, since it is the standard setting for much of modern mathematics. There are in fact several different, but equivalent, definitions of a manifold.

2.1. Coordinate covers. Take a Hausdorff topological (in fact metrizable) space M. A *coordinate patch* on M is an open set and a homeomorphism

$$M \supset \Omega \xrightarrow{F} \Omega' \subset \mathbb{R}^n$$

onto an open subset of $\mathbb{R}^n$. An atlas on M is a covering by such coordinate patches (Ω_a, F_a),

$$M = \bigcup_{a \in A} \Omega_a.$$

Since each $F_{ab} : \Omega'_a \to \Omega'_a$ is, by assumption, a homeomorphism, the transition maps

$$\begin{aligned} F_{ab} &: \Omega'_{ab} \to \Omega'_{ba}\,, \\ \Omega'_{ab} &= F_b(\Omega_a \cap \Omega_b)\,, \\ (\Rightarrow \Omega'_{ba} &= F_a(\Omega_a \cap \Omega_b)) \\ F_{ab} &= F_a \circ F_b^{-1} \end{aligned}$$

are also homeomorphisms of open subsets of $\mathbb{R}^n$ (in particular n is constructed on components of M). The atlas is $\mathcal{C}^k$, $\mathcal{C}^\infty$, real analytic, etc.) if each F_{ab} is $\mathcal{C}^k$, $\mathcal{C}^\infty$ or real analytic. A $\mathcal{C}^\infty$ ($\mathcal{C}^k$ or whatever) structure on M is usually taken to be a *maximal* $\mathcal{C}^\infty$ atlas (meaning any coordinate patch compatible with all elements of the atlas is already in the atlas).

2.2. Smooth functions. A second possible definition is to take again a Hausdorff topological space and a subspace $\mathcal{F} \subset C(M)$ of the continuous real-valued function on M with the following two properties.

1) For each $p \in M \exists f_1, \dots, f_n \in \mathcal{F}$ and an open set $\Omega \ni p$ such that $F = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ is a homeomorphism onto an open set, $\Omega' \subset \mathbb{R}^n$ and $(F^{-1})^* g \in \mathcal{C}^\infty(\Omega')\, \forall g \in \mathcal{F}$.
2) $\mathcal{F}$ is maximal with this property.

2.3. Embedding. Alternatively one can simply say that a ($\mathcal{C}^\infty$) manifold is a subset $M \subset \mathbb{R}^N$ such that $\forall p \in M \exists$ an open set $U \ni p$, $U \subset \mathbb{R}^N$, and $h_1, \dots, h_{N-n} \in \mathcal{C}^\infty(U)$ s.t.

$$\begin{aligned} M \cap U = \{q \in U;\ h_i(q) = 0,\ i = 1, \dots, N-n\} \\ dh_i(p) \text{ are linearly independent.} \end{aligned}$$

I leave it to you to show that these definitions are equivalent in an *appropriate sense*. If we weaken the various notions of coordinates in each case, for instance in the first case, by requiring that $\Omega' \in \mathbb{R}^{n-k} \times [0, \infty)^k$ for some k, with a corresponding version of smoothness, we arrive at the notion of a manifold with cones.[1]

[1] I always demand in addition that the boundary faces of a manifold with cones be a *embedded* but others differ on this. I call the more general object a *tied manifold*.

So I will assume that you are reasonably familiar with the notion of a smooth ($\mathcal{C}^\infty$) manifold M, equipped with the space $\mathcal{C}^\infty(M)$ — this is just $\mathcal{F}$ in the second definition and in the first

$$\mathcal{C}^\infty(M) = \{u : M \to \mathbb{R};\ u \circ F^{-1} \in \mathcal{C}^\infty(\Omega')\,\forall \text{ coordinate patches}\}.$$

Typically I will not distinguish between complex and real-valued functions unless it seems necessary in this context.

Manifolds are always paracompact — so have countable covers by compact sets — and admit partitions of unity.

Proposition 7. *If $M = \bigcup_{a\in A} U_a$ is a cover of a manifold by open sets then there exist $\rho_a \in \mathcal{C}^\infty(M)$ s.t.* $\operatorname{supp}(\rho_a) \Subset U_a$ *(i.e., $\exists K_a \Subset U_a$ s.t. $\rho_a = 0$ on $M\backslash K_a$), these supports are locally finite, so if $K \Subset M$ then*

$$\{a \in A; \rho_a(m) \neq 0 \text{ for some } m \in K\}$$

is finite, and finally

$$\sum_{a\in A} \rho_a(m) = 1,\ \forall\ m \in M.$$

It can also be arranged that

(1) $0 \le \rho_a(m) \le 1\ \forall\ a,\ \forall\ m \in M$.
(2) $\rho_a = \mu_a^2\,,\ \mu_a \in \mathcal{C}^\infty(M)$.
(3) $\exists\ \varphi_a \in \mathcal{C}^\infty(M),\ 0 \le \varphi_a \le 1,\ \varphi = 1$ in a neighborhood of $\operatorname{supp}(\rho_a)$ and the sets $\operatorname{supp}(\varphi_a)$ are locally finite.

Proof. Up to you. □

Using a partition of unity subordinate to a covering by coordinate patches we may transfer definitions from $\mathbb{R}^n$ to M, provided they are coordinate-invariant in the first place and preserved by multiplication by smooth functions of compact support. For instance:

Definition 2. If $u : M \longrightarrow \mathbb{C}$ and $s \ge 0$ then $u \in H^s_{\text{loc}}(M)$ if for some partition of unity subordinate to a cover of M by coordinate patches

$$\begin{gathered}(F_a^{-1})^*(\rho_a u) \in H^s(\mathbb{R}^n)\\ \text{or } (F_a^{-1})^*(\rho_a u) \in H^s_{\text{loc}}(\Omega'_a).\end{gathered} \tag{3.27}$$

Note that there are some abuses of notation here. In the first part of (3.27) we use the fact that $(F_a^{-1})^*(\rho_a u)$, defined really on Ω'_a (the image of the coordinate patch $F_a : \Omega_a \to \Omega'_a \in \mathbb{R}^n$), vanishes outside a compact subset and so can be unambiguously extended as zero outside Ω'_a to give a function on $\mathbb{R}^n$. The second form of (3.27) is better, but there is an equivalence relation, of equality off sets of measure zero, which is being ignored. The definition doesn't work well for $s < 0$ because u might then not be representable by a function so we don't know what u' is to start with.

The most sysetematic approach is to define distributions on M first, so we know what we are dealing with. However, there is a problem here too, because of the transformation law (3.5) that was forced on us by the local identification $\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^{-\infty}(\Omega)$. Namely, we really need *densities* on M before we can define distributions. I will discuss densities properly later; for the moment let me use a little ruse, sticking for simplicity to the compact case.

Definition 3. If M is a compact $\mathcal{C}^\infty$ manifold then $\mathcal{C}^0(M)$ is a Banach space with the supremum norm and a continuous linear functional

$$\mu : \mathcal{C}^0(M) \longrightarrow \mathbb{R} \tag{3.28}$$

is said to be a positive smooth measure if for every coordinate patch on M, $F : \Omega \longrightarrow \Omega'$ there exists $\mu_F \in \mathcal{C}^\infty(\Omega')$, $\mu_F > 0$, such that

$$\mu(f) = \int_{\Omega'} (F^{-1})^* f \mu_F dz \ \forall \ f \in \mathcal{C}^0(M) \text{ with } \operatorname{supp}(f) \subset \Omega. \tag{3.29}$$

Now if μ, $\mu' : \mathcal{C}^0(M) \longrightarrow \mathbb{R}$ is two such smooth measures then $\mu'_F = v_F \mu_F$ with $v_F \in \mathcal{C}^\infty(\Omega')$. In fact $\exists \ v \in \mathcal{C}^\infty(M)$, $v > 0$, such that $F^*_{v_F} = v$ on Ω. That is, the v's patch to a well-defined function globally on M. To see this, notice that every $g \in \mathcal{C}^0_{\mathrm{c}}(\Omega')$ is of the form $(F^{-1})^* g$ for some $g \in \mathcal{C}^0(M)$ (with support in Ω) so (3.29) certainly *determines* μ_F on Ω'. Thus, assuming we have two smooth measures, v_F is determined on Ω' for every coordinate patch. Choose a partition of unity ρ_a and *define*

$$v = \sum_a \rho_a F_a^* v_{F_a} \in \mathcal{C}^\infty(M).$$

Exercise. Show (using the transformation of integrals under diffeomorphisms) that

$$\mu'(f) = \mu(vf) \ \forall \ f \in \mathcal{C}^\infty(M). \tag{3.30}$$

Thus we have 'proved' half of

Proposition 8. *Any (compact) manifold admits a positive smooth density and any two positive smooth densities are related by* (3.30) *for some (uniquely determined)* $v \in \mathcal{C}^\infty(M)$, $v > 0$.

Proof. I have already unloaded the hard part on you. The extension is similar. Namely, chose a covering of M by coordinate patches and a corresponding partition of unity as above. Then simply *define*

$$\mu(f) = \sum_a \int_{\Omega'_a} (F_a^{-1})^* (\rho_a f) dz$$

using Lebesgue measure in each Ω'_a. The fact that this satisfies (3.29) is similar to the exercise above. $\square$

Now, for a compact manifold, we can define a *smooth positive density* $\mu' \in \mathcal{C}^\infty(M; \Omega)$ as a continuous linear functional of the form

$$\mu' : \mathcal{C}^0(M) \longrightarrow \mathbb{C}, \ \mu'(f) = \mu(\varphi f) \text{ for some } \varphi \in \mathcal{C}^\infty(M) \tag{3.31}$$

where φ is allowed to be complex-valued. For the moment the notation, $\mathcal{C}^\infty(M; \Omega)$, is not explained. However, the *choice* of a fixed positive $\mathcal{C}^\infty$ measure allows us to identify

$$\mathcal{C}^\infty(M; \Omega) \ni \mu' \longrightarrow \varphi \in \mathcal{C}^\infty(M),$$

meaning that this map is an isomorphism.

LEMMA 13. *For a compact manifold, M, $\mathcal{C}^\infty(M;\Omega)$ is a complete metric space with the norms and distance function*

$$\|\mu'\|_{(k)} = \sup_{|\alpha|\le k} |V_1^{\alpha_1}\cdots V_p^{\alpha_1}\varphi|$$
$$d(\mu_1',\mu_2') = \sum_{k=0}^{\infty} 2^{-k}\frac{\|\mu'\|_{(k)}}{1+\|\mu'\|_{(k)}}$$

where $\{V_1,\dots,V_p\}$ is a collection of vector fields spanning the tangent space at each point of M.

This is really a result of about $\mathcal{C}^\infty(M)$ itself. I have put it this way because of the current relevance of $\mathcal{C}^\infty(M;\Omega)$.

PROOF. First notice that there are indeed such vector fields on a compact manifold. Simply take a covering by coordinate patches and associated partitions of unity, φ_a, supported in the coordinate patch Ω_a. Then if $\Psi_a \in \mathcal{C}^\infty(M)$ has support in Ω_a and $\Psi_a \equiv 1$ in a neighborhood of $\operatorname{supp}(\varphi_a)$ consider

$$V_{a\ell} = \Psi_a(F_a^{-1})_*(\partial_{z_\ell}),\ \ell = 1,\dots,n,$$

just the coordinate vector fields cut off in Ω_a. Clearly, taken together, these span the tangent space at each point of M, i.e., the local coordinate vector fields are really linear combinations of the V_i given by renumbering the $V_{a\ell}$. It follows that

$$\|\mu'\|_{(k)} = \sup_{|\alpha\le k} |V_1^{\alpha_1}\cdots V_p^{\alpha_p}\varphi| \in M$$

is a norm on $\mathcal{C}^\infty(M;\Omega)$ locally equivalent to the $\mathcal{C}^k$ norm on φ_f on compact subsets of coordinate patches. It follows that (3.32) gives a distance function on $\mathcal{C}^\infty(M;\Omega)$ with respect to what is complete — just as for $\mathcal{S}(\mathbb{R}^n)$. □

Thus we can define the space of distributions on M as the space of continuous linear functionals $u \in \mathcal{C}^{-\infty}(M)$

$$u : \mathcal{C}^\infty(M;\Omega) \longrightarrow \mathbb{C},\ |u(\mu)| \le C_k\|\mu\|_{(k)}. \tag{3.32}$$

As in the Euclidean case smooth, and even locally integrable, functions embed in $\mathcal{C}^{-\infty}(M)$ by integration

$$L^1(M) \hookrightarrow \mathcal{C}^{-\infty}(M),\ f \mapsto f(\mu) = \int_M f\mu \tag{3.33}$$

where the integral is defined unambiguously using a partition of unity subordinate to a coordinate cover:

$$\int_M f\mu = \sum_a \int_{\Omega_a'} (F_a^{-1})^*(\varphi_a f\mu_a)dz$$

since $\mu = \mu_a dz$ in local coordinates.

DEFINITION 4. The Sobolev spaces on a compact manifold are defined by reference to a coordinate case, namely if $u \in \mathcal{C}^{-\infty}(M)$ then

$$u \in H^s(M) \Leftrightarrow u(\psi\mu) = u_a((F_a^{-1})^*\psi\mu_a),\ \forall\ \psi \in \mathcal{C}_c^\infty(\Omega_a) \text{ with } u_a \in H^s_{\text{loc}}(\Omega_a'). \tag{3.34}$$

Here the condition can be the requirement *for all* coordinate systems or for a covering by coordinate systems in view of the coordinate independence of the local Sobolev spaces on $\mathbb{R}^n$, that is the weaker condition implies the stronger.

Now we can transfer the properties of Sobolev for $\mathbb{R}^n$ to a compact manifold; in fact the compactness simplifies the properties

$$H^m(M) \subset H^{m'}(M),\ \forall\ m \geq m' \tag{3.35}$$

$$H^m(M) \hookrightarrow \mathcal{C}^k(M),\ \forall\ m > k + \frac{1}{2}\dim M \tag{3.36}$$

$$\bigcap_m H^m(M) = \mathcal{C}^\infty(M) \tag{3.37}$$

$$\bigcup_m H^m(M) = \mathcal{C}^{-\infty}(M). \tag{3.38}$$

These are indeed Hilbert(able) spaces — meaning they do not have a *natural* choice of Hilbert space structure, but they do have one. For instance

$$\langle u, v\rangle_s = \sum_a \langle (F_a^{-1})^* \varphi_a u, (F_a^{-1})^* \varphi_a v\rangle_{H^s(\mathbb{R}^n)}$$

where φ_a is a *square* partition of unity subordinate to coordinate covers.

3. Vector bundles

Although it is *not* really the subject of this course, it is important to get used to the coordinate-free language of vector bundles, etc. So I will insert here at least a minimum treatment of bundles, connections and differential operators on manifolds.

Addenda to Chapter 3

CHAPTER 4

Invertibility of elliptic operators

$\boxed{\boxed{\text{0.6Q; Revised: 6-8-2007; Run: February 7, 2008}}}$

Next we will use the local elliptic estimates obtained earlier on open sets in $\mathbb{R}^n$ to analyse the global invertibility properties of elliptic operators on compact manifolds. This includes at least a brief discussion of spectral theory in the self-adjoint case.

1. Global elliptic estimates

For a single differential operator acting on functions on a compact manifold we now have a relatively simple argument to prove global elliptic estimates.

PROPOSITION 9. *If M is a compact manifold and $P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is a differential operator with $\mathcal{C}^\infty$ coefficients which is* elliptic *(in the sense that $\sigma_m(P) \neq 0)$ on $T^*M\backslash 0)$ then for any s, $M \in \mathbb{R}$ there exist constants C_s, C'_M such that*

$$\begin{aligned} &u \in H^M(M),\ Pu \in H^s(M) \Longrightarrow u \in H^{s+m}(M) \\ &\|u\|_{s+m} \le C_s\|Pu\|_s + C'_M\|u\|_M, \end{aligned} \tag{4.1}$$

where m is the order of P.

PROOF. The regularity result in (4.1) follows directly from our earlier local regularity results. Namely, if $M = \bigcup_a \Omega_a$ is a (finite) covering of M by coordinate patches,

$$F_a : \Omega_a \longrightarrow \Omega'_a \subset \mathbb{R}^n$$

then

$$P_a v = (F_a^{-1})^* P F_a^* v,\ v \in \mathcal{C}_c^\infty(\Omega'_a) \tag{4.2}$$

defines $P_a \in \operatorname{Diff}^m(\Omega'_a)$ which is a differential operator in local coordinates with smooth coefficients; the invariant definition of ellipticity above shows that it is elliptic for each a. Thus if φ_a is a partition of unity subordinate to the open cover and $\psi_a \in \mathcal{C}_c^\infty(\Omega_a)$ are chosen with $\psi_a = 1$ in a neighbourhood of $\operatorname{supp}(\varphi_a)$ then

$$\|\varphi'_a v\|_{s+m} \le C_{a,s}\|\psi'_a P_a v\|_s + C'_{a,m}\|\psi'_a v\|_M \tag{4.3}$$

where $\varphi'_a = (F_a^{-1})^*\varphi_a$ and similarly for $\psi'_a(F_a^{-1})^*\varphi_a \in \mathcal{C}_c^\infty(\Omega'_a)$, are the local coordinate representations. We know that (4.3) holds for every $v \in \mathcal{C}^{-\infty}(\Omega'_a)$ such that $P_a v \in H^M_{\text{loc}}(\Omega'_a)$. Applying (4.3) to $(F_a^{-1})^* u = v_a$, for $u \in H^M(M)$, it follows that $Pu \in H^s(M)$ implies $P_a v_a \in H^M_{\text{loc}}(\Omega'_a)$, by coordinate-invariance of the Sobolev spaces and then conversely

$$v_a \in H^{s+m}_{\text{loc}}(\Omega'_a)\ \forall\ a \Longrightarrow u \in H^{s+m}(M).$$

The norm on $H^s(M)$ can be taken to be

$$\|u\|_s = \left(\sum_a \|(F_a^{-1})^*(\varphi_a u)\|_s^2\right)^{1/2}$$

so the estimates in (4.1) also follow from the local estimates:

$$\begin{aligned}\|u\|_{s+m}^2 &= \sum_a \|(F_a^{-1})^*(\varphi_a u)\|_{s+m}^2 \\ &\le \sum_a C_{a,s}\|\psi_a' P_a (F_a^{-1})^* u\|_s^2 \\ &\le C_s\|Pu\|_s^2 + C_M'\|u\|_M^2.\end{aligned}$$

□

Thus the elliptic regularity, and estimates, in (4.1) just follow by patching from the local estimates. The same argument applies to elliptic operators on vector bundles, once we prove the corresponding local results. This means going back to the beginning!

As discussed in Section 3, a differential operator between sections of the bundles E_1 and E_2 is represented in terms of local coordinates and local trivializations of the bundles, by a matrix of differential operators

$$P = \begin{bmatrix} P_{11}(z, D_z) & \cdots & P_{1\ell}(z, D_z) \\ \vdots & & \vdots \\ P_{n1}(z, D_z) & \cdots & P_{n\ell}(z, D_z) \end{bmatrix}.$$

The (usual) order of P is the maximum of the orders of the $P_{ij}(z, D_3)$ and the symbol is just the corresponding matrix of symbols

$$\sigma_m(P)(z,\zeta) = \begin{bmatrix} \sigma_m(P_{11})(z,\zeta) & \cdots & \sigma_m(P_{1\ell})(z,\zeta) \\ \vdots & & \vdots \\ \sigma_m(P_{n1})(z,\zeta) & \cdots & \sigma_m(P_{n\ell})(z,\zeta) \end{bmatrix}. \tag{4.4}$$

Such a P is said to be *elliptic* at z if this matrix is invariable for all $\zeta \neq 0$, $\zeta \in \mathbb{R}^n$. Of course this implies that the matrix is square, so the two vector bundles have the same rank, ℓ. As a differential operator, $P \in \mathrm{Diff}^m(M, \mathbb{E})$, $\mathbb{E} = E_1, E_2$, is *elliptic* if it is elliptic at each point.

Proposition 10. *If $P \in \mathrm{Diff}^m(M, \mathbb{E})$ is a differential operator between sections of vector bundles $(E_1, E_2) = \mathbb{E}$ which is* elliptic *of order m at every point of M then*

$$u \in \mathcal{C}^{-\infty}(M; E_1),\ Pu \in H^s(M, E_) \Longrightarrow u \in H^{s+m}(M; E_1) \tag{4.5}$$

and for all s, $t \in \mathbb{R}$ there exist constants $C = C_s$, $C' = C'_{s,t}$ such that

$$\|u\|_{s+m} \le C\|Pu\|_s + C'\|u\|_t. \tag{4.6}$$

Furthermore, there is an operator

$$Q : \mathcal{C}^\infty(M; E_2) \longrightarrow \mathcal{C}^\infty M; E_1) \tag{4.7}$$

such that

$$PQ - \mathrm{Id}_2 = R_2,\ QP - \mathrm{Id}_1 = R_1 \tag{4.8}$$

are smoothing operators.

Proof. As already remarked, we need to go back and carry the discussion through from the beginning for systems. Fortunately this requires little more than notational change.

Starting in the constant coefficient case, we first need to observe that ellipticity of a (square) matrix system is equivalent to the ellipticity of the determinant polynomial

$$D_p(\zeta) = \det \begin{bmatrix} P_{11}(\zeta) & \cdots & P_{1k}(\zeta) \\ \vdots & & \vdots \\ P_{k1}(\zeta) & \cdots & P_{kk}(\zeta) \end{bmatrix} \tag{4.9}$$

which is a polynomial degree km. If the P_i's are replaced by their leading parts, of homogeneity m, then D_p is replaced by its leading part of degree km. From this it is clear that the ellipticity at P is equivalent to the ellipticity at D_p. Furthermore the invertibility of matrix in (4.9), under the assumption of ellipticity, follows for $|\zeta| > C$. The inverse can be written

$$P(\zeta)^{-1} = \operatorname{cof}(P(\zeta))/D_p(\zeta).$$

Since the cofactor matrix represents the Fourier transform of a differential operator, applying the earlier discussion to D_p and then composing with this differential operator gives a generalized inverse etc.

For example, if $\Omega \subset \mathbb{R}^n$ is an open set and D_Ω is the parameterix constructed above for D_p on Ω then

$$Q_\Omega = \operatorname{cof}(P(D)) \circ D_\Omega$$

is a 2-sided parameterix for the matrix of operators P:

$$\begin{aligned} PQ_\Omega - \mathrm{Id}_{k\times k} &= R_R \\ Q_\Omega - \mathrm{Id}_{k\times k} &= R_L \end{aligned} \tag{4.10}$$

where R_L, R_R are $k \times k$ matrices of smoothing operators. Similar considerations apply to the variable coefficient case. To construct the global parameterix for an elliptic operator P we proceed as before to piece together the local parameterices Q_a for P with respect to a coordinate patch over which the bundles E_1, E_2 are trivial. Then

$$Qf = \sum_a F_a^* \psi_a' Q_a \varphi_a' (F_a)^{-1} f$$

is a global 1-sided parameterix for P; here φ_a is a partition of unity and $\psi_a \in \mathcal{C}_c^\infty(\Omega_a)$ is equal to 1 in a neighborhood of its support. □

(Probably should be a little more detail.)

2. Compact inclusion of Sobolev spaces

For any $R > 0$ consider the Sobolev spaces of elements with compact support in a ball:

$$\dot{H}^s(B) = \{u \in H^s(\mathbb{R}^n); u) = 0 \text{ in } |x| > 1\}. \tag{4.11}$$

Lemma 14. *Tthe inclusion map*

$$\dot{H}^s(B) \hookrightarrow \dot{H}^t(B) \textit{ is compact if } s > t. \tag{4.12}$$

Proof. Recall that compactness of a linear map between (separable) Hilbert (or Banach) spaces is the condition that the image of any bounded sequence has a convergent subsequence (since we are in separable spaces this is the same as the condition that the image of the unit ball have compact closure). So, consider a bounded sequence $u_n \in \dot{H}^s(B)$. Now $u \in \dot{H}^s(B)$ implies that $u \in H^s(\mathbb{R}^n)$ and that $\phi u = u$ where $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 in a neighbourhood of the unit ball. Thus the Fourier transform satifies

$$\hat{u} = \hat{\phi} * \hat{u} \Longrightarrow \hat{u} \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{4.13}$$

In fact this is true with uniformity. That is, one can bound any derivative of $\hat{u}$ on a compact set by the norm

$$\sup_{|z|\le R} |D_j \hat{u}| + \max_j \sup_{|z|\le R} |D_j \hat{u}| \le C(R)\|u\|_{H^s} \tag{4.14}$$

where the constant does not depend on u. By the Ascoli-Arzela theorem, this implies that for each R the sequence $\hat{u}_n$ has a convergent subsequence in $\mathcal{C}(\{|\zeta| \le R\})$. Now, by diagonalization we can extract a subsequence which converges in $\mathcal{V}_c(\{|\zeta| \le R\})$ for every R. This implies that the restriction to $\{|\zeta| \le R\}$ converges in the weighted L^2 norm corresponding to H^t, i.e. that $(1+|\zeta|^2)^{t/2}\chi_R \hat{u}_{n_j} \to (1+|\zeta|^2)^{t/2}\chi_R \hat{v}$ in L^2 where χ_R is the characteristic function of the ball of radius R. However the boundedness of u_n in H^s strengthens this to

$$(1+|\zeta|^2)^{t/2}\hat{u}_{n_j} \to (1+|\zeta|^2)^{t/2}\hat{v} \text{ in } L^2(\mathbb{R}^n).$$

Namely, the sequence is Cauchy in $L^(\mathbb{R}^n)$ and hence convergnet. To see this, just note that for $\epsilon > 0$ one can first choose R so large that the norm outside the ball is

$$\int_{|\zeta|\ge R} (1+|\zeta|^2)^t |u_n|^2 d\zeta \le (1+R^2)^{\frac{s-t}{2}} \int_{|\zeta|\ge R} (1+|\zeta|^2)^s |u_n|^2 d\zeta \le C(1+R^2)^{\frac{s-t}{2}} < \epsilon/2 \tag{4.15}$$

where C is the bound on the norm in H^s. Now, having chosen R, the subsequence converges in $|\zeta| \le R$. This proves the compactness. □

Once we have this local result we easily deduce the global result.

Proposition 11. *On a compact manifold the inclusion $H^s(M) \hookrightarrow H^t(M)$, for any $s > t$, is compact.*

Proof. If $\phi_i \in \mathcal{C}_c^\infty(U_i)$ is a partition of unity subordinate to an open cover of M by coordinate patches $g_i : U_i \longrightarrow U_i' \subset \mathbb{R}^n$, then

$$u \in H^s(M) \Longrightarrow (g_i^{-1})^* \phi_i u \in H^s(\mathbb{R}^n),\ \operatorname{supp}((g_i^{-1})^*\phi_i u) \Subset U_i'. \tag{4.16}$$

Thus if u_n is a bounded sequence in $H^s(M)$ then the $(g_i^{-1})^*\phi_i u_n$ form a bounded sequence in $H^s(\mathbb{R}^n)$ with fixed compact supports. It follows from Lemma 14 that we may choose a subsequence so that each $\phi_i u_{n_j}$ converges in $H^t(\mathbb{R}^n)$. Hence the subsequence u_{n_j} converges in $H^t(M)$. □

3. Elliptic operators are Fredholm

If V_1, V_2 are two vector spaces then a linear operator $P : V_1 \to V_2$ is said to be *Fredholm* if these are finite-dimensional subspaces $N_1 \subset V_1$, $N_2 \subset V_2$ such that

$$\begin{gathered} \{v \in V_1;\ Pv = 0\} \subset N_1 \\ \{w \in V_2;\ \exists\ v \in V_1,\ Pv = w\} + N_2 = V_2. \end{gathered} \tag{4.17}$$

The first condition just says that the null space is finite-dimensional and the second that the range has a finite-dimensional complement – by shrinking N_1 and N_2 if necessary we may arrange that the inclusion in (4.17) is an equality and that the sum is direct.

THEOREM 4. *For any elliptic operator, $P \in \mathrm{Diff}^m(M;\mathbb{E})$, acting between sections of vector bundles over a compact manifold,*

$$P : H^{s+m}(M; E_1) \longrightarrow H^s(M; E_2)$$
$$\text{and } P : \mathcal{C}^\infty(M; E_1) \longrightarrow \mathcal{C}^\infty(M; E_2)$$

are Fredholm for all $s \in \mathbb{R}$.

The result for the $\mathcal{C}^\infty$ spaces follows from the result for Sobolev spaces. To prove this, consider the notion of a Fredholm operator between Hilbert spaces,

$$P : H_1 \longrightarrow H_2. \tag{4.18}$$

In this case we can unwind the conditions (4.17) which are then equivalent to the three conditions

$$\begin{gathered} \mathrm{Nul}(P) \subset H_1 \text{ is finite-dimensional.} \\ \mathrm{Ran}(P) \subset H_2 \text{ is closed.} \\ \mathrm{Ran}(P))^\perp \subset H_2 \text{ is finite-dimensional.} \end{gathered} \tag{4.19}$$

Note that *any* subspace of a Hilbert space with a finite-dimensional complement is closed so (4.19) does follow from (4.17). On the other hand the ortho-complement of a subspace is the same as the ortho-complement of its closure so the first and the third conditions in (4.19) do *not* suffice to prove (4.17), in general. For instance the range of an operator can be dense but not closed.

The main lemma we need, given the global elliptic estimates, is a standard one:-

LEMMA 15. *If $R : H \longrightarrow H$ is a compact operator on a Hilbert space then $\mathrm{Id} - R$ is Fredholm.*

PROOF. A compact operator is one which maps the unit ball (and hence any bounded subset) of H into a precompact set, a set with compact closure. The unit ball in the null space of $\mathrm{Id} - R$ is

$$\{u \in H;\ \|u\| = 1,\ u = Ru\} \subset R\{u \in H;\ \|u\| = 1\}$$

and is therefore precompact. Since is it closed, it is compact and any Hilbert space with a compact unit ball is finite-dimensional. Thus the null space of $\mathrm{Id} - R$ is finite-dimensional.

Consider a sequence $u_n = v_n - Rv_n$ in the range of $\mathrm{Id} - R$ and suppose $u_n \to u$ in H; we need to show that u is in the range of $\mathrm{Id} - R$. We may assume $u \neq 0$, since 0 is in the range, and by passing to a subsequence suppose that $\|u_n\| \neq 0$; $\|u_n\| \to \|u\| \neq 0$ by assumption. Now consider $w_n = v_n/\|v_n\|$. Since $\|u_n\| \neq 0$, $\inf_n \|v_n\| \neq 0$, since other wise there is a subsequence converging to 0, and so w_n is well-defined and of norm 1. Since $w_n = Rw_n + u_n/\|v_n\|$ and $\|v_n\|$ is bounded below, w_n must have a convergence subsequence, by the compactness of R. Passing to such a subsequence, and relabelling, $w_n \to w$, $u_n \to u$, $u_n/\|v_n\| \to cu$, $c \in \mathbb{C}$. If $c = 0$ then $(\mathrm{Id} - R)w = 0$. However, we can assume in the first place that $u_n \perp \mathrm{Nul}(\mathrm{Id} - R)$, so the same is true of w_n. As $\|w\| = 1$ this is a contradiction, so $\|v_n\|$ is bounded above, $c \neq 0$, and hence there is a solution to $(\mathrm{Id} - R)w = u$. Thus the range of $\mathrm{Id} - R$ is closed.

The ortho-complement of the range $\mathrm{Ran}(\mathrm{Id} - R)^{\perp}$ is the null space at $\mathrm{Id} - R^*$ which is also finite-dimensional since R^* is compact. Thus $\mathrm{Id} - R$ is Fredholm. $\square$

PROPOSITION 12. *Any smoothing operator on a compact manifold is compact as an operator between (any) Sobolev spaces.*

PROOF. By definition a smoothing operator is one with a smooth kernel. For vector bundles this can be expressed in terms of local coordinates and a partition of unity with trivialization of the bundles over the supports as follows.

$$
\begin{gathered}
Ru = \sum_{a,b} \varphi_b R \varphi_a u \\
\varphi_b R \varphi_a u = F_b^* \varphi_b' R_{ab} \varphi_a' (F_a^{-1})^* u \qquad (4.20) \\
R_{ab} v(z) = \int_{\Omega_a'} R_{ab}(z, z') v(z'),\ z \in \Omega_b',\ v \in \mathcal{C}_c^\infty(\Omega_a'; E_1)
\end{gathered}
$$

where R_{ab} is a matrix of smooth sections of the localized (hence trivial by refinement) bundle on $\Omega_b' \times \Omega_a$. In fact, by inserting extra cutoffs in (4.20), we may assume that R_{ab} has compact support in $\Omega_b' \times \Omega_a'$. Thus, by the compactness of sums of compact operators, it suffices to show that a single smoothing operator of compact support compact support is compact on the standard Sobolev spaces. Thus if $R \in \mathcal{C}_c^\infty(\mathbb{R}^{2n}$

$$
H^{L'}(\mathbb{R}^n) \ni u \mapsto \int_{\mathbb{R}^n} R(z) \in H^L(\mathbb{R}^n) \qquad (4.21)
$$

is compact for any L, L'. By the continuous inclusion of Sobolev spaces it suffices to take $L' = -L$ with L a large even integer. Then $(\Delta + 1)^{L/2}$ is an isomorphism from $(L^2(\mathbb{R}^n))$ to $H^{-L}(\mathbb{R}^2)$ and from $H^L(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Thus the compactness of (4.21) is equivalent to the compactness of

$$
(\Delta + 1)^{L/2} R (\Delta + 1)^{L/2} \text{ on } L^2(\mathbb{R}^n). \qquad (4.22)
$$

This is still a smoothing operator with compactly supported kernel, then we are reduced to the special case of (4.21) for $L = L' = 0$. Finally then it suffices to use Sturm's theorem, that R is uniformly approximated by polynomials on a large ball. Cutting off on left and right then shows that

$$
\rho(z) R_i(z, z') \rho(z') \to Rz, z') \text{ uniformly on } \mathbb{R}^{2n}
$$

the R_i is a polynomial (and $\rho(z)\rho(z') = 1$ on $\mathrm{supp}(R)$) with $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. The uniform convergence of the kernels implies the convergence of the operators on $L^2(\mathbb{R}^n)$ in the norm topology, so R is in the norm closure of the finite rank operators on $L^2(\mathbb{R}^n)$, hence is compact. $\square$

PROOF OF THEOREM 4. We know that P has a 2-sided parameterix $Q : H^s(M; E_2) \longrightarrow H^{s+m}(M; E_1)$ (for any s) such that

$$
PQ - \mathrm{Id}_2 = R_2,\ QP - \mathrm{Id}_2 = R_1,
$$

are both smoothing (or at *least* C^N for arbitrarily large N) operators. Then we can apply Proposition 12 and Lemma 15. First

$$
QP = \mathrm{Id} - R_1 : H^{s+m}(M; E_1) \longrightarrow H^{s+m}(M; E_2)
$$

have finite-dimensional null spaces. However, the null space of P is certainly contained in the null space of $\mathrm{Id}-R$, so it too is finite-dimensional. Similarly,

$$PQ = \mathrm{Id}-R_1 : H^s(M;E_2) \longrightarrow H^s(M;E_2)$$

has closed range of finite codimension. But the range of P certainly *contains* the range of $\mathrm{Id}-R$ so it too must be closed and of finite codimension. Thus P is Fredholm as an operator from $H^{s+m}(M;E_2)$ to $H^s(M;E_2)$ for any $s \in \mathbb{R}$.

So consider P as an operator on the $\mathcal{C}^\infty$ spaces. The null space of $P : H^m(M;E_1) \longrightarrow H^0(M;E_2)$ consists of $\mathcal{C}^\infty$ sections, by elliptic regularity, so must be equal to the null space on $\mathcal{C}^\infty(M;E_1)$ — which is therefore finite-dimensional. Similarly consider the range of $P : H^m(M;E_1) \longrightarrow H^0(M;E_2)$. We know this to have a finite-dimensional complement, with basis $v_1, \dots, v_n \in H^0(M;E_2)$. By the density of $\mathcal{C}^\infty(M;E_2)$ in $L^2(M;E_2)$ we can approximate the v_i's closely by $w_i \in \mathcal{C}^\infty(M;E_2)$. On close enough approximation, the w_i must span the complement. Thus $PH^m(M;E_1)$ has a complement in $L^2(M;E_2)$ which is a finite-dimensional subspace of $\mathcal{C}^\infty(M;E_2)$; call this N_2. If $f \in \mathcal{C}^\infty(M;E_2) \subset L^2(M;E_2)$ then there are constants c_i such that

$$f - \sum_{i=1}^{N} c_i w_i = Pu,\ u \in H^m(M;E_1).$$

Again by elliptic regularity, $u \in \mathcal{C}^\infty(M;E_1)$ thus N_2 is a complement to $P\mathcal{C}^\infty(M;E_1)$ in $\mathcal{C}^\infty(M;E_2)$ and P is Fredholm. $\square$

The point of Fredholm operators is that they are 'almost invertible' — in the sense that they are invertible up to finite-dimensional obstructions. However, a Fredholm operator may not itself be *close* to an invertible operator. This defect is measured by the index

$$\begin{gathered}\mathrm{ind}(P) = \dim \mathrm{Nul}(P) - \dim(\mathrm{Ran}(P)^\perp)\\ P : H^m(M;E_1) \longrightarrow L^2(M;E_2).\end{gathered}$$

4. Generalized inverses

Written, at least in part, by Chris Kottke.

As discussed above, a bounded operator between Hilbert spaces,

$$T : H_1 \longrightarrow H_2$$

is Fredholm if and only if it has a parametrix up to compact errors, that is, there exists an operator

$$S : H_2 \longrightarrow H_1$$

such that

$$TS - \mathrm{Id}_2 = R_2,\ ST - \mathrm{Id}_1 = R_1$$

are both compact on the respective Hilbert spaces H_1 and H_2. In this case of Hilbert spaces there is a "preferred" parametrix or generalized inverse.

Recall that the adjoint

$$T^* : H_2 \longrightarrow H_1$$

of any bounded operator is defined using the Riesz Representation Theorem. Thus, by the continuity of T, for any $u \in H_2$,

$$H_1 \ni \phi \longrightarrow \langle T\phi, u\rangle \in \mathbb{C}$$

is continuous and so there exists a unique $v \in H_1$ such that

$$\langle T\phi, u\rangle_2 = \langle \phi, v\rangle_1,\ \forall\ \phi \in H_1.$$

Thus v is determined by u and the resulting map

$$H_2 \ni u \mapsto v = T^* u \in H_1$$

is easily seen to be continuous giving the adjoint identity

$$\langle T\phi, u\rangle = \langle \phi, T^* u\rangle,\ \forall\ \phi \in H_1,\ u \in H_2 \tag{4.23}$$

In particular it is always the case that

$$\operatorname{Nul}(T^*) = (\operatorname{Ran}(T))^{\perp} \tag{4.24}$$

as follows directly from (4.23). As a useful consequence, if $\operatorname{Ran}(T)$ is closed, then $H_2 = \operatorname{Ran}(T) \oplus \operatorname{Nul}(T^*)$ is an orthogonal direct sum.

PROPOSITION 13. *If $T : H_1 \longrightarrow H_2$ is a Fredholm operator between Hilbert spaces then T^* is also Fredholm,* $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$*, and T has a unique generalized inverse $S : H_2 \longrightarrow H_1$ satisfying*

$$TS = \operatorname{Id}_2 - \Pi_{\operatorname{Nul}(P^*)},\ ST = \operatorname{Id}_1 - \Pi_{\operatorname{Nul}(P)} \tag{4.25}$$

PROOF. A straightforward exercise, but it should probably be written out! $\square$

Notice that $\operatorname{ind}(T)$ is the difference of the two non-negative integers $\dim \operatorname{Nul}(T)$ and $\dim \operatorname{Nul}(T^*)$. Thus

$$\dim \operatorname{Nul}(T) \geq \operatorname{ind}(T) \tag{4.26}$$

$$\dim \operatorname{Nul}(T^*) \geq -\operatorname{ind}(T) \tag{4.27}$$

so if $\operatorname{ind}(T) \neq 0$ then T is definitely *not* invertible. In fact it cannot then be made invertible by small bounded perturbations.

PROPOSITION 14. *If H_1 and H_2 are two seperable, infinite-dimensional Hilbert spaces then for all $k \in \mathbb{Z}$,*

$$\operatorname{Fr}_k = \{T : H_1 \longrightarrow H_2;\ T \textit{ is Fredholm and } \operatorname{ind}(T) = k\}$$

is a non-empty subset of $B(H_1, H_2)$, the Banach space of bounded operators from H_1 to H_2.

PROOF. All separable Hilbert spaces of infinite dimension are isomorphic, so Fr_0 is non-empty. More generally if $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of H_1, then the shift operator, determined by

$$S_k e_i = \begin{cases} e_{i+k}, & i \geq 1,\ k \geq 0 \\ e_{i+k}, & i \geq -k,\ k \leq 0 \\ 0, & i < -k \end{cases}$$

is easily seen to be Fredholm of index k in H_1. Composing with an isomorphism to H_2 shows that $\operatorname{Fr}_k \neq \emptyset$ for all $k \in \mathbb{Z}$. $\square$

One important property of the spaces $\operatorname{Fr}_k(H_1, H_2)$ is that they are stable under compact perturbations; that is, if $K : H_1 \longrightarrow H_2$ is a compact operator and $T \in \operatorname{Fr}_k$ then $(T + K) \in \operatorname{Fr}_k$. That $(T + K)$ is Fredholm is clear, sinces a parametrix for T is a parametrix for $T + K$, but it remains to show that the index itself is stable and we do this in steps. In what follows, take $T \in \mathrm{Fr}_k(H_1, H_2)$ with kernel $N_1 \subset H_1$. Define $\tilde{T}$ by the factorization

$$T : H_1 \longrightarrow \tilde{H}_1 = H_1/N_1 \xrightarrow{\tilde{T}} \operatorname{Ran} T \hookrightarrow H_2, \tag{4.28}$$

so that $\tilde{T}$ is invertible.

Lemma 16. *Suppose $T \in \mathrm{Fr}_k(H_1, H_2)$ has kernel $N_1 \subset H_1$ and $M_1 \supset N_1$ is a finite dimensional subspace of H_1 then defining $T' = T$ on $M_1^\perp$ and $T' = 0$ on M_1 gives an element $T' \in \mathrm{Fr}_k$.*

Proof. Since $N_1 \subset M_1$, T' is obtained from (4.28) by replacing $\tilde{T}$ by $\tilde{T}'$ which is defined in essentially the same way as T', that is $\tilde{T}' = 0$ on M_1/N_1, and $\tilde{T}' = \tilde{T}$ on the orthocomplement. Thus the range of $\tilde{T}'$ in $\operatorname{Ran}(T)$ has complement $\tilde{T}(M_1/N_1)$ which has the same dimension as M_1/N_1. Thus T' has null space M_1 and has range in H_2 with complement of dimension that of $M_1/N_1 + N_2$, and hence has index k. $\square$

Lemma 17. *If A is a finite rank operator $A : H_1 \longrightarrow H_2$ such that $\operatorname{Ran} A \cap \operatorname{Ran} T = \{0\}$, then $T + A \in \mathrm{Fr}_k$.*

Proof. First note that $\operatorname{Nul}(T + A) = \operatorname{Nul} T \cap \operatorname{Nul} A$ since

$$x \in \operatorname{Nul}(T + A) \Leftrightarrow Tx = -Ax \in \operatorname{Ran} T \cap \operatorname{Ran} A = \{0\} \Leftrightarrow x \in \operatorname{Nul} T \cap \operatorname{Nul} A.$$

Similarly the range of $T + A$ restricted to $\operatorname{Nul} T$ meets the range of $T + A$ restricted to $(\operatorname{null} T)^\perp$ only in 0 so the codimension of the $\operatorname{Ran}(T + A)$ is the codimension of $\operatorname{Ran} A_N$ where A_N is A as a map from $\operatorname{Nul} T$ to $H_2/\operatorname{Ran} T$. So, the equality of row and column rank for matrices,

$$\operatorname{codim} \operatorname{Ran}(T+A) = \operatorname{codim} \operatorname{Ran} T - \dim \operatorname{Nul}(A_N) = \dim \operatorname{Nul}(T) - k - \dim \operatorname{Nul}(A_N) = \dim \operatorname{Nul}(T+A) - k.$$

Thus $T + A \in \mathrm{Fr}_k$. $\square$

Proposition 15. *If $A : H_1 \longrightarrow H_2$ is any finite rank operator, then $T + A \in \mathrm{Fr}_k$.*

Proof. Let $E_2 = \operatorname{Ran} A \cap \operatorname{Ran} T$, which is finite dimensional, then $E_1 = \tilde{T}^{-1}(E_2)$ has the same dimension. Put $M_1 = E_1 \oplus N_1$ and apply Lemma 16 to get $T' \in \mathrm{Fr}_k$ with kernel M_1. Then

$$T + A = T' + A' + A$$

where $A' = T$ on E_1 and $A' = 0$ on $E_1^\perp$. Then $A' + A$ is a finite rank operator and $\operatorname{Ran}(A' + A) \cap \operatorname{Ran} T' = \{0\}$ and Lemma 17 applies. Thus

$$T + A = T' + (A' + A) \in \mathrm{Fr}_k(H_1, H_2).$$

$\square$

Proposition 16. *If $B : H_1 \longrightarrow H_2$ is compact then $T + B \in \mathrm{Fr}_k$.*

Proof. A compact operator is the sum of a finite rank operator and an operator of arbitrarily small norm so it suffices to show that $T + C \in \mathrm{Fr}_k$ where $\|C\| < \epsilon$ for ϵ small enough and then apply Proposition 15. Let $P : H_1 \longrightarrow \tilde{H}_1 = H_1/N_1$ and $Q : H_2 \longrightarrow \operatorname{Ran} T$ be projection operators. Then

$$C = QCP + QC(\operatorname{Id} - P) + (\operatorname{Id} - Q)CP + (\operatorname{Id} - Q)C(\operatorname{Id} - P)$$

the last three of which are finite rank operators. Thus it suffices to show that

$$\tilde{T} + QC : \tilde{H}_1 \longrightarrow \operatorname{Ran} T$$

is invertible. The set of invertible operators is open, by the convergence of the Neumann series so the result follows. □

REMARK 1. In fact the Fr_k are all *connected* although I will not use this below. In fact this follows from the multiplicativity of the index:-

$$\mathrm{Fr}_k \circ \mathrm{Fr}_l = \mathrm{Fr}_{k+l} \tag{4.29}$$

and the connectedness of the group of invertible operators on a Hilbert space. The topological type of the Fr_k is actually a point of some importance. A fact, which you should know but I am not going to prove here is:-

THEOREM 5. *The open set* $\mathrm{Fr} = \bigcup_k \mathrm{Fr}_k$ *in the Banach space of bounded operators on a separable Hilbert space is a* classifying space *for even K-theory.*

That is, if X is a reasonable space – for instance a compact manifold – then the space of homotopy classes of continuous maps into Fr may be canonically identified as an Abelian group with the (complex) K-theory of X :

$$\mathrm{K}^0(X) = [X; \mathrm{Fr}]. \tag{4.30}$$

5. Self-adjoint elliptic operators

Last time I showed that elliptic differential operators, acting on functions on a compact manifold, are Fredholm on Sobolev spaces. Today I will first quickly discuss the rudiments of spectral theory for self-adjoint elliptic operators and then pass over to the general case of operators between sections of vector bundles (which is really only notationally different from the case of operators on functions).

To define self-adjointness of an operator we need to define the adjoint! To do so requires invariant integration. I have already talked about this a little, but recall from 18.155 (I hope) Riesz' theorem identifying (appropriately behaved, i.e. Borel outer continuous and inner regular) measures on a locally compact space with continuous linear functionals on $\mathcal{C}^0_0(M)$ (the space of continuous functions 'vanishing at infinity'). In the case of a manifold we define a smooth positive measure, also called a positive density, as one given in local coordinates by a smooth positive multiple of the Lebesgue measure. The existence of such a density is guaranteed by the existence of a partition of unity subordinate to a coordinate cover, since the we can take

$$\nu = \sum_j \phi_j f_j^* |dz| \tag{4.31}$$

where $|dz|$ is Lebesgue measure in the local coordinate patch corresponding to $f_j : U_j \longrightarrow U'_j$. Since we know that a smooth coordinate transforms $|dz|$ to a positive smooth multiple of the new Lebesque measure (namely the absolute value of the Jacobian) and two such positive smooth measures are related by

$$\nu' = \mu\nu,\ 0 < \mu \in \mathcal{C}^\infty(M). \tag{4.32}$$

In the case of a compact manifold this allows one to define integration of functions and hence an inner product on $L^2(M)$,

$$\langle u, v\rangle_\nu = \int_M u(z)\overline{v(z)}\nu. \tag{4.33}$$

It is with respect to such a choice of smooth density that adjoints are defined.

LEMMA 18. *If $P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is a differential operator with smooth coefficients and ν is a smooth positive measure then there exists a unque differential operator with smooth coefficients $P^* : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ such that*

$$\langle Pu, v\rangle_\nu = \langle u, P^*v\rangle_\nu \ \forall\ u,\ v \in \mathcal{C}^\infty(M). \tag{4.34}$$

PROOF. First existence. If ϕ_i is a partition of unity subordinate to an open cover of M by coordinate patches and $\phi_i' \in \mathcal{C}^\infty(M)$ have supports in the same coordinate patches, with $\phi_i' = 1$ in a neighbourhood of $\operatorname{supp}(\phi_i)$ then we know that

$$Pu = \sum_i \phi_i' P \phi_i u = \sum_i f_i^* P_i (f_i^{-1})^* u \tag{4.35}$$

where $f_i : U)i \longrightarrow U_i'$ are the coordinate charts and P_i is a differential operator on U_i' with smooth coefficients, all compactly supported in U_i'. The existence of P^* follows from the existence of $(\phi_i' P \phi_i)^*$ and hence P_i^* in each coordinate patch, where the P_i^* should satisfy

$$\int_{U_i'} (P_i) u' \overline{v'} \mu' dz = \int_{U_i'} u' \overline{P_i^* v'} \mu' dz,\ \forall\ u', v' \in \mathcal{C}^\infty(U_i'). \tag{4.36}$$

Here $\nu = \mu' |dz|$ with $0 < \mu' \in \mathcal{C}^\infty(U_i')$ in the local coordinates. So in fact P_i^* is unique and given by

$$P_i^*(z, D)v' = \sum_{|\alpha| \le m} (\mu')^{-1} D^\alpha \overline{p_\alpha(z)} \mu' v' \text{ if } P_i = \sum_{|\alpha| \le m} p_\alpha(z) D^\alpha. \tag{4.37}$$

The uniqueness of P^* follows from (4.34) since the difference of two would be an operator $Q : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ satisfying

$$\langle u, Qv\rangle_\nu = 0 \ \forall\ u,\ v \in \mathcal{C}^\infty(M) \tag{4.38}$$

and this implies that $Q = 0$ as an operator. □

PROPOSITION 17. *If $P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is an elliptic differential operator of order $m > 0$ which is (formally) self-adjoint with respect to some smooth positive density then*

$$\operatorname{spec}(P) = \{\lambda \in \mathbb{C}; (P - \lambda) : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M) \text{ is not an isomorphism}\} \tag{4.39}$$

is a discrete subset of $\mathbb{R}$, for each $\lambda \in \operatorname{spec}(P)$

$$E(\lambda) = \{u \in \mathcal{C}^\infty(M); Pu = \lambda u\} \tag{4.40}$$

is finite dimensional and

$$L^2(M) = \sum_{\lambda \in \operatorname{spec}(P)} E(\lambda) \text{ is orthogonal.} \tag{4.41}$$

Formal self-adjointness just means that $P^* = P$ as differential operators acting on $\mathcal{C}^\infty(M)$. Actual self-adjointness means a little more but this follows easily from formal self-adjointness and ellipticity.

PROOF. First notice that $\operatorname{spec}(P) \subset \mathbb{R}$ since if $Pu = \lambda u$ with $u \in \mathcal{C}^\infty(M)$ then

$$\lambda \|u\|\nu^2 = \langle Pu, u\rangle = \langle u, Pu\rangle = \bar{\lambda} \|u\|\nu^2 \tag{4.42}$$

so $\lambda \notin \mathbb{R}$ implies that the null space of $P-\lambda$ is trivial. Since we know that the range is closed and has complement the null space of $(P-\lambda)^* = P - \bar{\lambda}$ it follows that $P-\lambda$ is an isomorphism on $\mathcal{C}^\infty(M)$ if $\lambda \notin \mathbb{R}$.

If $\lambda \in \mathbb{R}$ then we also know that $E(\lambda)$ is finite dimensional. For any $\lambda \in \mathbb{R}$ suppose that $(P-\lambda)u = 0$ with $u \in \mathcal{C}^\infty(M)$. Then we know that $P-\lambda$ is an isomorphism from $E(\lambda)^\perp$ to itself which extends by continuity to an isomorphism from the closure of $E^\perp(\lambda)$ in $H^m(M)$ to $E^\perp(\lambda) \subset L^2(M)$. It follows that $P-\lambda'$ defines such an isomorphism for $|\lambda = l'| < \epsilon$ for some $\epsilon > 0$. However acting on $E(\lambda)$, $P-\lambda' = (\lambda-\lambda')$ is also an isomorphism for $\lambda' \neq \lambda$ so $P-\lambda'$ is an isomorphism. This shows that $E(\lambda') = \{0\}$ for $|\lambda'-\lambda| < \epsilon$.

This leaves the completeness statement, (4.41). In fact this really amounts to the existence of a non-zero eigenvalue as we shall see. Consider the generalized inverse of P acting on $L^2(M)$. It maps the orthocomplement of the null space to itself and is a compact operator, as follows from the a priori estimats for P and the compactness of the embedding of $H^m(M)$ in $L^2(M)$ for $m > 0$. Futhermore it is self-adjoint. A standard result shows that a compact self-adjoint operator either has a non-zero eigenvalue or is itself zero. For the completeness it is enough to show that the generalized inverse maps the orthocomplement of the span of the $E(\lambda)$ in $L^2(M)$ into itself and is compact. It is therefore either zero or has a non-zero eigenvalue. Any corresponding eigenfunction would be an eigenfunction of P and hence in one of the $E(\lambda)$ so this operator must be zero, meaning that (4.41) holds. □

For single differential operators we first considered constant coefficient operators, then extended this to variable coefficient operators by a combination of perturbation (to get the a priori estimates) and construction of parametrices (to get approximation) and finally used coordinate invariance to transfer the discussion to a (compact) manifold. If we consider matrices of operators we can follow the same path, so I shall only comment on the changes needed.

A $k \times l$ matrix of differential operators (so with k rows and l columns) maps l-vectors of smooth functions to k vectors:

$$P_{ij}(D) = \sum_{|\alpha| \le m} c_{\alpha,i,j} D^\alpha,\ (P(D)u)_i(z) = \sum_j P_{ij}(D)u_j(z). \tag{4.43}$$

The matrix $P_{ij}(\zeta)$ is invertible if and only if $k = l$ and the polynomial of order mk, $\det P(\zeta) \neq 0$. Such a matrix is said to be elliptic if $\det P(\zeta)$ is elliptic. The cofactor matrix defines a matrix P' of differential operators of order $(k-1)m$ and we may construct a parametrix for P (assuming it to be elliptic) from a parametrix for $\det P$:

$$Q_P = Q_{\det P} P'(D). \tag{4.44}$$

It is then easy to see that it has the same mapping properties as in the case of a single operator (although notice that the product is no longer commutative because of the non-commutativity of matrix multiplication)

$$Q_P P = \operatorname{Id} - R_L,\ P Q_P = \operatorname{Id} - R_R \tag{4.45}$$

where R_L and R_R are given by matrices of convolution operators with all elements being Schwartz functions. For the action on vector-valued functions on an open subset of $\mathbb{R}^n$ we may proceed exactly as before, cutting off the kernel of Q_P with a properly supported function which is 1 near the diagonal

$$Q_\Omega f(z) = \int_\Omega q(z - z')\chi(z, z')f(z')dz'. \tag{4.46}$$

The regularity estimates look exactly the same as before if we define the local Sobolev spaces to be simply the direct sum of k copies of the usual local Sobolev spaces

$$Pu = f \in H^s_{\text{loc}}(\Omega) \Longrightarrow \|\psi u\|_{s+m} \le C\|\psi P(D)u\|_s + C'\|\phi u\|_{m-1} \text{ or } \|\psi u\|_{s+m} \le C\|\phi P(D)u\|_s + C''\|\phi u\|_M \tag{4.47}$$

where ψ, $\phi \in \mathcal{C}^\infty_c(\Omega)$ and $\phi = 1$ in a neighbourhood of ψ (and in the second case C'' depends on M.

Now, the variable case proceed again as before, where now we are considering a $k \times k$ matrix of differential operators of order m. I will not go into the details. A priori estimates in the first form in (4.47), for functions ψ with small support near a point, follow by perturbation from the constant coefficient case and then in the second form by use of a partition of unity. The existence of a parametrix for the variable coefficient matrix of operators also goes through without problems – the commutativity which disappears in the matrix case was not used anyway.

As regards coordinate transformations, we get the same results as before. It is also notural to allow transformations by variable coefficient matrices. Thus if $G_i(z) \in \mathcal{C}^\infty(\Omega; \mathrm{GL}(k, \mathbb{C})$ $i = 1, 2$, are smooth family of invertible matrices we may consider the composites PG_2 or $G_1^{-1}P$, or more usually the 'conjugate' operator

$$G_1^{-1}P(z, D)G)2 = P'(z, D). \tag{4.48}$$

This is also a variable coefficient differential operator, elliptic if and only if $P(z, D)$ is elliptic. The Sobolev spaces $H^s_{\text{loc}}(\Omega; \mathbb{R}^k)$ are invariant under composition with such matrices, since they are the same in each variable.

Combining coordinate transformations and such matrix conjugation allows us to consider not only manifolds but also vector bundles over manifolds. Let me briefly remind you of what this is about. Over an open subset $\Omega \subset \mathbb{R}^n$ one can introduce a vector bundle as just a subbundle of some trivial N-dimensional bundle. That is, consider a smooth $N \times N$ matrix $\Pi \in \mathcal{C}^\infty(\Omega; M(N, \mathbb{C}))$ on Ω which is valued in the projections (i.e. idempotents) meaning that $\Pi(z)\Pi(z) = \Pi(z)$ for all $z \in \Omega$. Then the range of $\Pi(z)$ defines a linear subspace of $\mathbb{C}^N$ for each $z \in \Omega$ and together these form a vector bundle over Ω. Namely these spaces fit together to define a manifold of dimension $n+k$ where k is the rank of $\Pi(z)$ (constant if Ω is connected, otherwise require it be the same on all components)

$$E_\Omega = \bigcup_{z \in \Omega} E_z,\ E_z = \Pi(z)\mathbb{C}^N. \tag{4.49}$$

If $\bar{z} \in \Omega$ then we may choose a basis of $E_{\bar{z}}$ and so identify it with $\mathbb{C}^k$. By the smoothness of $\Pi(z)$ in z it follows that in some small ball $B(\bar{z}, r)$, so that $\|\Pi(z)(\Pi(z) - \Pi(\bar{z}))\Pi(z)\| < \frac{1}{2}$) the map

$$E_{B(\bar{z},r)} = \bigcup_{z \in B(\bar{z},r)} E_z,\ E_z = \Pi(z)\mathbb{C}^N \ni (z, u) \longmapsto (z, E(\bar{z})u) \in B(\bar{z}, r) \times E_{\bar{z}} \simeq B(\bar{z}, r) \times \mathbb{C}^k \tag{4.50}$$

is an isomorphism. Injectivity is just injectivity of each of the maps $E_z \longrightarrow E_{\bar{z}}$ and this follows from the fact that $\Pi(z)\Pi(\bar{z})\Pi(z)$ is invertible on E_z; this also implies surjectivity.

6. Index theorem

Addenda to Chapter 4

CHAPTER 5

Suspended families and the resolvent

$\boxed{\boxed{\text{0.6Q; Revised: 6-8-2007; Run: February 7, 2008}}}$

For a compact manifold, M, the Sobolev spaces $H^s(M;E)$ (of sections of a vector bundle E) are defined above by reference to local coordinates and local trivializations of E. If M is not compact (but is paracompact, as is demanded by the definition of a manifold) the same sort of definition leads either to the spaces of sections with compact support, or the "local" spaces:

$$H^s_{\mathrm{c}}(M;E) \subset H^s_{\mathrm{loc}}(M;E),\ s \in \mathbb{R}. \tag{5.1}$$

Thus, if $F_a : \Omega_a \longrightarrow \Omega'_a$ is a covering of M, for $a \in A$, by coordinate patches over which E is trivial, $T_a : (F_a^{-1})^* E \cong \mathbb{C}^N$, and $\{\rho_a\}$ is a partition of unity subordinate to this cover then

$$\mu \in H^s_{\mathrm{loc}}(M;E) \Leftrightarrow T_a(F_a^{-1})^*(\rho_a\mu) \in H^s(\Omega'_a;\mathbb{C}^N)\ \forall\ a. \tag{5.2}$$

Practically, these spaces have serious limitations; for instance they are not Hilbert or even Banach spaaces. On the other hand they certainly have their uses and differential operators act on them in the usual way,

$$\begin{aligned} &P \in \mathrm{Diff}^m(M;\mathbb{E}) \Rightarrow \\ &\qquad P : H^{s+m}_{\mathrm{loc}}(M;E_+) \to H^s_{\mathrm{loc}}(M;E_-), \\ &\qquad P : H^{s+m}_{\mathrm{c}}(M;E_+) \to H^s_{\mathrm{c}}(M;E_-). \end{aligned} \tag{5.3}$$

However, without some limitations on the growth of elements, as is the case in $H^s_{\mathrm{loc}}(M;E)$, it is not reasonable to expect the null space of the first realization of P above to be finite dimensional. Similarly in the second case it is not reasonable to expect the operator to be even close to surjective.

1. Product with a line

Some corrections from Fang Wang added, 25 July, 2007.

Thus, for non-compact manifolds, we need to find intermediate spaces which represent some growth constraints on functions or distributions. Of course this is precisely what we have done for $\mathbb{R}^n$ in defining the weighted Sobolev spaces,

$$H^{s,t}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); \langle z\rangle^{-t} u \in H^s(\mathbb{R}^n)\right\}. \tag{5.4}$$

However, it turns out that even these spaces are not always what we want.

To lead up to the discussion of other spaces I will start with the simplest sort of non-compact space, the real line. To make things more interesting (and useful) I will conisider

$$X = \mathbb{R} \times M \tag{5.5}$$

where M is a compact manifold. The new Sobolev spaces defined for this product will combine the features of $H^s(\mathbb{R})$ and $H^s(M)$. The Sobolev spaces on $\mathbb{R}^n$ are associated with the translation action of $\mathbb{R}^n$ on itself, in the sense that this fixes the "uniformity" at infinity through the Fourier transform. What happens on X is quite similar.

First we can define "tempered distributions" on X. The space of Schwartz functions of rapid decay on X can be fixed in terms of differential operators on M and differentiation on $\mathbb{R}$.

(5.6)

$$\mathcal{S}(\mathbb{R}\times M)=\left\{u:\mathbb{R}\times M\longrightarrow\mathbb{C};\ \sup_{\mathbb{R}\times M}\left|t^lD_t^kPu(t,\cdot)\right|<\infty\ \forall\ l,\ k,\ P\in\operatorname{Diff}^*(M)\right\}.$$

Exercise 1. Define the corresponding space for sections of a vector bundle E over M lifted to X and then put a topology on $\mathcal{S}(\mathbb{R}\times M;E)$ corresponding to these estimates and check that it is a complete metric space, just like $\mathcal{S}(\mathbb{R})$ in Chapter 1.

There are several different ways to look at

$$\mathcal{S}(\mathbb{R}\times M)\subset\mathcal{C}^\infty(\mathbb{R}\times M).$$

Namely we can think of either $\mathbb{R}$ or M as "coming first" and see that

$$\mathcal{S}(\mathbb{R}\times M)=\mathcal{C}^\infty(M;\mathcal{S}(\mathbb{R}))=\mathcal{S}(\mathbb{R};\mathcal{C}^\infty(M)). \tag{5.7}$$

The notion of a $\mathcal{C}^\infty$ function on M with values in a topological vector space is easy to define, since $\mathcal{C}^0(M;\mathcal{S}(\mathbb{R}))$ is defined using the metric space topology on $\mathcal{S}(\mathbb{R})$. In a coordinate patch on M higher derivatives are defined in the usual way, using difference quotients and these definitions are coordinate-invariant. Similarly, continuity and differentiability for a map $\mathbb{R}\to\mathcal{C}^\infty(M)$ are easy to define and then

$$\mathcal{S}(\mathbb{R};\mathcal{C}^\infty(M))=\left\{u:\mathbb{R}\to\mathcal{C}^\infty(M);\ \sup_t\left\|t^kD_t^pu\right\|_{\mathcal{C}^l(M)}<\infty,\ \forall\ k,\ p,\ l\right\}. \tag{5.8}$$

Using such an interpretation of $\mathcal{S}(\mathbb{R}\times M)$, or directly, it follows easily that the 1-dimensional Fourier transform gives an isomorphism $\mathcal{F}:\mathcal{S}(\mathbb{R}\times M)\to\mathcal{S}(\mathbb{R}\times M)$ by

$$\mathcal{F}:u(t,\cdot)\longmapsto\hat{u}(\tau,\cdot)=\int_{\mathbb{R}}e^{-it\tau}u(t,\cdot)\,dt. \tag{5.9}$$

So, one might hope to use $\mathcal{F}$ to define Sobolev spaces on $\mathbb{R}\times M$ with uniform behavior as $t\to\infty$ in $\mathbb{R}$. However this is not so straightforward, although I will come back to it, since the 1-dimensional Fourier transform in (5.9) does *nothing* in the variables in M. Instead let us think about $L^2(\mathbb{R}\times M)$, the definition of which requires a choice of measure.

Of course there is an obvious class of product measures on $\mathbb{R}\times M$, namely $dt\cdot\nu_M$, where ν_M is a positive smooth density on M and dt is Lebesgue measure on $\mathbb{R}$. This corresponds to the functional

$$\int:\mathcal{C}^0_{\mathrm{c}}(\mathbb{R}\times M)\ni u\longmapsto\int u(t,\cdot)\,dt\cdot\nu\in\mathbb{C}. \tag{5.10}$$

The analogues of (5.7) correspond to Fubini's Theorem.

(5.11)
$$L^2_{\mathrm{ti}}(\mathbb{R}\times M)=\left\{u:\mathbb{R}\times M\to\mathbb{C}\text{ measurable; }\int|u(t,z)|^2\,dt\,\nu_z<\infty\right\}/\sim\ \text{a.e.}$$
$$L^2_{\mathrm{ti}}(\mathbb{R}\times M)=L^2(\mathbb{R};L^2(M))=L^2(M;L^2(\mathbb{R})).$$

Here the subscript "ti" is supposed to denote translation-invariance (of the measure and hence the space).

We can now easily define the Sobolev spaces of positive integer order:

$$(5.12)\quad H^m_{\mathrm{ti}}(\mathbb{R}\times M)=\Big\{u\in L^2_{\mathrm{ti}}(\mathbb{R}\times M);\\ D^j_tP_ku\in L^2_{\mathrm{ti}}(\mathbb{R}\times M)\ \forall\,j\le m-k,\ 0\le k\le m,\ P_k\in\mathrm{Diff}^k(M)\Big\}.$$

In fact we can write them more succinctly by defining

(5.13)
$$\mathrm{Diff}^k_{\mathrm{ti}}(\mathbb{R}\times M)=\left\{Q\in\mathrm{Diff}^m(\mathbb{R}\times M);\ Q=\sum_{0\le j\le m}D^j_tP_j,\ P_j\in\mathrm{Diff}^{m-j}(M)\right\}.$$

This is the space of "t-translation-invariant" differential operators on $\mathbb{R}\times M$ and (5.12) reduces to

(5.14)
$$H^m_{\mathrm{ti}}(\mathbb{R}\times M)=\left\{u\in L^2_{\mathrm{ti}}(\mathbb{R}\times M);\ Pu\in L^2_{\mathrm{ti}}(\mathbb{R}\times M),\ \forall\ P\in\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M)\right\}.$$

I will discuss such operators in some detail below, especially the elliptic case. First, we need to consider the Sobolev spaces of non-integral order, for completeness sake if nothing else. To do this, observe that on $\mathbb{R}$ itself (so for $M=\{\mathrm{pt}\}$), $L^2_{\mathrm{ti}}(\mathbb{R}\times\{\mathrm{pt}\})=L^2(\mathbb{R})$ in the usual sense. Let us consider a special partition of unity on $\mathbb{R}$ consisting of integral translates of *one* function.

Definition 5. An element $\mu\in\mathcal{C}^\infty_c(\mathbb{R})$ generates a "**ti-partition of unity**" (a non-standard description) on $\mathbb{R}$ if $0\le\mu\le1$ and $\sum_{k\in\mathbb{Z}}\mu(t-k)=1$.

It is easy to construct such a μ. Just take $\mu_1\in\mathcal{C}^\infty_c(\mathbb{R})$, $\mu_1\ge0$ with $\mu_1(t)=1$ in $|t|\le1/2$. Then let
$$F(t)=\sum_{k\in\mathbb{Z}}\mu_1(t-k)\in\mathcal{C}^\infty(\mathbb{R})$$
since the sum is finite on each bounded set. Moreover $F(t)\ge1$ and is itself invariant under translation by any integer; set $\mu(t)=\mu_1(t)/F(t)$. Then μ generates a ti-partition of unity.

Using such a function we can easily decompose $L^2(\mathbb{R})$. Thus, setting $\tau_k(t)=t-k$,

$$(5.15)\qquad f\in L^2(\mathbb{R})\iff(\tau_k^*f)\mu\in L^2_{\mathrm{loc}}(\mathbb{R})\ \forall\,k\in\mathbb{Z}\text{ and }\sum_{k\in\mathbb{Z}}\int\left|\tau_k^*f\mu\right|^2\,dt<\infty.$$

Of course, saying $(\tau_k^*f)\mu\in L^2_{\mathrm{loc}}(\mathbb{R})$ is the same as $(\tau_k^*f)\mu\in L^2_{\mathrm{c}}(\mathbb{R})$. Certainly, if $f\in L^2(\mathbb{R})$ then $(\tau_k^*f)\mu\in L^2(\mathbb{R})$ and since $0\le\mu\le1$ and $\mathrm{supp}(\mu)\subset[-R,R]$ for some R,
$$\sum_k\int\left|(\tau_k^*f)\mu\right|^2\le C\int|f|^2\,dt.$$

Conversely, since $\sum_{|k|\le T}\mu = 1$ on $[-1,1]$ for some T, it follows that

$$\int |f|^2\,dt \le C'\sum_k \int \left|(\tau_k^* f)\mu\right|^2\,dt.$$

Now, $D_t\tau_k^* f = \tau_k^*(D_t f)$, so we can use (5.15) to rewrite the definition of the spaces $H^k_{\mathrm{ti}}(\mathbb{R}\times M)$ in a form that extends to *all* orders. Namely

$$u\in H^s_{\mathrm{ti}}(\mathbb{R}\times M) \iff (\tau_k^* u)\mu \in H^s_{\mathrm{c}}(\mathbb{R}\times M) \text{ and } \sum_k \|\tau_k^* u\|_{H^s} < \infty \tag{5.16}$$

provided we choose a fixed norm on $H^s_{\mathrm{c}}(\mathbb{R}\times M)$ giving the usual topology for functions supported in a fixed compact set, for example by embedding $[-T,T]$ in a torus $\mathbb{T}$ and then taking the norm on $H^s(\mathbb{T}\times M)$.

LEMMA 19. *With* $\mathrm{Diff}^m_{ti}(\mathbb{R}\times M)$ *defined by* (5.13) *and the translation-invariant Sobolev spaces by* (5.16),

$$\begin{aligned} &P\in \mathrm{Diff}^m_{ti}(\mathbb{R}\times M) \Longrightarrow \\ &\qquad P: H^{s+m}_{ti}(\mathbb{R}\times M) \longrightarrow H^s_{ti}(\mathbb{R}\times M)\ \forall\, s\in\mathbb{R}. \end{aligned} \tag{5.17}$$

PROOF. This is basically an exercise. Really we also need to check a little more carefully that the two definitions of $H^(_{\mathrm{ti}}\mathbb{R}\times M)$ for k a positive integer, are the same. In fact this is similar to the proof of (5.17) so is omitted. So, to prove (5.17) we will proceed by induction over m. For $m=0$ there is nothing to prove. Now observe that the translation-invariant of P means that $P\tau_k^* u = \tau_k^*(Pu)$ so

$$\begin{aligned} &u\in H^{s+m}_{\mathrm{ti}}(\mathbb{R}\times M) \Longrightarrow \\ &\qquad P(\tau_k^* u\mu) = \tau_k^*(Pu) + \sum_{m'<m} \tau_k^*(P_{m'}u)D_t^{m-m'}\mu,\ P_{m'}\in \mathrm{Diff}^{m'}_{\mathrm{ti}}(\mathbb{R}\times M). \end{aligned} \tag{5.18}$$

The left side is in $H^s_{\mathrm{ti}}(\mathbb{R}\times M)$, with the sum over k of the squares of the norms bounded, by the regularity of u. The same is easily seen to be true for the sum on the right by the inductive hypothesis, and hence for the first term on the right. This proves the mapping property (5.17) and continuity follows by the same argument or the closed graph theorem. □

We can, and shall, extend this in various ways. If $\mathbb{E} = (E_1, E_2)$ is a pair of vector bundles over M then it lifts to a pair of vector bundles over $\mathbb{R}\times M$, which we can again denote by $\mathbb{E}$. It is then straightforward to define $\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M);\mathbb{E})$ and the Sobolev spaces $H^s_{\mathrm{ti}}(\mathbb{R}\times M; E_i)$ and to check that (5.17) extends in the obvious way.

Then main question we want to understand is the *invertibility* of an operator such as P in (5.17). However, let me look first at these Sobolev spaces a little more carefully. As already noted we really have two definitions in the case of positive integral order. Thinking about these we can also make the following provisional definitions in terms of the 1-dimensional Fourier transform discussed above – where the '$\tilde{H}$' notation is only temporary since these will turn out to be the same as the spaces just considered.

For any compact manifold define

$$\tilde{H}^s_{\mathrm{ti}}(\mathbb{R}\times M) = \Big\{u\in L^2(\mathbb{R}\times M); \quad \|u\|_s^2 = \int_{\mathbb{R}}\left(\langle\tau\rangle^s|\hat{u}(\tau,\cdot)|^2_{L^2(M)} + \int_{\mathbb{R}}|\hat{u}(\tau,\cdot)|^2_{H^s(M)}\right)d\tau<\infty\Big\},\ s\ge 0 \tag{5.19}$$

$$\tilde{H}^s_{\mathrm{ti}}(\mathbb{R}\times M) = \big\{u\in\mathcal{S}'(\mathbb{R}\times M); u=u_1+u_2, \quad u_1\in L^2(\mathbb{R};H^s(M)),\ u_2\in L^2(M;H^s(\mathbb{R}))\big\},\ \|u\|_s^2=\inf\|u_1\|^2+\|u_2\|^2,\ s<0. \tag{5.20}$$

The following interpolation result for Sobolev norms on M should be back in Chapter 3.

Lemma 20. *If M is a compact manifold or $\mathbb{R}^n$ then for any $m_1\ge m_2\ge m_3$ and any R, the Sobolev norms are related by*

$$\|u\|_{m_2}\le C\left((1+R)^{m_2-m_1}\|u\|_{m_1}+(1+R)^{m_2-m_3}\|u\|_{m_3}\right). \tag{5.21}$$

Proof. On $\mathbb{R}^n$ this follows directly by dividing Fourier space in two pieces

$$\begin{aligned}\|u\|^2_{m_2} &= \int_{|\zeta|>R}\langle\zeta\rangle^{2m_2}|\hat{u}|d\zeta+\int_{|\zeta|\le R}\langle\zeta\rangle^{2m_2}|\hat{u}|d\zeta\\ &\le\langle R\rangle^{2(m_1-m_2)}\int_{|\zeta|>R}\langle\zeta\rangle^{2m_1}|\hat{u}|d\zeta+\langle R\rangle^{2(m_2-m_3)}\int_{|\zeta|\le R}\langle\zeta\rangle^{2m_3}|\hat{u}|d\zeta\\ &\le\langle R\rangle^{2(m_1-m_2)}\|u\|^2_{m_1}+\langle R\rangle^{2(m_2-m_3)}\|u\|^2_{m_3}.\end{aligned} \tag{5.22}$$

On a compact manifold we have defined the norms by using a partition ϕ_i of unity subordinate to a covering by coordinate patches $F_i: Y_i\longrightarrow U_i'$:

$$\|u\|^2_m=\sum_i\|(F_i)^*(\phi_i u)\|^2_m \tag{5.23}$$

where on the right we are using the Sobolev norms on $\mathbb{R}^n$. Thus, applying the estimates for Euclidean space to each term on the right we get the same estimate on any compact manifold. □

Corollary 1. *If $u\in\tilde{H}^s_{ti}(\mathbb{R}\times M)$, for $s>0$, then for any $0<t<s$*

$$\int_{\mathbb{R}}\langle\tau\rangle^{2t}\|\hat{u}(\tau,\cdot)\|^2_{H^{s-t}(M)}d\tau<\infty \tag{5.24}$$

which we can interpret as meaning '$u\in H^t(\mathbb{R};H^{s-t}(M))$ or $u\in H^{s-t}(M;H^s(\mathbb{R}))$.'

Proof. Apply the estimate to $\hat{u}(\tau,\cdot)\in H^s(M)$, with $R=|\tau|$, $m_1=s$ and $m_3=0$ and integrate over τ. □

Lemma 21. *The Sobolev spaces $\tilde{H}^s_{ti}(\mathbb{R}\times M)$ and $H^s_{ti}(\mathbb{R}\times M)$ are the same.*

Proof. □

Lemma 22. *For $0<s<1$ $u\in H^s_{ti}(\mathbb{R}\times M)$ if and only if $u\in L^2(\mathbb{R}\times M)$ and*

$$\int_{\mathbb{R}^2\times M}\frac{|u(t,z)-u(t',z)|^2}{|t-t'|^{2s+1}}dtdt'\nu+\int_{\mathbb{R}\times M^2}\frac{|u(t,z')-u(t,z)|^2}{\rho(z,z')^{s+\frac{n}{2}}}dt\nu(z)\nu(z')<\infty, \quad n=\dim M, \tag{5.25}$$

where $0 \le \rho \in \mathcal{C}^\infty(M^2)$ *vanishes exactly quadratically at* $\mathrm{Diag} \subset M^2$.

Proof. This follows as in the cases of $\mathbb{R}^n$ and a compact manifold discussed earlier since the second term in (5.25) gives (with the L^2 norm) a norm on $L^2(\mathbb{R}; H^s(M))$ and the first term gives a norm on $L^2(M; H^s(\mathbb{R}))$. □

Using these results we can see directly that the Sobolev spaces in (5.19) have the following 'obvious' property as in the cases of $\mathbb{R}^n$ and M.

Lemma 23. *Schwartz space* $\mathcal{S}(\mathbb{R}\times M) = \mathcal{C}^\infty(M; \mathcal{S}(\mathbb{R}))$ *is dense in each* $H^s_{ti}(\mathbb{R}\times M)$ *and the* L^2 *pairing extends by continuity to a jointly continuous non-degenerate pairing*

$$H^s_{ti}(\mathbb{R}\times M) \times H^{-s}_{ti}(\mathbb{R}\times M) \longrightarrow \mathbb{C} \tag{5.26}$$

which identifies $H^{-s}_{ti}(\mathbb{R}\times M)$ *with the dual of* $H^s_{ti}(\mathbb{R}\times M)$ *for any* $s \in \mathbb{R}$.

Proof. I leave the density as an exercise – use convolution in $\mathbb{R}$ and the density of $\mathcal{C}^\infty(M)$ in $H^s(M)$ (explicity, using a partition of unity on M and convolution on $\mathbb{R}^n$ to get density in each coordinate patch).

Then the existence and continuity of the pairing follows from the definitions and the corresponding pairings on $\mathbb{R}$ and M. We can assume that $s > 0$ in (5.26) (otherwise reverse the factors). Then if $u \in H^s_{\mathrm{ti}}(\mathbb{R}\times M)$ and $v = v_1 + v_2 \in H^{-s}_{\mathrm{ti}}(\mathbb{R}\times M)$ as in (5.20),

$$(u, v) = \int_{\mathbb{R}} (u(t,\cdot), u_1(t,\cdot))\, dt + \int_M (u(\cdot, z), v_2(\cdot, z))\, \nu_z \tag{5.27}$$

where the first pairing is the extension of the L^2 pairing to $H^s(M) \times H^{-s}(M)$ and in the second case to $H^s(\mathbb{R})\times H^{-s}(\mathbb{R})$. The continuity of the pairing follows directly from (5.27).

So, it remains only to show that the pairing is non-degenerate – so that

$$H^{-s}_{\mathrm{ti}}(\mathbb{R}\times M) \ni v \longmapsto \sup_{\|u\|_{H^s_{\mathrm{ti}}(\mathbb{R}\times M)}=1} |(u, v)| \tag{5.28}$$

is equivalent to the norm on $H^{-s}_{\mathrm{ti}}(\mathbb{R}\times M)$. We already know that this is bounded above by a multiple of the norm on H^{-s}_{ti} so we need the estimate the other way. To see this we just need to go back to Euclidean space. Take a partition of unity ψ_i with our usual ϕ_i on M subordinate to a coordinate cover and consider with $\phi_i = 1$ in a neighbourhood of the support of ψ_i. Then

$$(u, \psi_i v) = (\phi_i u, \psi_i v) \tag{5.29}$$

allows us to extend $\psi_i v$ to a continuous linear functional on $H^s(\mathbb{R}^n)$ by reference to the local coordinates and using the fact that for $s > 0$ $(F_i^{-1})^*(\phi_i u) \in H^s(\mathbb{R}^{n+1})$. This shows that the coordinate representative of $\psi_i v$ is a sum as desired and summing over i gives the desired bound. □

2. Translation-invariant Operators

Some corrections from Fang Wang added, 25 July, 2007.

Next I will characterize those operators $P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M; \mathbb{E})$ which give invertible maps (5.17), or rather in the case of a pair of vector bundles $\mathbb{E} = (E_1, E_2)$ over M :

$$P : H^{s+m}_{\mathrm{ti}}(\mathbb{R}\times M; E_1) \longrightarrow H^s_{\mathrm{ti}}(\mathbb{R}\times M; E_2),\ P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M; \mathbb{E}). \tag{5.30}$$

This is a generalization of the 1-dimensional case, $M = \{\mathrm{pt}\}$ which we have already discussed. In fact it will become clear how to generalize some parts of the discussion below to products $\mathbb{R}^n \times M$ as well, but the case of a 1-dimensional Euclidean factor is both easier and more fundamental.

As with the constant coefficient case, there is a basic dichotomy here. A t-translation-invariant differential operator as in (5.30) is Fredholm if and only if it is invertible. To find necessary and sufficient conditons for invertibility we will we use the 1-dimensional Fourier transform as in (5.9).

If

$$P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M); \mathbb{E}) \Longleftrightarrow P = \sum_{i=0}^{m} D_t^i P_i,\ P_i \in \mathrm{Diff}^{m-i}(M; \mathbb{E}) \tag{5.31}$$

then

$$P : \mathcal{S}(\mathbb{R} \times M; E_1) \longrightarrow \mathcal{S}(\mathbb{R} \times M; E_2)$$

and

$$\widehat{Pu}(\tau, \cdot) = \sum_{i=0}^{m} \tau^i P_i \hat{u}(\tau, \cdot) \tag{5.32}$$

where $\hat{u}(\tau, \cdot)$ is the 1-dimensional Fourier transform from (5.9). So we clearly need to examine the "suspended" family of operators

$$P(\tau) = \sum_{i=0}^{m} \tau^i P_i \in \mathcal{C}^\infty\left(\mathbb{C}; \mathrm{Diff}^m(M; \mathbb{E})\right). \tag{5.33}$$

I use the term "suspended" to denote the addition of a parameter to $\mathrm{Diff}^m(M; \mathbb{E})$ to get such a family—in this case polynomial. They are sometimes called "operator pencils" for reasons that escape me. Anyway, the main result we want is

THEOREM 6. *If $P \in \mathrm{Diff}^m_{ti}(M; \mathbb{E})$ is elliptic then the suspended family $P(\tau)$ is invertible for all $\tau \in \mathbb{C} \setminus D$ with inverse*

$$P(\tau)^{-1} : H^s(M; E_2) \longrightarrow H^{s+m}(M; E_1) \tag{5.34}$$

where

$$D \subset \mathbb{C} \text{ is discrete and } D \subset \{\tau \in \mathbb{C};\ |\operatorname{Re}\tau| \le c|\operatorname{Im}\tau| + 1/c\} \tag{5.35}$$

for some $c > 0$ (see Fig. 1 – still not quite right).

In fact we need some more information on $P(\tau)^{-1}$ which we will pick up during the proof of this result. The translation-invariance of P can be written in operator form as

$$Pu(t + s, \cdot) = (Pu)(t + s, \cdot)\ \forall\ s \in \mathbb{R} \tag{5.36}$$

LEMMA 24. *If $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ is elliptic then it has a parametrix*

$$Q : \mathcal{S}(\mathbb{R} \times M; E_2) \longrightarrow \mathcal{S}(\mathbb{R} \times M; E_1) \tag{5.37}$$

which is translation-invariant in the sense of (5.36) *and preserves the compactness of supports in $\mathbb{R}$,*

$$Q : \mathcal{C}^\infty_c(\mathbb{R} \times M; E_2) \longrightarrow \mathcal{C}^\infty_c(\mathbb{R} \times M; E_1) \tag{5.38}$$

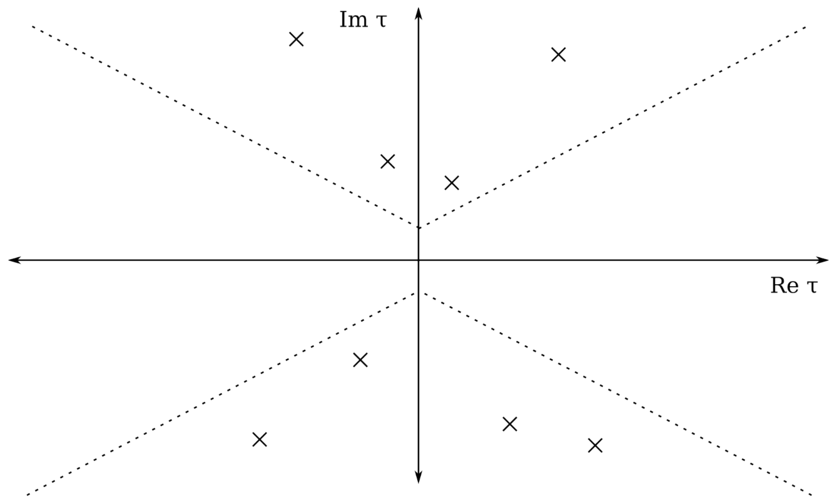

Figure 1. The region D.

Proof. In the case of a compact manifold we contructed a global parametrix by patching local parametricies with a partition of unity. Here we do the same thing, treating the variable $t \in \mathbb{R}$ globally throughout. Thus if $F_a : \Omega_a \to \Omega'_a$ is a coordinate patch in M over which E_1 and (hence) E_2 are trivial, P becomes a square matrix of differential operators

$$P_a = \begin{bmatrix} P_{11}(z, D_t, D_z) & \cdots & P_{l1}(z, D_t, D_z) \\ \vdots & & \vdots \\ P_{1l}(z, D_t, D_z) & \cdots & P_{ll}(z, D_t, D_z) \end{bmatrix} \tag{5.39}$$

in which the coefficients do *not* depend on t. As discussed in Sections 2 and 3 above, we can construct a local parametrix in Ω'_a using a properly supported cutoff χ. In the t variable the parametrix is global anyway, so we use a fixed cutoff $\tilde{\chi} \in \mathcal{C}^\infty_c(\mathbb{R})$, $\tilde{\chi} = 1$ in $|t| < 1$, and so construct a parametrix

$$Q_a f(t, z) = \int_{\Omega'_a} q(t - t', z, z')\tilde{\chi}(t - t')\chi(z, z')f(t', z')\, dt'\, dz'. \tag{5.40}$$

This satisfies

$$P_a Q_a = \operatorname{Id} - R_a, \quad Q_a P_a = \operatorname{Id} - R'_a \tag{5.41}$$

where R_a and R'_a are smoothing operators on Ω'_a with kernels of the form

$$\begin{gathered} R_a f(t, z) = \int_{\Omega'_a} R_a(t - t', z, z')f(t', z')\, dt'\, dz' \\ R_a \in \mathcal{C}^\infty(\mathbb{R} \times \Omega'^2_a),\ R_a(t, z, z') = 0 \text{ if } |t| \ge 2 \end{gathered} \tag{5.42}$$

with the support proper in Ω'_a.

Now, we can sum these local parametricies, which are all t-translation-invariant to get a global parametrix with the same properties

$$Qf = \sum_a \chi_a (F_a^{-1})^* (T_a^{-1})^* Q_a T_a^* F_a^* f \tag{5.43}$$

where T_a denotes the trivialization of bundles E_1 and E_2. It follows that Q satisfies (5.38) and since it is translation-invariant, also (5.37). The global version of (5.41) becomes

$$
\begin{aligned}
&PQ = \mathrm{Id} - R_2, \quad QP = \mathrm{Id} - R_1,\\
&R_i : \mathcal{C}_c^\infty(\mathbb{R}\times M; E_i) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R}\times M; E_i),\\
&R_i f = \int_{\mathbb{R}\times M} R_i(t-t', z, z') f(t', z')\, dt'\, \nu_{z'}
\end{aligned}
\tag{5.44}
$$

where the kernels

$$
R_i \in \mathcal{C}_c^\infty\left(\mathbb{R}\times M^2; \mathrm{Hom}(E_i)\right),\ i = 1, 2. \tag{5.45}
$$

□

In fact we can deduce directly from (5.40) the boundedness of Q.

Lemma 25. *The properly-supported parametrix Q constructed above extends by continuity to a bounded operator*

$$
\begin{aligned}
&Q : H^s_{ti}(\mathbb{R}\times M; E_2) \longrightarrow H^{s+m}_{ti}(\mathbb{R}\times M; E_1)\ \forall\, s\in\mathbb{R}\\
&Q : \mathcal{S}(\mathbb{R}\times M; E_2) \longrightarrow \mathcal{S}(\mathbb{R}\times M; E_1).
\end{aligned}
\tag{5.46}
$$

Proof. This follows directly from the earlier discussion of elliptic regularity for each term in (5.43) to show that

$$
\begin{aligned}
Q : \{f \in H^s_{\mathrm{ti}}(\mathbb{R}\times M; E_2;\ \mathrm{supp}(f) \subset [-2,2]\times M\}&\\
\longrightarrow \left\{u \in H^{s+m}_{\mathrm{ti}}(\mathbb{R}\times M; E_1;\ \mathrm{supp}(u) \subset [-2-R, 2+R]\times M\right\}&
\end{aligned}
\tag{5.47}
$$

for some R (which can in fact be taken to be small and positive). Indeed on compact sets the translation-invariant Sobolev spaces reduce to the usual ones. Then (5.46) follows from (5.47) and the translation-invariance of Q. Using a $\mu \in \mathcal{C}_c^\infty(\mathbb{R})$ generating a ti-paritition of unity on $\mathbb{R}$ we can decompose

$$
H^s_{\mathrm{ti}}(\mathbb{R}\times M; E_2) \ni f = \sum_{k\in\mathbb{Z}} \tau_k^*(\mu\tau_{-k}^* f). \tag{5.48}
$$

Then

$$
Qf = \sum_{k\in\mathbb{Z}} \tau_k^*\left(Q(\mu\tau_{-k}^* f)\right). \tag{5.49}
$$

The estimates corresponding to (5.47) give

$$
\|Qf\|_{H^{s+m}_{\mathrm{ti}}} \le C\|f\|_{H^s_{\mathrm{ti}}}
$$

if f has support in $[-2,2]\times M$. The decomposition (5.48) then gives

$$
\sum \|\mu\tau_{-k}^* f\|^2_{H^s} = \|f\|^2_{H_s} < \infty \implies \|Qf\|^2 \le C'\|f\|^2_{H^s}.
$$

This proves Lemma 25. □

Going back to the remainder term in (5.44), we can apply the 1-dimensional Fourier transform and find the following uniform results.

LEMMA 26. *If R is a compactly supported, t-translation-invariant smoothing operator as in* (5.44) *then*

$$\widehat{Rf}(\tau,\cdot)=\widehat{R}(\tau)\widehat{f}(\tau,\cdot) \tag{5.50}$$

where $\widehat{R}(\tau)\in\mathcal{C}^\infty\big(\mathbb{C}\times M^2;\operatorname{Hom}(E)\big)$ is entire in $\tau\in\mathbb{C}$ and satisfies the estimates

$$\forall\, k,p\ \exists\, C_{p,k} \text{ such that } \|\tau^k\widehat{R}(\tau)\|_{\mathcal{C}^p}\le C_{p,k}\exp(A|\operatorname{Im}\tau|). \tag{5.51}$$

Here A is a constant such that

$$\operatorname{supp}R(t,\cdot)\subset[-A,A]\times M^2. \tag{5.52}$$

PROOF. This is a parameter-dependent version of the usual estimates for the Fourier-Laplace transform. That is,

$$\widehat{R}(\tau,\cdot)=\int e^{-i\tau t}R(t,\cdot)\,dt \tag{5.53}$$

from which all the statements follow just as in the standard case when $R\in\mathcal{C}^\infty_c(\mathbb{R})$ has support in $[-A,A]$. □

PROPOSITION 18. *If R is as in Lemma 26 then there exists a discrete subset $D\subset\mathbb{C}$ such that $\Big(\operatorname{Id}-\widehat{R}(\tau)\Big)^{-1}$ exists for all $\tau\in\mathbb{C}\setminus D$ and*

$$\Big(\operatorname{Id}-\widehat{R}(\tau)\Big)^{-1}=\operatorname{Id}-\widehat{S}(\tau) \tag{5.54}$$

where $\widehat{S}:\mathbb{C}\longrightarrow\mathcal{C}^\infty(M^2;\operatorname{Hom}(E))$ is a family of smoothing operators which is meromorphic in the complex plane with poles of finite order and residues of finite rank at D. Furthermore,

$$D\subset\{\tau\in\mathbb{C};\ \log(|\operatorname{Re}\tau|)<c|\operatorname{Im}\tau|+1/c\} \tag{5.55}$$

for some $c>0$ and for any $C>0$, there exists C' such that

$$|\operatorname{Im}\tau|<C,\ |\operatorname{Re}\tau|>C'\implies\|\tau^k\widehat{S}(\tau)\|_{\mathcal{C}^p}\le C_{p,k}. \tag{5.56}$$

PROOF. This is part of "Analytic Fredholm Theory" (although usually done with compact operators on a Hilbert space). The estimates (5.51) on $\widehat{R}(\tau)$ show that, in some region as on the right in (5.55),

$$\|\widehat{R}(\tau)\|_{L^2}\le 1/2. \tag{5.57}$$

Thus, by Neumann series,

$$\widehat{S}(\tau)=\sum_{k=1}^{\infty}\Big(\widehat{R}(\tau)\Big)^k \tag{5.58}$$

exists as a bounded operator on $L^2(M;E)$. In fact it follows that $\widehat{S}(\tau)$ is itself a family of smoothing operators in the region in which the Neumann series converges. Indeed, the series can be rewritten

$$\widehat{S}(\tau)=\widehat{R}(\tau)+\widehat{R}(\tau)^2+\widehat{R}(\tau)\widehat{S}(\tau)\widehat{R}(\tau) \tag{5.59}$$

The smoothing operators form a "corner" in the bounded operators in the sense that products like the third here are smoothing if the outer two factors are. This follows from the formula for the kernel of the product

$$\int_{M\times M}\widehat{R}_1(\tau;z,z')\widehat{S}(\tau;z',z'')\widehat{R}_2(\tau;z'',\tilde{z})\,\nu_{z'}\,\nu_{z''}.$$

Thus $\widehat{S}(\tau) \in \mathcal{C}^\infty(M^2; \mathrm{Hom}(E))$ exists in a region as on the right in (5.55). To see that it extends to be meromorphic in $\mathbb{C} \setminus D$ for a discrete divisor D we can use a finite-dimensional approximation to $\widehat{R}(\tau)$.

Recall — if neccessary from local coordinates — that given any $p \in \mathbb{N}, R > 0, q > 0$ there are finitely many sections $f_i^{(\tau)} \in \mathcal{C}^\infty(M; E'), g_i^{(\tau)} \in \mathcal{C}^\infty(M; E)$ and such that

$$\|\widehat{R}(\tau) - \sum_i g_i(\tau, z) \cdot f_i(\tau, z')\|_{\mathcal{C}^p} < \epsilon, \quad |\tau| < R. \tag{5.60}$$

Writing this difference as $M(\tau)$,

$$\mathrm{Id} - \widehat{R}(\tau) = \mathrm{Id} - M(\tau) + F(\tau)$$

where $F(\tau)$ is a finite rank operator. In view of (5.60), $\mathrm{Id} - M(\tau)$ is invertible and, as seen above, of the form $\mathrm{Id} - \widehat{M}(\tau)$ where $\widehat{M}(\tau)$ is holomorphic in $|\tau| < R$ as a smoothing operator.

Thus

$$\mathrm{Id} - \widehat{R}(\tau) = (\mathrm{Id} - M(\tau))(\mathrm{Id} + F(\tau) - \widehat{M}(\tau)F(\tau))$$

is invertible if and only if the finite rank perturbation of the identity by $(\mathrm{Id} - \widehat{M}(\tau))F(\tau)$ is invertible. For R large, by the previous result, this finite rank perturbation must be invertible in an open set in $\{|\tau| < R\}$. Then, by standard results for finite dimensional matrices, it has a meromorphic inverse with finite rank (generalized) residues. The same is therefore true of $\mathrm{Id} - \widehat{R}(\tau)$ itself.

Since $R > 0$ is arbitrary this proves the result. $\square$

Proof. Proof of Theorem 6 We have proved (5.44) and the corresponding form for the Fourier transformed kernels follows:

$$\widehat{P}(\tau)\widehat{Q}'(\tau) = \mathrm{Id} - \widehat{R}_2(\tau), \ \widehat{Q}'(\tau)\widehat{P}(\tau) = \mathrm{Id} - \widehat{R}_1(\tau) \tag{5.61}$$

where $\widehat{R}_1(\tau), \widehat{R}_2(\tau)$ are families of smoothing operators as in Proposition 18. Applying that result to the first equation gives a new meromorphic right inverse

$$\widehat{Q}(\tau) = \widehat{Q}'(\tau)(\mathrm{Id} - \widehat{R}_2(\tau))^{-1} = \widehat{Q}'(\tau) - \widehat{Q}'(\tau)M(\tau)$$

where the first term is entire and the second is a meromorphic family of smoothing operators with finite rank residues. The same argument on the second term gives a left inverse, but his shows that $\widehat{Q}(\tau)$ must be a two-sided inverse.

This we have proved everything except the locations of the poles of $\widehat{Q}(\tau)$ — which are only constrained by (5.55) instead of (5.35). However, we can apply the same argument to $P_\theta(z, D_t, D_z) = P(z, e^{i\theta}D_t, D_z)$ for $|\theta| < \delta$, $\delta > 0$ small, since P_θ stays elliptic. This shows that the poles of $\widehat{Q}(\tau)$ lie in a set of the form (5.35). $\square$

3. Invertibility

We are now in a position to characterize those t-translation-invariant differential operators which give isomorphisms on the translation-invariant Sobolev spaces.

Theorem 7. *An element $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; E)$ gives an isomorphism* (5.30) *(or equivalently is Fredholm) if and only if it is elliptic and $D \cap \mathbb{R} = \emptyset$, i.e. $\hat{P}(\tau)$ is invertible for all $\tau \in \mathbb{R}$.*

PROOF. We have already done most of the work for the important direction for applications, which is that the ellipticity of P and the invertibility at $\hat{P}(\tau)$ for all $\tau \in \mathbb{R}$ together imply that (5.30) is an isomorphism for any $s \in \mathbb{R}$.

Recall that the ellipticity of P leads to a parameterix Q which is translation-invariant and has the mapping property we want, namely (5.46).

To prove the same estimate for the true inverse (and its existence) consider the difference

$$\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{\mathbb{R}}(\tau),\ \tau \in \mathbb{R}. \tag{5.62}$$

Since $\hat{P}(\tau) \in \mathrm{Diff}^m(M;\mathbb{E})$ depends smoothly on $\tau \in \mathbb{R}$ and $\hat{Q}(\tau)$ is a paramaterix for it, we know that

$$\hat{R}(\tau) \in \mathcal{C}^\infty(\mathbb{R};\Psi^{-\infty}(M;\mathbb{E})) \tag{5.63}$$

is a smoothing operator on M which depends smoothly on $\tau \in \mathbb{R}$ as a parameter. On the other hand, from (5.61) we also know that for large real τ,

$$\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{Q}(\tau)M(\tau)$$

where $M(\tau)$ satisfies the estimates (5.56). It follows that $\hat{Q}(\tau)M(\tau)$ also satisfies these estimates and (5.63) can be strengthened to

$$\sup_{\tau\in\mathbb{R}} \|\tau^k \hat{R}(\tau,\cdot,\cdot)\|_{\mathcal{C}^p} < \infty\ \forall\ p, k. \tag{5.64}$$

That is, the kernel $\hat{R}(\tau) \in \mathcal{S}(\mathbb{R};\mathcal{C}^\infty(M^2;\mathrm{Hom}(\mathbb{E})))$. So if we define the t-translation-invariant operator

$$Rf(t,z) = (2\pi)^{-1}\int e^{it\tau}\hat{R}(\tau)\hat{f}(\tau,\cdot)d\tau \tag{5.65}$$

by inverse Fourier transform then

$$R : H^s_{\mathrm{ti}}(\mathbb{R}\times M;E_2) \longrightarrow H^\infty_{\mathrm{ti}}(\mathbb{R}\times M;E_1)\ \forall\ s\in\mathbb{R}. \tag{5.66}$$

It certainly suffices to show this for $s < 0$ and then we know that the Fourier transform gives a map

$$\mathcal{F} : H^s_{\mathrm{ti}}(\mathbb{R}\times M;E_2) \longrightarrow \langle\tau\rangle^{|s|}L^2(\mathbb{R};H^{-|s|}(M;E_2)). \tag{5.67}$$

Since the kernel $\hat{R}(\tau)$ is rapidly decreasing in τ, as well as being smooth, for every $N > 0$,

$$\hat{R}(\tau) : \langle\tau\rangle^{|s|}L^2(\mathbb{R};H^{-|s|}M;E_2) \longrightarrow \langle\tau\rangle^{-N}L^2(\mathbb{R};H^N(M;E_2)) \tag{5.68}$$

and inverse Fourier transform maps

$$\mathcal{F}^{-1} : \langle\tau\rangle^{-N}H^N(M;E_2) \longrightarrow H^N_{\mathrm{ti}}(\mathbb{R}\times M;E_2)$$

which gives (5.66).

Thus $Q+R$ has the same property as Q in (5.46). So it only remains to check that $Q+R$ is the two-sided version of P and it is enough to do this on $\mathcal{S}(\mathbb{R}\times M;E_i)$ since these subspaces are dense in the Sobolev spaces. This in turn follows from (5.62) by taking the Fourier transform. Thus we have shown that the invertibility of P follows from its ellipticity and the invertibility of $\hat{P}(\tau)$ for $\tau \in \mathbb{R}$.

The converse statement is less important but certainly worth knowing! If P is an isomorphism as in (5.30), even for one value of s, then it must be elliptic — this follows as in the compact case since it is everywhere a local statement. Then if $\hat{P}(\tau)$ is not invertible for some $\tau \in \mathbb{R}$ we know, by ellipticity, that it is Fredholm and, by the stability of the index, of index zero (since $\hat{P}(\tau)$ *is* invertible for a dense set of $\tau \in \mathbb{C}$). There is therefore some $\tau_0 \in \mathbb{R}$ and $f_0 \in \mathcal{C}^\infty(M; E_2)$, $f_0 \neq 0$, such that

$$\hat{P}(\tau_0)^* f_0 = 0. \tag{5.69}$$

It follows that f_0 is *not* in the range of $\hat{P}(\tau_0)$. Then, choose a cut off function, $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\rho(\tau_0) = 1$ (and supported sufficiently close to τ_0) and define $f \in \mathcal{S}(\mathbb{R} \times M; E_2)$ by

$$\hat{f}(\tau, \cdot) = \rho(\tau) f_0(\cdot). \tag{5.70}$$

Then $f \notin P \cdot H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_1)$ for any $s \in \mathbb{R}$. To see this, suppose $u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_1)$ has

$$Pu = f \Rightarrow \hat{P}(\tau)\hat{u}(\tau) = \hat{f}(\tau) \tag{5.71}$$

where $\hat{u}(\tau) \in \langle\tau\rangle^{|s|} L^2(\mathbb{R}; H^{-|s|}(M; E_1))$. The invertibility of $P(\tau)$ for $\tau \neq \tau_0$ on $\operatorname{supp}(\rho)$ (chosen with support close enough to τ_0) shows that

$$\hat{u}(\tau) = \hat{P}(\tau)^{-1} \hat{f}(\tau) \in \mathcal{C}^\infty((\mathbb{R} \backslash \{\tau_0\}) \times M; E_1).$$

Since we know that $\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{R}(\tau)$ is a meromorphic family of smoothing operators it actually follows that $\hat{u}(\iota)$ is meromorphic in τ near τ_0 in the sense that

$$\hat{u}(\tau) = \sum_{j=1}^{k} (\tau - \tau_0)^{-j} u_j + v(\tau) \tag{5.72}$$

where the $u_j \in \mathcal{C}^\infty(M; E_1)$ and $v \in \mathcal{C}^\infty((\tau - \epsilon, \tau + \epsilon) \times M; E_1)$. Now, one of the u_j is not identically zero, since otherwise $\hat{P}(\tau_0) v(\tau_0) = f_0$, contradicting the choice of f_0. However, a function such as (5.72) is *not* locally in L^2 with values in any Sobolev space on M, which contradicts the existence of $u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_1)$.

This completes the proof for invertibility of P. To get the Fredholm version it suffices to prove that if P is Fredholm then it is invertible. Since the arguments above easily show that the null space of P is empty on any of the $H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_1)$ spaces and the same applies to the adjoint, we easily conclude that P is an isomorphism if it is Fredholm. $\square$

This result allows us to deduce similar invertibility conditions on exponentially-weighted Sobolev spaces. Set

$$e^{at} H^s_{\mathrm{ti}}(\mathbb{R} \times M; E) = \{u \in H^s_{\mathrm{loc}}(\mathbb{R} \times M; E);\ e^{-at} u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E)\} \tag{5.73}$$

for any $\mathcal{C}^\infty$ vector bundle E over M. The translation-invariant differential operators also act on these spaces.

Lemma 27. *For any $a \in \mathbb{R}$, $P \in \operatorname{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ defines a continuous linear operator*

$$P : e^{at} H^{s+m}_{ti}(\mathbb{R} \times M; E_1) \longrightarrow e^{at} H^{s+m}_{ti}(\mathbb{R} \times M; E_2). \tag{5.74}$$

Proof. We already know this for $a = 0$. To reduce the general case to this one, observe that (5.74) just means that

$$P \cdot e^{at} u \in e^{at} H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2)\ \forall\ u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_1) \tag{5.75}$$

with continuity meaning just continuous dependence on u. However, (5.75) in turn means that the conjugate operator

$$P_a = e^{-at} \cdot P \cdot e^{at} : H^{s+m}_{\mathrm{ti}}(\mathbb{R} \times M; E_1) \longrightarrow H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2). \tag{5.76}$$

Conjugation by an exponential is actually an isomorphism

$$\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M; \mathbb{E}) \ni P \longmapsto e^{-at} P e^{at} \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M; \mathbb{E}). \tag{5.77}$$

To see this, note that elements of $\mathrm{Diff}^j(M; \mathbb{E})$ commute with multiplication by e^{at} and

$$e^{-at} D_t e^{at} = D_t - ia \tag{5.78}$$

which gives (5.77)).

The result now follows. $\square$

Proposition 19. *If $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ is elliptic then as a map* (5.74) *it is invertible precisely for*

$$a \notin -\operatorname{Im}(D),\ D = D(P) \subset \mathbb{C}, \tag{5.79}$$

that is, a is not the negative of the imaginary part of an element of D.

Note that the set $-\operatorname{Im}(D) \subset \mathbb{R}$, for which invertibility fails, is discrete. This follows from the discreteness of D and the estimate (5.35). Thus in Fig 1 invertibility on the space with weight e^{at} correspond exactly to the horizonatal line with $\operatorname{Im} \tau = -a$ missing D.

Proof. This is direct consequence of (**??**) and the discussion around (5.76). Namely, P is invertible as a map (5.74) if and only if P_a is invertible as a map (5.30) so, by Theorem 7, if

and only if

$$D(P_a) \cap \mathbb{R} = \emptyset.$$

From (5.78), $D(P_a) = D(P) + ia$ so this condition is just $D(P) \cap (\mathbb{R} - ia) = \emptyset$ as claimed. $\square$

Although this is a characterization of the Fredholm properties on the standard Sobolev spaces, it is not the end of the story, as we shall see below.

One important thing to note is that $\mathbb{R}$ has *two* ends. The exponential weight e^{at} treats these differently – since if it is big at one end it is small at the other – and in fact we (or rather you) can easily define doubly-exponentially weighted spaces and get similar results for those. Since this is rather an informative extended exercise, I will offer some guidance.

Definition 6. Set

$$\begin{aligned} H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R} \times M; E) = \{u \in H^s_{\mathrm{loc}}(\mathbb{R} \times M; E);\\ \chi(t) e^{-at} u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E)(1 - \chi(t)) e^{bt} u \in H^s_{ti}(\mathbb{R} \times M; E)\} \end{aligned} \tag{5.80}$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R})$, $\chi = 1$ in $t > 1$, $\chi = 0$ in $t < -1$.

Exercises.

(1) Show that the spaces in (5.80) are independent of the choice of χ, are all Hilbertable (are complete with respect to a Hilbert norm) and show that if $a+b \geq 0$

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E) = e^{at}H^s_{\mathrm{ti}}(\mathbb{R}\times M;E) + e^{-bt}H^s_{\mathrm{ti}}(\mathbb{R}\times M;E) \tag{5.81}$$

whereas if $a+b \leq 0$ then

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E) = e^{at}H^s_{\mathrm{ti}}(\mathbb{R}\times M;E) \cap e^{-bt}H^s_{\mathrm{ti}}(\mathbb{R}\times M;E). \tag{5.82}$$

(2) Show that any $P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M;\mathbb{E})$ defines a continuous linear map for any s, a, $b \in \mathbb{R}$

$$P : H^{s+m,a,b}_{\mathrm{ti\text{-}exp}}(\mathbb{R}\times M;E_1) \longrightarrow H^{s,a,b}_{\mathrm{ti\text{-}exp}}(\mathbb{R}\times M;E_2). \tag{5.83}$$

(3) Show that the standard L^2 pairing, with respect to dt, a smooth positive density on M and an inner product on E extends to a non-degenerate bilinear pairing

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E) \times H^{-s,-a,-b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E) \longrightarrow \mathbb{C} \tag{5.84}$$

for any s, a and b. Show that the adjoint of P with respect to this pairing is P^* on the 'negative' spaces – you can use this to halve the work below.

(4) Show that if P is elliptic then (5.83) is Fredholm precisely when

$$a \notin -\operatorname{Im}(D) \text{ and } b \notin \operatorname{Im}(D). \tag{5.85}$$

Hint:- Assume for instance that $a+b \geq 0$ and use (5.81). Given (5.85) a parametrix for P can be constructed by combining the inverses on the single exponential spaces

$$Q_{a,b} = \chi' P_a^{-1}\chi + (1-\chi'')P_{-b}^{-1}(1-\chi) \tag{5.86}$$

where χ is as in (5.80) and χ' and χ'' are similar but such that $\chi'\chi = 1$, $(1-\chi'')(1-\chi) = 1-\chi$.

(5) Show that P is an isomorphism if and only if

$$a+b \leq 0 \text{ and } [a,-b]\cap -\operatorname{Im}(D) = \emptyset \text{ or } a+b \geq 0 \text{ and } [-b,a]\cap -\operatorname{Im}(D) = \emptyset.$$

(6) Show that if $a+b \leq 0$ and (5.85) holds then

$$\operatorname{ind}(P) = \dim \operatorname{null}(P) = \sum_{\tau_i \in D\cap(\mathbb{R}\times[b,-a])} \operatorname{Mult}(P,\tau_i)$$

where $\operatorname{Mult}(P,\tau_i)$ is the *algebraic* multiplicity of τ as a 'zero' of $\hat{P}(\tau)$, namely the dimension of the generalized null space

$$\operatorname{Mult}(P,\tau_i) = \dim\left\{u = \sum_{p=0}^{N} u_p(z)D^p_\tau\delta(\tau-\tau_i); P(\tau)u(\tau) \equiv 0\right\}.$$

(7) Characterize these multiplicities in a more algebraic way. Namely, if τ' is a zero of $P(\tau)$ set $E_0 = \operatorname{null} P(\tau')$ and $F_0 = \mathcal{C}^\infty(M;E_2)/P(\tau')\mathcal{C}^\infty(M;E_1)$. Since $P(\tau)$ is Fredholm of index zero, these are finite dimensional vector spaces of the same dimension. Let the derivatives of P be $T_i = \partial^i P/\partial\tau^i$ at $\tau = \tau'$ Then define $R_1 : E_0 \longrightarrow F_0$ as T_1 restricted to E_0 and projected to F_0. Let E_1 be the null space of R_1 and $F_1 = F_0/R_1E_0$. Now proceed inductively and define for each i the space E_i as the null space of R_i, $F_i = F_{i-1}/R_iE_{i-1}$ and $R_{i+1} : E_i \longrightarrow F_i$ as T_i restricted to E_i and projected to F_i. Clearly E_i and F_i have the same, finite, dimension which

is non-increasing as i increases. The properties of $P(\tau)$ can be used to show that for large enough i, $E_i = F_i = \{0\}$ and

$$\mathrm{Mult}(P,\tau') = \sum_{i=0}^{\infty} \dim(E_i) \tag{5.87}$$

where the sum is in fact finite.

(8) Derive, by duality, a similar formula for the index of P when $a+b \geq 0$ and (5.85) holds, showing in particular that it is injective.

4. Resolvent operator

Addenda to Chapter 5

More?

- Why – manifold with boundary later for Euclidean space, but also resolvent (Photo-C5-01)
- Hölder type estimates – Photo-C5-03. Gives interpolation.

As already noted even a result such as Proposition 19 and the results in the exercises above by no means exhausts the possibile realizations of an element $P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M;\mathbb{E})$ as a Fredholm operator. Necessarily these other realization cannot simply be between spaces like those in (5.80). To see what else one can do, suppose that the condition in Theorem 7 is violated, so

$$D(P) \cap \mathbb{R} = \{\tau_1,\dots,\tau_N\} \neq \emptyset. \tag{5.88}$$

To get a Fredholm operator we need to change either the domain or the range space. Suppose we want the range to be $L^2(\mathbb{R}\times M;E_2)$. Now, the condition (5.85) guarantees that P is Fredholm as an operator (5.83). So in particular

$$P : H^{m,\epsilon,\epsilon}_{\mathrm{ti-exp}}(\mathbb{R}\times M;E_1) \longrightarrow H^{0,\epsilon,\epsilon}_{\mathrm{ti\text{-}exp}}(\mathbb{R}\times M;E_2) \tag{5.89}$$

is Fredholm for all $\epsilon > 0$ sufficiently small (becuase D is discrete). The image space (which is necessarily the range in this case) just consists of the sections of the form $\exp(a|t|)f$ with f in L^2. So, in this case the range certainly contains L^2 so we can define

(5.90)
$$\mathrm{Dom}_{\mathrm{AS}}(P) = \{u \in H^{m,\epsilon,\epsilon}_{\mathrm{ti\text{-}exp}}(\mathbb{R}\times M;E_1); Pu \in L^2(\mathbb{R}\times M;E_2)\},\ \epsilon > 0 \text{ sufficiently small.}$$

This space is independent of $\epsilon > 0$ if it is taken smalle enough, so the same space arises by taking the intersection over $\epsilon > 0$.

Proposition 20. *For any elliptic element $P \in \mathrm{Diff}^m_{ti}(\mathbb{R}\times M;\mathbb{E})$ the space in (5.90) is Hilbertable space and*

$$P : \mathrm{Dom}_{AS}(P) \longrightarrow L^2(\mathbb{R}\times M;E_2) \text{ is Fredholm.} \tag{5.91}$$

I have not made the assumption (5.88) since it is relatively easy to see that if $D\cap\mathbb{R} = \emptyset$ then the domain in (5.90) reduces again to $H^m_{\mathrm{ti}}(\mathbb{R}\times M;E_1)$ and (5.91) is just the standard realization. Conversely of course under the assumption (5.88) the domain in (5.91) is strictly larger than the standard Sobolev space. To see what it actually is requires a bit of work but if you did the exercises above you are in a position to work this out! Here is the result when there is only one pole of $\hat{P}(\tau)$ on the real line and it has order one.

PROPOSITION 21. *Suppose $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ is elliptic, $\hat{P}(\tau)$ is invertible for $\tau \in \mathbb{R} \setminus \{0\}$ and in addition $\tau \hat{P}(\tau)^{-1}$ is holomorphic near 0. Then the Atiyah-Singer domain in (5.91) is*

$$\begin{gathered}\mathrm{Dom}_{AS}(P) = \big\{u = u_1 + u_2; u_1 \in H^m_{ti}(\mathbb{R} \times M; E_1), \\ u_2 = f(t)v,\ v \in \mathcal{C}^\infty(M; E_1),\ \hat{P}(0)v = 0,\ f(t) = \int_0^t g(t)dt,\ g \in H^{m-1}(\mathbb{R})\big\}.\end{gathered} \tag{5.92}$$

Notice that the 'anomalous' term here, u_2, need not be square-integrable. In fact for any $\delta > 0$ the power $\langle t\rangle^{\frac{1}{2}-\delta} v \in \langle t\rangle^{1-\delta} L^2(\mathbb{R} \times M; E_1)$ is included and conversely

$$f \in \bigcap_{\delta > 0} \langle t\rangle^{1+\delta} H^{m-1}(\mathbb{R}). \tag{5.93}$$

One can say a lot more about the growth of f if desired but it is generally quite close to $\langle t\rangle L^2(\mathbb{R})$.

Domains of this sort are sometimes called 'extended L^2 domains' – see if you can work out what happens more generally.

CHAPTER 6

Manifolds with boundary

$\boxed{\text{0.6Q; Revised: 6-8-2007; Run: February 7, 2008}}$

- Dirac operators – Photos-C5-16, C5-17.
- Homogeneity etc Photos-C5-18, C5-19, C5-20, C5-21, C5-23, C5-24.

1. Compactifications of $\mathbb{R}$.

As I will try to show by example later in the course, there are I believe considerable advantages to looking at compactifications of non-compact spaces. These advantages show up last in geometric and analytic considerations. Let me start with the simplest possible case, namely the real line. There are two standard compactifications which one can think of as 'exponential' and 'projective'. Since there is only one connected compact manifold with boundary compactification corresponds to the choice of a diffeomorphism onto the interior of $[0,1]$:

$$\begin{gathered}\gamma:\mathbb{R}\longrightarrow[0,1],\ \gamma(\mathbb{R})=(0,1),\\ \gamma^{-1}:(0,1)\longrightarrow\mathbb{R},\ \gamma,\gamma^{-1}\mathcal{C}^\infty.\end{gathered}\tag{6.1}$$

In fact it is not particularly pleasant to have to think of the global maps γ, although we can. Rather we can think of separate maps

$$\begin{gathered}\gamma_+:(T_+,\infty)\longrightarrow[0,1]\\ \gamma_-:(T_-,-\infty)\longrightarrow[0,1]\end{gathered}\tag{6.2}$$

which *both* have images $(0,x_\pm)$ and as diffeomorphism other than signs. In fact if we want the two ends to be the 'same' then we can take $\gamma_-(t)=\gamma_+(-t)$. I leave it as an exercise to show that γ then exists with

$$\begin{cases}\gamma(t)=\gamma_+(t) & t\gg 0\\ \gamma(t)=1-\gamma_-(t) & t\ll 0.\end{cases}\tag{6.3}$$

So, all we are really doing here is identifying a 'global coordinate' γ_+^*x near ∞ and another near $-\infty$. Then two choices I refer to above are

$$\begin{matrix}x=e^{-t} & \text{exponential compactification}\\ x=1/t & \text{projective compactification}\,.\end{matrix}\tag{CR.4}$$

Note that these are *alternatives*!

Rather than just consider $\mathbb{R}$, I want to consider $\mathbb{R}\times M$, with M compact, as discussed above.

Lemma 28. *If $R:H\longrightarrow H$ is a compact operator on a Hilbert space then $\operatorname{Id}-R$ is Fredholm.*

PROOF. A compact operator is one which maps the unit ball (and hence any bounded subset) of H onto a precompact set, a set with compact closure. The unit ball in the null space of $\mathrm{Id} - R$ is

$$\{u \in H; \|u\| = 1\,,\ u = Ru\} \subset R\{u \in H; \|u\| = 1\}$$

and is therefore precompact. Since it is closed, it is compact and any Hilbert space with a compact unit ball is finite dimensional. Thus the null space of $(\mathrm{Id} - R)$ is finite dimensional.

Consider a sequence $u_n = v_n - Rv_n$ in the range of $\mathrm{Id} - R$ and suppose $u_n \to u$ in H. We may assume $u \neq 0$, since 0 is in the range, and by passing to a subsequence suppose that of γ on ?? fields. Clearly

$$\begin{array}{lll} \gamma(t) = e^{-t} & \Rightarrow & \gamma_*(\partial_t) = -x(\partial_x) \\ \tilde{\gamma}(t) = 1/t & \Rightarrow & \tilde{\gamma}_*(\partial_t) = -s^2\partial_s \end{array} \tag{CR.5}$$

where I use 's' for the variable in the second case to try to reduce confusion, it is just a variable in $[0,1]$. Dually

$$\begin{aligned} \gamma^*\left(\frac{dx}{x}\right) &= -dt \\ \tilde{\gamma}^*\left(\frac{ds}{s^2}\right) &= -dt \end{aligned} \tag{CR.6}$$

in the two cases. The minus signs just come from the fact that both γ's reverse orientation.

PROPOSITION 22. *Under exponential compactification the translation-invariant Sobolev spaces on $\mathbb{R} \times M$ are identified with*

$$\begin{aligned} H^k_b([0,1] \times M) = \Big\{ u \in L^2\left([0,1] \times M; \frac{dx}{x}V_M\right)\ ; \forall\ \ell, p \leq k \\ P_p \in \mathrm{Diff}^p(M)\,,\ (xD_x)^\ell P_p u \in L^2\big([0,1] \times M; \frac{dx}{x}V_M\big)\Big\} \end{aligned} \tag{6.4}$$

for k a positive integer, $\dim M = n$,

$$\begin{aligned} H^s_b([0,1] \times M) = \big\{ u \in L^2\left([0,1] \times M; \frac{dx}{x}V_M\right); \\ \iint \frac{|u(x,z) - u(x',z')|^2}{\big(|\log\frac{x}{x'}|^2 + \rho(z,z')\big)^{\frac{n+s+1}{2}}}\frac{dx}{x}\frac{dx'}{x'}\nu\nu' < \infty\big\} 0 < s < 1 \end{aligned} \tag{6.5}$$

and for $s < 0$, $k \in \mathbb{N}$ s.t., $0 \leq s + k < 1$,

$$\begin{aligned} H^s_b([0,1] \times M) = \big\{ u = \sum_{0 \leq j+p \leq k} (Xd^J_X)p_P u_{j,p}, \\ P_p \in \mathrm{Diff}^p(M)\,,\ u_{j,p} \in H^{s+k}_b([0,1] \times M)\big\}. \end{aligned} \tag{6.6}$$

Moreover the L^2 pairing with respect to the measure $\frac{dx}{x}\nu$ extends by continuity from the dense subspaces $\mathcal{C}^\infty_c((0,1) \times M)$ to a non-degenerate pairing

$$H^s_b([0,1] \times M) \times H^{-s}_b([0,1] \times M) \ni (n,u) \longmapsto \int u \cdot v \frac{dx}{x}\nu \in \mathbb{C}. \tag{6.7}$$

$\square$

PROOF. This is all just translation of the properties of the space $H^s_{\mathrm{ti}}(\mathbb{R}\times M)$ to the new coordinates. $\square$

Note that there are other properties I have not translated into this new setting. There is one additional fact which it is easy to check. Namely $\mathcal{C}^\infty([0,1]\times M)$ acts as multipliers on all the spaces $H^s_{\mathrm{b}}([0,1]\times M)$. This follows directly from Proposition 22;

$$\mathcal{C}^\infty([0,1]\times M)\times H^s_{\mathrm{b}}([0,1]\times M)\ni(\varphi,u)\mapsto\varphi u\in H^s_{\mathrm{b}}([0,1]\times M)\,. \tag{CR.12}$$

What about the 'b' notation? Notice that $(1-x)x\partial_x$ and the smooth vector fields on M span, over $\mathcal{C}^\infty(X)$, for $X=[0,1]\times M$, *all* the vector fields tangent to $\{x=0|u|x=1\}$. Thus we can define the 'boundary differential operators' as

$$\operatorname{Diff}^m_{\mathrm{b}}([0,1]\times M_i)^E=\Bigg\{P=\sum_{0\le j+p\le m}a_{j,p}(x_j)((1-x)xD_x)^jP_p\ , \\ P_p\in\operatorname{Diff}^p(M_i)^E\Big\} \tag{CR.13}$$

and conclude from (CR.12) and the earlier properties that

$$P\in\operatorname{Diff}^m_{\mathrm{b}}(X;E)\Rightarrow \\ P:H^{s+m}_{\mathrm{b}}(X;E)\to H^s_{\mathrm{b}}(X;E)\,\forall s\in\mathbb{R}\,. \tag{CR.14}$$

THEOREM 8. *A differential operator as in* (8) *is Fredholm if and only if it is elliptic in the interior and the two "normal operators'*

$$I_\pm(P)=\sum_{0\le j+p\le m}a_{j,p}(x_{\pm1})(\pm D_k)^iP_p\quad x_+=0\,,\,x_-=1 \tag{CR.16}$$

derived from (CR.13)*, are elliptic and invertible on the translation-invariant Sobolev spaces.*

PROOF. As usual we are more interested in the sufficiency of these conditions than the necessity. To prove this result by using the present (slightly low-tech) methods requires going back to the beginning and redoing most of the proof of the Fredholm property for elliptic operators on a compact manifold.

The first step then is *a priori* bounds. What we want to show is that if the conditions of the theorem hold then for $u\in H^{s+m}_{\mathrm{b}}(X;E)$, $x=\mathbb{R}\times M$, $\exists C>0$ s.t.

$$\|u\|_{m+s}\le C_s\|Pu\|_s+C_s\|x(1-x)u\|_{s-1+m}\,. \tag{CR.17}$$

Notice that the norm on the right has a factor, $x(1-x)$, which vanishes at the boundary. Of course this is supposed to come from the invertibility of $I_\pm(P)$ in $\mathbb{R}(0)$ and the ellipticity of P.

By comparison $I_\pm(P):H^{s+m}_{\hbar}(\mathbb{R}\times M)\to H^s_{\hbar}(\mathbb{R}\times M)$ are isomorphisms — necessary and sufficient conditions for this are given in Theorem ???. We can use the compactifying map γ to convert this to a statement as in (CR.17) for the operators

$$P_\pm\in\operatorname{Diff}^m_{\mathrm{b}}(X)\,,\,P_\pm=I_\pm(P)(\gamma_*D_t,\cdot)\,. \tag{CR.18}$$

Namely

$$\|u\|_{m+s}\le C_s\|P_\pm u\|_s \tag{CR.19}$$

where these norms, as in (CR.17) are in the H^s_{b} spaces. Note that near $x=0$ or $x=1$, $P_\pm$ are obtained by substituting $D_t \mapsto xD_x$ or $(1-x)D_x$ in (CR.17). Thus

$$P - P_\pm \in (x - x_\pm)\,\mathrm{Diff}^m_{\mathrm{b}}(X)\,, \quad x_\pm = 0, 1 \tag{CR.20}$$

have coefficients which *vanish* at the appropriate boundary. This is precisely how (CR.16) is derived from (CR.13). Now choose $\varphi \in \mathcal{C}^\infty, (0,1) \times M$ which is equal to 1 on a sufficiently large set (and has $0 \le \varphi \le 1$) so that

$$1 - \varphi = \varphi_+ + \varphi_-\,,\ \varphi_\pm \in \mathcal{C}^\infty([0,1] \times M) \tag{CR.21}$$

have $\mathrm{supp}(\varphi_\pm) \subset \{|x - x_\pm| \le \epsilon)$, $0 \le \varphi_+ 1$.

By the interim elliptic estimate,

$$\|\varphi u\|_{s+m} \le C_s \|\varphi P u\|_s + C'_s \|\psi u\|_{s-1+m} \tag{CR.22}$$

where $\psi \in \mathcal{C}^\infty_c((0,1) \times M)$. On the other hand, because of (CR.20)

$$\begin{aligned} \|\varphi_\pm u\|_{m+s} &\le C_s \|\varphi_\pm P_\pm u\|_s + C_s \|[\varphi_\pm, P_\pm u]\|_s \\ &\le C_s \|\varphi_\pm P u\|_s + C_s \varphi_\pm (P - P_\pm) u\|_s + C_s \|[\varphi_\pm, P_\pm] u\|_s\,. \end{aligned} \tag{CR.23}$$

Now, if we can choose the support at $\varphi_\pm$ small enough — recalling that C_s truly depends on $I_\pm(P_t)$ and s — then the second term on the right in (CR.23) is bounded by $\frac{1}{4}\|u\|_{m+s}$, since *all* the coefficients of $P - P_\pm$ are small on the support off $\varphi_\pm$. Then (CR.24) ensures that the final term in (CR.17), since the coefficients vanish at $x = x_\pm$.

The last term in (CR.22) has a similar bound since ψ has compact support in the interim. This combining (CR.2) and (CR.23) gives the desired bound (CR.17).

To complete the proof that P is Fredholm, we need another property of these Sobolev spaces.

LEMMA 29. *The map*

$$Xx(1-x) : H^s_b(X) \longrightarrow H^{s-1}_b(X) \tag{6.8}$$

is compact.

PROOF. Follow it back to $\mathbb{R} \times M$!

□

Now, it follows from the *a priori* estimate (CR.17) that, as a map (CR.14), P has finite dimensional null space and closed range. This is really the proof of Proposition ?? again. Moreover the adjoint of P with respect to $\frac{dx}{x}V, P^*$, is again elliptic and satisfies the condition of the theorem, so it too has finite-dimensional null space. Thus the range of P has finite codimension so it is Fredholm.

□

A corresponding theorem, with similar proof follows for the cusp compactification. I will formulate it later.

2. Basic properties

A discussion of manifolds with boundary goes here.

3. Boundary Sobolev spaces

Generalize results of Section 1 to arbitrary compact manifolds with boundary.

4. Dirac operators

Euclidean and then general Dirac operators

5. Homogeneous translation-invariant operators

One application of the results of Section 3 is to homogeneous constant-coefficient operators on $\mathbb{R}^n$, including the Euclidean Dirac operators introduced in Section 4. Recall from Chapter 2 that an elliptic constant-coefficient operator is Fredholm, on the standard Sobolev spaces, if and only if its characteristic polynomial has no real zeros. If P is homogeneous

$$P_{ij}(t\zeta) = t^m P_{ij}(\zeta) \ \forall \ \zeta \in \mathbb{C}^n \,,\ t \in \mathbb{R}\,, \tag{6.9}$$

and elliptic, then the only real zero (of the determinant) is at $\zeta = 0$. We will proceed to discuss the radial compactification of Euclidean space to a ball, or more conveniently a half-sphere

$$\gamma_R : \mathbb{R}^n \hookrightarrow \mathbb{S}^{n,1} = \{Z \in \mathbb{R}^{n+1}\,;\, |Z| = 1\,,\ Z_0 \geq 0\}\,. \tag{6.10}$$

Transferring P to $\mathbb{S}^{n,1}$ gives

$$P_R \in Z_0^m \operatorname{Diff}^m_{\mathrm{b}}(\mathbb{S}^{n,1}\,;\, \mathbb{C}^N) \tag{6.11}$$

which is elliptic and to which the discussion in Section 3 applies.

In the 1-dimensional case, the map (6.10) reduces to the second 'projective' compactification of $\mathbb{R}$ discussed above. It can be realized globally by

$$\gamma_R(z) = \left(\frac{1}{\sqrt{1+|z|^2}}\,,\ \frac{z}{\sqrt{1+|z|^2}}\right) \in \mathbb{S}^{n,1}\,. \tag{6.12}$$

Geometrically this corresponds to a form of stereographic projection. Namely, if $\mathbb{R}^n \ni z \mapsto (1, z) \in \mathbb{R}^{n+1}$ is embedded as a 'horizontal plane' which is then projected radially onto the sphere (of radius one around the origin) one arrives at (6.12). It follows easily that γ_R is a diffeomorphism onto the open half-sphere with inverse

$$z = Z'/Z_0\,,\ Z' = (Z_1, \dots, Z_n)\,. \tag{6.13}$$

Whilst (6.12) is concise it is not a convenient form of the compactification as far as computation is concerned. Observe that

$$x \mapsto \frac{x}{\sqrt{1+x^2}}$$

is a diffeomorphism of neighborhoods of $0 \in \mathbb{R}$. It follows that Z_0, the first variable in (6.12) can be replaced, near $Z_0 = 0$, by $1/|z| = x$. That is, there is a diffeomorphism

$$\{0 \leq Z_0 \leq \epsilon\} \cap \mathbb{S}^{n,1} \leftrightarrow [0, \delta]_x \times \mathbb{S}^{n-1}_\theta \tag{6.14}$$

which composed with (6.12) gives $x = 1/|z|$ and $\theta = z/|z|$. In other words the compactification (6.12) is equivalent to the introduction of polar coordinates near infinity on $\mathbb{R}^n$ followed by inversion of the radial variable.

LEMMA 30. *If $P = (P_{ij}(D_z))$ is an $N \times N$ matrix of constant coefficient operators in $\mathbb{R}^n$ which is homogeneous of degree $-m$ then* (6.11) *holds after radial compactification. If P is elliptic then P_R is elliptic.*

Proof. This is a bit tedious if one tries to do it by direct computation. However, it is really only the homogeneity that is involved. Thus if we use the coordinates $x = 1/|z|$ and $\theta = z/|z|$ valid near the boundary of the compactification (i.e., near ∞ on $\mathbb{R}^n$) then

$$P_{ij} = \sum_{0\le\ell\le m} D_x^\ell P_{\ell,i,j}(x,\theta,D_\theta)\,,\ P_{\ell,i,j}\in\mathcal{C}^\infty(0,\delta)_x\,;\,\mathrm{Diff}^{m-\ell}(\mathbb{S}^{n-1}). \tag{6.15}$$

Notice that we *do* know that the coefficients are smooth in $0 < x < \delta$, since we are applying a diffeomorphism there. Moreover, the operators $P_{\ell,i,j}$ are uniquely determined by (6.15).

So we can exploit the assumed homogeneity of P_{ij}. This means that for any $t > 0$, the transformation $z \mapsto tz$ gives

$$P_{ij}f(tz) = t^m(P_{ij}f)(tz)\,. \tag{6.16}$$

Since $|tz| = t|z|$, this means that the transformed operator must satisfy

$$\sum_\ell D_x^\ell P_{\ell,i,j}(x,\theta,D_\theta)f(x/t,\theta) = t^m(\sum_\ell D^\ell P_{\ell,i,j}(\cdot,\theta,D_\theta)f(\cdot,\theta))(x/t)\,. \tag{6.17}$$

Expanding this out we conclude that

$$x^{-m-\ell}P_{\ell,i,j}(x,\theta,D_\theta) = P_{\ell,i,j}(\theta,D_\theta) \tag{6.18}$$

is independent of x. Thus in fact (6.15) becomes

$$P_{ij} = x^m \sum_{0\le j\le \ell} x^\ell D_x^\ell P_{\ell,j,i}(\theta,D_\theta)\,. \tag{6.19}$$

Since we can rewrite

$$x^\ell D_x = \sum_{0\le j\le\ell} C_{\ell,j}(xD_x)^j \tag{6.20}$$

(with explicit coefficients if you want) this gives (6.11). Ellipticity in this sense, meaning that

$$x^{-m}P_R \in \mathrm{Diff}^m_{\mathrm{b}}(\mathbb{S}^{n,1};\mathbb{C}^N) \tag{6.21}$$

(6.19) and the original ellipticity at P. Namely, when expressed in terms of xD_x the coefficients of 6.21 are independent of x (this of course just reflects the homogeneity), ellipticity in $x > 0$ follows by the coordinate independence of ellipticity, and hence extends down to $x = 0$. □

Now the coefficient function Z_0^{w+m} in (6.11) always gives an isomorphism

$$\times\, Z_0^m : Z_0^w H^s_{\mathrm{b}}(\mathbb{S}^{n,1}) \longrightarrow Z_0^{w+m}H^s_{\mathrm{b}}(\mathbb{S}^{n,1})\,. \tag{6.22}$$

Combining this with the results of Section 3 we find most of

Theorem 9. *If P is an $N\times N$ matrix of constant coefficient differential operators on $\mathbb{R}^n$ which is elliptic and homogeneous of degree $-m$ then there is a discrete set $-\operatorname{Im}(D(P))\subset\mathbb{R}$ such that*

$$P : Z_0^w H_b^{m+s}(\mathbb{S}^{n,1}) \longrightarrow Z_0^{w+m}H_b^s(\mathbb{S}^{n,1}) \text{ is Fredholm } \forall\ w\notin -\operatorname{Im}(D(P)) \tag{6.23}$$

where (6.12) *is used to pull these spaces back to $\mathbb{R}^n$. Moreover,*

$$\begin{gathered} P \text{ is injective for } w\in[0,\infty) \text{ and}\\ P \text{ is surjective for } w\in(-\infty,n-m]\cap(-\operatorname{Im}(D)(P))\,. \end{gathered} \tag{6.24}$$

Proof. The conclusion (6.23) is exactly what we get by applying Theorem X knowing (6.11).

To see the specific restriction (6.24) on the null space and range, observe that the domain spaces in (6.23) are tempered. Thus the null space is contained in the null space on $\mathcal{S}'(\mathbb{R}^n)$. Fourier transform shows that $P(\zeta)\hat{u}(\zeta) = 0$. From the assumed ellipticity of P and homogeneity it follows that $\operatorname{supp}(\hat{u}(\zeta)) \subset \{0\}$ and hence $\hat{u}$ is a sum of derivatives of delta functions and finally that u itself is a polynomial. If $w \geq 0$ the domain in (6.23) contains no polynomials and the first part of (6.24) follows.

The second part of (6.24) follows by a duality argument. Namely, the adjoint of P with respect to $L^2(\mathbb{R}^n)$, the usual Lebesgue space, is P^* which is another elliptic homogeneous differential operator with constant coefficients. Thus the first part of (6.24) applies to P^*. Using the homogeneity of Lebesgue measure,

$$|dz| = \frac{dx}{x^{n+1}} \cdot \nu_\theta \text{ near } \infty \tag{6.25}$$

and the shift in weight in (6.23), the second part of (6.24) follows. □

One important consequence of this is a result going back to Nirenberg and Walker (although expressed in different language).

Corollary 2. *If P is an elliptic $N \times N$ matrix constant coefficient differential operator which is homogeneous of degree m, with $n > m$, the the map* (6.23) *is an isomorphism for $w \in (0, n - m)$.*

In particular this applies to the Laplacian in dimensions $n > 2$ and to the constant coefficient Dirac operators discussed above in dimensions $n > 1$. In these cases it is also straightforward to compute the index and to identify the surjective set. Namely, for a constant coefficient Dirac operator

$$D(P) = i\mathbb{N}_0 \cup i(n - m + \mathbb{N}_0)\,. \tag{6.26}$$

Figure goes here.

6. Scattering structure

Let me briefly review how the main result of Section 5 was arrived at. To deal with a constant coefficient Dirac operator we first radially compactified $\mathbb{R}^n$ to a ball, then peeled off a multiplicative factor Z_0 from the operator showed that the remaining operator was Fredholm by identifing a neighbourhood of the boundary with part of $\mathbb{R} \times \mathbb{S}^{n-1}$ using the exponential map to exploit the results of Section 1 near infinity. Here we will use a similar, but different, procedure to treat a different class of operators which are Fredholm on the *standard* Sobolev spaces.

Although we will only apply this in the case of a ball, coming from $\mathbb{R}^n$, I cannot resist carrying out the discussed for a general compact manifolds — since I think the generality clarifies what is going on. Starting from a compact manifold with boundary, M, the first step is essentially the reverse of the radial compactification of $\mathbb{R}^n$.

Near any point on the boundary, $p \in \partial M$, we can introduce 'admissible' coordinates, $x, y_1, \ldots, y_{n-1}$ where $\{x = 0\|$ is the local form of the boundary and $y_1, \ldots, y_{n-1}$ are tangential coordinates; we normalize $y_1 = \cdots = y_{n-1} = 0$ at p. By

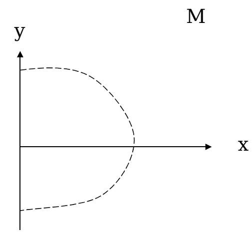

FIGURE 1. Boundary coordinate patch

reversing the radial compactification of $\mathbb{R}^n$ I mean we can introduce a diffeomorphism of a neighbourhood of p to a conic set in $\mathbb{R}^n$:

$$z_n = 1/x\,,\ z_j = y_j/x\,,\ j = 1, \dots, n-1\,. \tag{6.27}$$

Clearly the 'square' $|y| < \epsilon$, $0 < x < \epsilon$ is mapped onto the truncated conic set

$$z_n \geq 1/\epsilon\,,\ |z'| < \epsilon|z_n|\,,\ z' = (z_1, \dots, z_{n-1})\,. \tag{6.28}$$

DEFINITION 7. We define spaces $H^s_{\text{sc}}(M)$ for any compact manifold with boundary M by the requirements

$$u \in H^s_{\text{sc}}(M) \Longleftrightarrow u \in H^s_{\text{loc}}(M \setminus \partial M) \text{ and } R^*_j(\varphi_j u) \in H^s(\mathbb{R}^n) \tag{6.29}$$

for $\varphi_j \in \mathcal{C}^\infty(M)$, $0 \leq \varphi_i \leq 1$, $\sum \varphi_i = 1$ in a neighbourhood of the boundary and where each φ_j is supported in a coordinate patch (6), (6.28) with R given by (6.27).

Of course such a definition would not make much sense if it depended on the choice of the partition of unity near the boundary $\{\varphi_i\|$ or the choice of coordinate. So really (7) should be preceded by such an invariance statement. The key to this is the following observation.

PROPOSITION 23. *If we set $\mathcal{V}_{\text{sc}}(M) = x\mathcal{V}_{\text{b}}(M)$ for any compact manifold with boundary then for any $\psi \in \mathcal{C}^\infty(M)$ supported in a coordinate patch* (6)*, and any*

$\mathcal{C}^\infty$ vector field V on M

$$\psi V \in \mathcal{V}_{\mathrm{sc}}(M) \Longleftrightarrow \psi V = \sum_{j=1}^{n} \mu_j (R^{-1})_*(D_{z_j})\,,\ \mu_j \in \mathcal{C}^\infty(M)\,. \tag{6.30}$$

PROOF. The main step is to compute the form of D_{z_j} in terms of the coordinate obtained by inverting (6.27). Clearly

$$D_{z_n} = x^2 D_x\,,\ D_{z_j} = x D_{y_j} - y_i x^2 D_x\,,\ j < n\,. \tag{6.31}$$

Now, as discussed in Section 3, xD_x and D_{y_j} locally span $\mathcal{V}_{\mathrm{b}}(M)$, so $x^2 D_x$, xD_{y_j} locally span $\mathcal{V}_{\mathrm{sc}}(M)$. Thus (6.31) shows that in the singular coordinates (6.27), $\mathcal{V}_{\mathrm{sc}}(M)$ is spanned by the D_{z_ℓ}, which is exactly what (6.30) claims. $\square$

Next let's check what happens to Euclidean measure under R, actually we did this before:

$$|dz| = \frac{|dx|}{x^{n+1}} \nu_y\,. \tag{SS.9}$$

Thus we can first identify what (6.29) means in the case of $s = 0$.

LEMMA 31. *For $s = 0$, Definition (7) unambiguously defines*

$$H^0_{sc}(M) = \left\{ u \in L^2_{\mathrm{loc}}(M)\,;\ \int |u|^2 \frac{\nu_M}{x^{n+1}} < \infty \right\} \tag{6.32}$$

where ν_M is a positive smooth density on M (smooth up to the boundary of course) and $x \in \mathcal{C}^\infty(M)$ is a boundary defining function.

PROOF. This is just what (6.29) and (SS.9) mean. $\square$

Combining this with Proposition 23 we can see directly what (6.29) means for $kin\mathbb{N}$.

LEMMA 32. *If (6.29) holds for $s = k \in \mathbb{N}$ for any one such partition of unity then $u \in H^0_{sc}(M)$ in the sense of (6.32) and*

$$V_1 \dots V_j u \in H^0_{sc}(M)\ \forall\ V_i \in \mathcal{V}_{\mathrm{sc}}(M) \text{ if } j \le k\,, \tag{6.33}$$

and conversely.

PROOF. For clarity we can proceed by induction on k and replace (6.33) by the statements that $u \in H^{k-1}_{\mathrm{sc}}(M)$ and $Vu \in H^{k-1}_{\mathrm{sc}}(M)$ $\forall V \in \mathcal{V}_{\mathrm{sc}}(M)$. In the interior this is clear and follows immediately from Proposition 23 provided we carry along the inductive statement that

$$\mathcal{C}^\infty(M) \text{ acts by multiplication on } H^k_{\mathrm{sc}}(M)\,. \tag{6.34}$$

$\square$

As usual we can pass to general $s \in \mathbb{R}$ by treating the cases $0 < s < 1$ first and then using the action of the vector fields.

PROPOSITION 24. *For $0 < s < 1$ the condition (6.29) (for any one partition of unity) is equivalent to requiring $u \in H^0_{sc}(M)$ and*

$$\iint_{M\times M} \frac{|u(p) - u(p')|^2}{\rho_{sc}^{n+2s}} \frac{\nu_M}{x^{n+1}} \frac{\nu'_M}{(x')^{n+1}} < \infty \tag{6.35}$$

where $\rho_{sc}(p,p') = \chi\chi' p(p,p') + \sum_j \varphi_j \varphi'_j \langle z - z'\rangle$.

PROOF. Use local coordinates. □

Then for $s \geq 1$ if k is the integral part of s, so $0 \leq s - k < 1$, $k \in \mathbb{N}$,

$$u \in H^s_{\mathrm{sc}}(M) \Longleftrightarrow V_1, \dots, V_j u \in H^{s-k}_{\mathrm{sc}}(M)\,, V_i \in \mathcal{V}_{\mathrm{sc}}(M)\,,\ j \leq k \tag{6.36}$$

and for $s < 0$ if $k \in \mathbb{N}$ is chosen so that $0 \leq k + s < 1$, then

$$\begin{gathered} u \in H^s_{\mathrm{sc}}(M) \Leftrightarrow \exists\ V_j \in H^{s+k}_{\mathrm{sc}}(M)\,,\ j = 1, \dots, \mathbb{N}\,, \\ u_j \in H^{s-k}_{\mathrm{sc}}(M)\,, V_{j,i}(M)\,,\ 1 \leq i \leq \ell_j \leq k \text{ s.t.} \\ u = u_0 + \sum_{j=1}^{N} V_{j,i} \cdots V_{j,\ell_j} u_j. \end{gathered} \tag{6.37}$$

All this complexity is just because we are preceding in such a 'low-tech' fashion. The important point is that these Sobolev spaces are determined by the choice of 'structure vector fields', $V \in \mathcal{V}_{\mathrm{sc}}(M)$. I leave it as an important exercise to check that

LEMMA 33. *For the ball, or half-sphere,*

$$\gamma_R^* H^s_{sc}(\mathbb{S}^{n,1}) = H^s(\mathbb{R}^n)\,.$$

Thus on Euclidean space we have done *nothing*. However, my claim is that we understand things better by doing this! The idea is that we should Fourier analysis on $\mathbb{R}^n$ to analyse differential operators which are made up out of $\mathcal{V}_{\mathrm{sc}}(M)$ on any compact manifold with boundary M, and this includes $\mathbb{S}^{n,1}$ as the radial compactification of $\mathbb{R}^n$. Thus set

$$\begin{aligned} \mathrm{Diff}^m_{\mathrm{sc}}(M) = \big\{ P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M);\ \exists\ f \in \mathcal{C}^\infty(M) \text{ and} \\ V_{i,j} \in \mathcal{V}_{\mathrm{sc}}(M) \text{ s.t. } P = f + \sum_{i,1 \leq j \leq m} V_{i,1} \dots V_{i,j} \big\}. \end{aligned} \tag{6.38}$$

In local coordinates this is just a differential operator and it is smooth up to the boundary. Since only scattering vector fields are allowed in the definition such an operator is quite degenerate at the boundary. It always looks like

$$P = \sum_{k+|\alpha| \leq m} a_{k,\alpha}(x, y)(x^2 D_x)^k (x D_y)^\alpha, \tag{6.39}$$

with smooth coefficients in terms of local coordinates (6).

Now, if we *freeze* the coefficients at a point, p, on the boundary of M we get a polynomial

$$\sigma_{\mathrm{sc}}(P)(p) = \sum_{k+|\alpha| \leq m} a_{k,\alpha}(p) \tau^k \eta^\alpha. \tag{6.40}$$

Note that this is *not* in general homogeneous since the lower order terms are retained. Despite this one gets essentially the same polynomial at each point, independent of the admissible coordinates chosen, as will be shown below. Let's just assume this for the moment so that the condition in the following result makes sense.

THEOREM 10. *If* $P \in \mathrm{Diff}^m_{sc}(M;\mathbb{E})$ *acts between vector bundles over* M*, is elliptic in the interior and each of the polynomials (matrices)* (6.40) *is elliptic and has no real zeros then*

$$P : H^{s+m}_{sc}(M, E_1) \longrightarrow H^s_{sc}(M; E_2) \text{ is Fredholm} \tag{6.41}$$

for each $s \in \mathbb{R}$ *and conversely.*

7. Manifolds with corners

8. Blow up

Last time at the end I gave the following definition and theorem.

DEFINITION 8. We define weighted (non-standard) Sobolev spaces for $(m, w) \in \mathbb{R}^2$ on $\mathbb{R}^n$ by

$$\tilde{H}^{m,w}(\mathbb{R}^n) = \{u \in M^m_{\mathrm{loc}}(\mathbb{R}^n); F^*\left((1-\chi)r^{-w}u\right) \in H^m_{\mathrm{ti}}(\mathbb{R} \times \mathbb{S}^{n-1})\} \tag{6.42}$$

where $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\chi(y) = 1$ in $|y| < 1$ and

$$F : \mathbb{R} \times \mathbb{S}^{n-1} \ni (t, \theta) \longrightarrow (e^t, e^t\theta) \in \mathbb{R}^n \setminus \{0\}. \tag{6.43}$$

THEOREM 11. *If* $P = \sum\limits_{i=1}^n \Gamma_i D_i$, $\Gamma_i \in M(N, \mathbb{C})$*, is an elliptic, constant coefficient, homogeneous differential operator of first order then*

$$P : \tilde{H}^{m,w}(\mathbb{R}^n) \longrightarrow \tilde{H}^{m-1,w+1}(\mathbb{R}^n) \ \forall\ (m, w) \in \mathbb{R}^2 \tag{6.44}$$

is continuous and is Fredholm for $w \in \mathbb{R} \setminus \tilde{D}$ *where* $\tilde{D}$ *is discrete.*

If P *is a Dirac operators, which is to say explicitly here that the coefficients are 'Pauli matrices' in the sense that*

$$\Gamma_i^* = \Gamma_i,\ \Gamma_i^2 = \mathrm{Id}_{N\times N},\ \forall\ i,\ \Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 0,\ i \neq j, \tag{6.45}$$

then

$$\tilde{D} = -\mathbb{N}_0 \cup (n - 2 + \mathbb{N}_0) \tag{6.46}$$

and if $n > 2$ *then for* $w \in (0, n-2)$ *the operator* P *in* (6.44) *is an isomorphism.*

I also proved the following result from which this is derived

LEMMA 34. *In polar coordinates on* $\mathbb{R}^n$ *in which* $\mathbb{R}^n \setminus \{0\} \simeq (0, \infty) \times \mathbb{S}^{n-1}$, $y = r\theta$,

$$D_{y_j} = \tag{6.47}$$

CHAPTER 7

Electromagnetism

0.6Q; Revised: 6-8-2007; Run: February 7, 2008

1. Maxwell's equations

Maxwell's equations in a vacuum take the standard form

$$\operatorname{div}\mathbf{E} = \rho \qquad \operatorname{div}\mathbf{B} = 0$$
$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \operatorname{curl}\mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \tag{7.1}$$

where $\mathbf{E}$ is the electric and $\mathbf{B}$ the magnetic field strength, both are 3-vectors depending on position $z \in \mathbb{R}^3$ and time $t \in \mathbb{R}$. The external quantities are ρ, the charge density which is a scalar, and $\mathbf{J}$, the current density which is a vector.

We will be interested here in stationary solutions for which $\mathbf{E}$ and $\mathbf{B}$ are independent of time and with $\mathbf{J} = 0$, since this also represents motion in the standard description. Thus we arrive at

$$\operatorname{div}\mathbf{E} = \rho \qquad \operatorname{div}\mathbf{B} = 0$$
$$\operatorname{curl}\mathbf{E} = 0 \qquad \operatorname{curl}\mathbf{B} = 0. \tag{7.2}$$

The simplest interesting solutions represent charged particles, say with the charge at the origin, $\rho = c\delta_0(z)$, and with no magnetic field, $\mathbf{B} = 0$. By identifying $\mathbf{E}$ with a 1-form, instead of a vector field on $\mathbb{R}^3$,

$$\mathbf{E} = (E_1, E_2, E_3) \Longrightarrow e = E_1 dz_1 + E_2 dz_2 + E_3 dz_3 \tag{7.3}$$

we may identify $\operatorname{curl}\mathbf{E}$ with the 2-form de,

$$de = \left(\frac{\partial E_2}{\partial z_1} - \frac{\partial E_1}{\partial z_2}\right) dz_1 \wedge dz_2 + \left(\frac{\partial E_3}{\partial z_2} - \frac{\partial E_2}{\partial z_3}\right) dz_2 \wedge dz_3 + \left(\frac{\partial E_1}{\partial z_3} - \frac{\partial E_3}{\partial z_1}\right) dz_3 \wedge dz_1. \tag{7.4}$$

Thus (7.2) implies that e is a closed 1-form, satisfying

$$\frac{\partial E_1}{\partial z_1} + \frac{\partial E_2}{\partial z_2} + \frac{\partial E_3}{\partial z_3} = c\delta_0(z). \tag{7.5}$$

By the Poincaré Lemma, a closed 1-form on $\mathbb{R}^3$ is exact, $e = dp$, with p determined up to an additive constant. If e is smooth (which it cannot be, because of (7.5)), then

$$p(z) - p(z') = \int_0^1 \gamma^* e \qquad \text{along } \gamma : [0,1] \longrightarrow \mathbb{R}^3, \gamma(0) = z', \gamma(1) = z. \tag{7.6}$$

It is reasonable to look for a particular p and 1-form e which satisfy (7.5) and are smooth outside the origin. Then (7.6) gives a potential which is well defined, up to an additive constant, outside 0, once z' is fixed, since $de = 0$ implies that the integral of $\gamma^* e$ along a closed curve vanishes. This depends on the fact that $\mathbb{R}^3 \backslash \{0\}$ is simply connected.

So, modulo confirmation of these simple statements, it suffices to look for $p \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash \{0\})$ satisfying $e = dp$ and (7.5), so

$$\Delta p = -\left(\frac{\partial^2 p}{\partial z_1^2} + \frac{\partial^2 p}{\partial z_2^2} + \frac{\partial^2 p}{\partial z_3^2}\right) = -c\delta_0(z). \tag{7.7}$$

Then $\mathbf{E}$ is recovered from $e = dp$.

The operator 'div' can also be understood in terms of de Rham d together with the Hodge star $*$. If we take $\mathbb{R}^3$ to have the standard orientation and Euclidean metric $dz_1^2 + dz_2^2 + dz_3^2$, the Hodge star operator is given on 1-forms by

$$* \, dz_1 = dz_2 \wedge dz_3, \quad *dz_2 = dz_3 \wedge dz_1, \quad *dz_3 = dz_1 \wedge dz_2. \tag{7.8}$$

Thus $*e$ is a 2-form,

$$\begin{aligned} & * \, e = E_1 \, dz_2 \wedge dz_3 + E_2 \, dz_3 \wedge dz_1 + E_3 \, dz_1 \wedge dz_2 \\ & \Longrightarrow d * e = \left(\frac{\partial E_1}{\partial z_1} + \frac{\partial E_2}{\partial z_2} + \frac{\partial E_3}{\partial z_3}\right) dz_1 \wedge dz_2 \wedge dz_3 = (\operatorname{div} \mathbf{E}) \, dz_1 \wedge dz_2 \wedge dz_3. \end{aligned} \tag{7.9}$$

The stationary Maxwell's equations on e become

$$d * e = \rho \, dz_1 \wedge dz_2 \wedge dz_3, \ de = 0. \tag{7.10}$$

There is essential symmetry in (7.1) except for the appearance of the "source" terms, ρ and $\mathbf{J}$. To reduce (7.1) to two equations, analogous to (7.10) but in 4-dimensional (Minkowski) space requires $\mathbf{B}$ to be identified with a 2-form on $\mathbb{R}^3$, rather than a 1-form. Thus, set

$$\beta = B_1 \, dz_2 \wedge dz_3 + B_2 \, dz_3 \wedge dz_1 + B_3 \, dz_1 \wedge dz_2. \tag{7.11}$$

Then

$$d\beta = \operatorname{div} \mathbf{B} \, dz_1 \wedge dz_2 \wedge dz_3 \tag{7.12}$$

as follows from (7.9) and the second equation in (7.1) implies β is closed.

Thus e and β are respectively a closed 1-form and a closed 2-form on $\mathbb{R}^3$. If we return to the general time-dependent setting then we may define a 2-form on $\mathbb{R}^4$ by

$$\lambda = e \wedge dt + \beta \tag{7.13}$$

where e and β are pulled back by the projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$. Computing directly,

$$d\lambda = d'e \wedge dt + d'\beta + \frac{\partial \beta}{\partial t} \wedge dt \tag{7.14}$$

where d' is now the differential on $\mathbb{R}^3$. Thus

$$d\lambda = 0 \Leftrightarrow d'e + \frac{\partial \beta}{\partial t} = 0, \quad d'\beta = 0 \tag{7.15}$$

recovers two of Maxwell's equations. On the other hand we can define a 4-dimensional analogue of the Hodge star but corresponding to the Minkowski metric, not the Euclidean one. Using the natural analogue of the 3-dimensional Euclidean Hodge by formally inserting an i into the t-component, gives

$$\begin{cases} *_4 dz_1 \wedge dz_2 = i dz_3 \wedge dt \\ *_4 dz_1 \wedge dz_3 = i dt \wedge dz_2 \\ *_4 dz_1 \wedge dt = -i dz_2 \wedge dz_3 \\ *_4 dz_2 \wedge dz_3 = i dz_1 \wedge dt \\ *_4 dz_2 \wedge dt = -i dz_3 \wedge dz_1 \\ *_4 dz_3 \wedge dt = -i dz_1 \wedge dz_2. \end{cases} \tag{7.16}$$

The other two of Maxwell's equations then become

$$d *_4 \lambda = d(-i * e + i(*\beta) \wedge dt) = -i(\rho\, dz_1 \wedge dz_2 \wedge dz_3 + j \wedge dt) \tag{7.17}$$

where j is the 1-form associated to $\mathbf{J}$ as in (7.3). For our purposes this is really just to confirm that it is best to think of $\mathbf{B}$ as the 2-form β rather than try to make it into a 1-form. There are other good reasons for this, related to behaviour under linear coodinate changes.

Returning to the stationary setting, note that (7.7) has a 'preferred' solution

$$p = \frac{1}{4\pi|z|}. \tag{7.18}$$

This is in fact the only solution which vanishes at infinity.

Proposition 25. *The only tempered solutions of* (7.7) *are of the form*

$$p = \frac{1}{4\pi|z|} + q,\ \Delta q = 0,\ q \text{ a polynomial.} \tag{7.19}$$

Proof. The only solutions are of the form (7.19) where $q \in \mathcal{S}'(\mathbb{R}^3)$ is harmonic. Thus $\widehat{q} \in \mathcal{S}'(\mathbb{R}^3)$ satisfies $|\xi|^2\widehat{q} = 0$, which implies that q is a polynomial. $\square$

2. Hodge Theory

The Hodge $*$ operator discussed briefly above in the case of $\mathbb{R}^3$ (and Minkowski 4-space) makes sense in any oriented real vector space, V, with a Euclidean inner product—that is, on a finite dimensional real Hilbert space. Namely, if $e_1, \dots, e_n$ is an oriented orthonormal basis then

$$*\,(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \operatorname{sgn}(i_*) e_{i_{k+1}} \wedge \cdots e_{i_n} \tag{7.20}$$

extends by linearity to

$$* : \bigwedge^k V \longrightarrow \bigwedge^{n-k} V. \tag{7.21}$$

Proposition 26. *The linear map* (7.21) *is independent of the oriented orthonormal basis used to define it and so depends only on the choice of inner product and orientation of V. Moreover,*

$$*^2 = (-1)^{k(n-k)},\ \text{on } \bigwedge^k V. \tag{7.22}$$

Proof. Note that $\operatorname{sgn}(i_*)$, the sign of the permutation defined by $\{i_1, \dots, i_n\}$ is fixed by

$$e_{i_1} \wedge \cdots \wedge e_{i_n} = \operatorname{sgn}(i_*) e_1 \wedge \cdots \wedge e_n. \tag{7.23}$$

Thus, on the basis $e_{i_1} \wedge \ldots \wedge e_{i_n}$ of $\bigwedge^k V$ given by strictly increasing sequences $i_1 < i_2 < \cdots < i_k$ in $\{1, \dots, n\}$,

$$e_* \wedge *e_* = \operatorname{sgn}(i_*)^2 e_1 \wedge \cdots \wedge e_n = e_1 \wedge \cdots \wedge e_n. \tag{7.24}$$

The standard inner product on $\bigwedge^k V$ is chosen so that this basis is orthonormal. Then (7.24) can be rewritten

$$e_I \wedge *e_J = \langle e_I, e_J \rangle e_1 \wedge \cdots \wedge e_n. \tag{7.25}$$

This in turn fixes $*$ uniquely since the pairing given by

$$\bigwedge^k V \times \bigwedge^{k-1} V \ni (u, v) \longmapsto (u \wedge v)/_{e_1 \wedge \cdots \wedge e_n} \tag{7.26}$$

is non-degenerate, as can be checked on these bases.

Thus it follows from (7.25) that $*$ depends only on the choice of inner product and orientation as claimed, provided it is shown that the inner product on $\bigwedge^k V$ only depends on that of V. This is a standard fact following from the embedding

$$\bigwedge^k V \hookrightarrow V^{\otimes k} \tag{7.27}$$

as the totally antisymmetric part, the fact that $V^{\otimes k}$ has a natural inner product and the fact that this induces one on $\bigwedge^k V$ after normalization (depending on the convention used in (7.27). These details are omitted. $\square$

Since $*$ is uniquely determined in this way, it necessarily depends smoothly on the data, in particular the inner product. On an oriented Riemannian manifold the induced inner product on $T_p^* M$ varies smoothly with p (by assumption) so

$$* : \bigwedge_p^k M \longrightarrow \bigwedge_p^{n-k} M,\ \bigwedge_p^k M = \bigwedge_p^k (T_p^* M) \tag{7.28}$$

varies smoothly and so defines a smooth bundle map

$$* \in \mathcal{C}^\infty(M; \bigwedge^k M, \bigwedge^{n-k} M). \tag{7.29}$$

An oriented Riemannian manifold carries a natural volume form $\nu \in \mathcal{C}^\infty(M, \bigwedge^n M)$, and this allows (7.25) to be written in integral form:

$$\int_M \langle \alpha, \beta \rangle \, \nu = \int_M \alpha \wedge *\beta \ \forall\, \alpha, \beta \in \mathcal{C}^\infty(M, \bigwedge^k M). \tag{7.30}$$

LEMMA 35. *On an oriented, (compact) Riemannian manifold the adjoint of d with respect to the Riemannian inner product and volume form is*

$$d^* = \delta = (-1)^{k+n(n-k+1)} * d * \ \text{ on } \bigwedge^k M. \tag{7.31}$$

PROOF. By definition,

$$\begin{aligned} d : \mathcal{C}^\infty(M, \bigwedge^k M) &\longrightarrow \mathcal{C}^\infty(M, \bigwedge^{k+1} M) \\ &\Longrightarrow \delta : \mathcal{C}^\infty(M, \bigwedge^{k+1} M) \longrightarrow \mathcal{C}^\infty(M, \bigwedge^k M), \\ \int_M \langle d\alpha, \alpha' \rangle \, \nu = \int_M \langle \alpha, \delta\alpha' \rangle \, \nu \ &\forall\, \alpha \in \mathcal{C}^\infty(M, \bigwedge^k M), \alpha' \in \mathcal{C}^\infty(M, \bigwedge^{k+1} M). \end{aligned} \tag{7.32}$$

Applying (7.30) and using Stokes' theorem, (and compactness of either M or the support of at least one of α, α'),

$$\begin{aligned} \int_M \langle \delta\alpha, \alpha' \rangle \, \nu &= \int_M d\alpha \wedge *\alpha' \\ &= \int_M d(\alpha \wedge *\alpha') + (-1)^{k+1} \int_M \alpha \wedge d * \alpha' = 0 + (-1)^{k+1} \int_M \langle \alpha, *^{-1} d * \alpha' \rangle \, \nu. \end{aligned}$$

Taking into account (7.22) to compute $*^{-1}$ on $n-k$ forms shows that

$$\delta\alpha' = (-1)^{k+1+n(n-k)} * d * \quad \text{ on } (k+1)\text{-forms} \tag{7.33}$$

which is just (7.31) on k-forms. $\square$

Notice that changing the orientation simply changes the sign of $*$ on all forms. Thus (7.31) does not depend on the orientation and as a local formula is valid even if M is not orientable — since the existence of $\delta = d^*$ does *not* require M to be orientable.

THEOREM 12 (Hodge/Weyl). *On any compact Riemannian manifold there is a canonical isomorphism*

$$H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{Ho}}(M) = \left\{u \in L^2(M; \textstyle\bigwedge^k M);\ (d+\delta)u = 0\right\} \tag{7.34}$$

where the left-hand side is either the $\mathcal{C}^\infty$ or the distributional de Rham cohomology

$$\begin{aligned}\left\{u \in \mathcal{C}^\infty(M; \textstyle\bigwedge^k M);\ du = 0\right\} \Big/ d\,\mathcal{C}^\infty(M; \textstyle\bigwedge^k M) \\ \cong \left\{u \in \mathcal{C}^{-\infty}(M; \textstyle\bigwedge^k M);\ du = 0\right\} \Big/ d\,\mathcal{C}^{-\infty}(M; \textstyle\bigwedge^k M).\end{aligned} \tag{7.35}$$

PROOF. The critical point of course is that

$$d + \delta \in \mathrm{Diff}^1(M; \textstyle\bigwedge^* M) \text{ is elliptic.} \tag{7.36}$$

We know that the symbol of d at a point $\zeta \in T^*_pM$ is the map

$$\textstyle\bigwedge^k M \ni \alpha \mapsto i\zeta \wedge \alpha. \tag{7.37}$$

We are only interested in $\zeta \neq 0$ and by homogeneity it is enough to consider $|\zeta| = 1$. Let $e_1 = \zeta$, e_2, ..., e_n be an orthonormal basis of T^*_pM, then from (7.31) with a fixed sign throughout:

$$\sigma(\delta,\zeta)\alpha = \pm * (i\zeta \wedge \cdot) * \alpha. \tag{7.38}$$

Take $\alpha = e_I$, $*\alpha = \pm e_{I'}$ where $I \cup I' = \{1, \dots, n\}$. Thus

$$\sigma(\delta,\zeta)\alpha = \left\{ \begin{array}{ll} 0 & 1 \notin I \\ \pm i\alpha_{I\setminus\{1\}} & 1 \in I \end{array}\right. . \tag{7.39}$$

In particular, $\sigma(d+\delta)$ is an isomorphism since it satisfies

$$\sigma(d+\delta)^2 = |\zeta|^2 \tag{7.40}$$

as follows from (7.37) and (7.39) or directly from the fact that

$$(d+\delta)^2 = d^2 + d\delta + \delta d + \delta^2 = d\delta + \delta d \tag{7.41}$$

again using (7.37) and (7.39).

Once we know that $d+\delta$ is elliptic we conclude from the discussion of Fredholm properties above that the distributional null space

$$\left\{u \in \mathcal{C}^{-\infty}(M, \textstyle\bigwedge^* M);\ (d+\delta)u = 0\right\} \subset \mathcal{C}^\infty(M, \textstyle\bigwedge^* M) \tag{7.42}$$

is finite dimensional. From this it follows that

$$\begin{aligned} H^k_{\mathrm{Ho}} =& \{u \in \mathcal{C}^{-\infty}(M, \textstyle\bigwedge^k M);\ (d+\delta)u = 0\} \\ =& \{u \in \mathcal{C}^\infty(M, \textstyle\bigwedge^k M);\ du = \delta u = 0\}\end{aligned} \tag{7.43}$$

and that the null space in (7.42) is simply the direct sum of these spaces over k. Indeed, from (7.42) the integration by parts in

$$0 = \int \langle du, (d+\delta)u\rangle\,\nu = \|du\|^2_{L^2} + \int \langle u, \delta^2 u\rangle\,\nu = \|du\|^2_{L^2}$$

is justified.

Thus we can consider $d+\delta$ as a Fredholm operator in three forms

$$\begin{aligned} d+\delta :&\mathcal{C}^{-\infty}(M,\textstyle\bigwedge^* M) \longrightarrow \mathcal{C}^{-\infty}(M,\textstyle\bigwedge^* M),\\ d+\delta :&H^1(M,\textstyle\bigwedge^* M) \longrightarrow H^1(M,\textstyle\bigwedge^* M),\\ d+\delta :&\mathcal{C}^{\infty}(M,\textstyle\bigwedge^* M) \longrightarrow \mathcal{C}^{\infty}(M,\textstyle\bigwedge^* M)\end{aligned} \tag{7.44}$$

and obtain the three direct sum decompositions

$$\begin{aligned} \mathcal{C}^{-\infty}(M,\textstyle\bigwedge^* M) &= H^*_{\mathrm{Ho}} \oplus (d+\delta)\mathcal{C}^{-\infty}(M,\textstyle\bigwedge^* M),\\ L^2(M,\textstyle\bigwedge^* M) &= H^*_{\mathrm{Ho}} \oplus (d+\delta)L^2(M,\textstyle\bigwedge^* M),\\ \mathcal{C}^{\infty}(M,\textstyle\bigwedge^* M) &= H^*_{\mathrm{Ho}} \oplus (d+\delta)\mathcal{C}^{\infty}(M,\textstyle\bigwedge^* M).\end{aligned} \tag{7.45}$$

The same complement occurs in all three cases in view of (7.43).

From (7.43) directly, all the "harmonic" forms in $H^k_{\mathrm{Ho}}(M)$ are closed and so there is a natural map

$$H^k_{\mathrm{Ho}}(M) \longrightarrow H^k_{\mathrm{dR}}(M) \longrightarrow H^k_{\mathrm{dR},\mathcal{C}^{-\infty}}(M) \tag{7.46}$$

where the two de Rham spaces are those in (7.35), not yet shown to be equal.

We proceed to show that the maps in (7.46) are isomorphisms. First to show injectivity, suppose $u \in H^k_{\mathrm{Ho}}(M)$ is mapped to zero in either space. This means $u = dv$ where v is either $\mathcal{C}^\infty$ or distributional, so it suffices to suppose $v \in \mathcal{C}^{-\infty}(M,\bigwedge^{k-1} M)$. Since u is smooth the integration by parts in the distributional pairing

$$\|u\|^2_{L^2} = \int_M \langle u, dv\rangle\,\nu = \int_M \langle \delta u, v\rangle\,\nu = 0$$

is justified, so $u = 0$ and the maps are injective.

To see surjectivity, use the Hodge decomposition (7.45). If $u' \in \mathcal{C}^{-\infty}(M,\bigwedge^k M)$ or $\mathcal{C}^{\infty}(M,\bigwedge^k M)$, we find

$$u' = u_0 + (d+\delta)v \tag{7.47}$$

where correspondingly, $v \in \mathcal{C}^{-\infty}(M,\bigwedge^* M)$ or $\mathcal{C}^{\infty}(M,\bigwedge^* M)$ and $u_0 \in H^k_{\mathrm{Ho}}(M)$. If u' is closed, $du' = 0$, then $d\delta v = 0$ follows from applying d to (7.47) and hence $(d+\delta)\delta v = 0$, since $\delta^2 = 0$. Thus $\delta v \in H^*_{\mathrm{Ho}}(M)$ and in particular, $\delta v \in \mathcal{C}^\infty(M,\bigwedge^* M)$. Then the integration by parts in

$$\|\delta v\|^2_{L^2} = \int \langle \delta v, \delta v\rangle\,\nu = \int \langle v, (d+\delta)\delta v\rangle\,\nu = 0$$

is justified, so $\delta v = 0$. Then (7.47) shows that any closed form, smooth or distributional, is cohomologous in the same sense to $u_0 \in H^k_{\mathrm{Ho}}(M)$. Thus the natural maps (7.46) are isomorphisms and the Theorem is proved. $\square$

Thus, on a compact Riemannian manifold (whether orientable or not), each de Rham class has a unique harmonic representative.

3. Coulomb potential

4. Dirac strings

Addenda to Chapter 7

CHAPTER 8

Monopoles

$\boxed{\boxed{\text{0.6Q; Revised: 6-8-2007; Run: February 7, 2008}}}$

1. Gauge theory

2. Bogomolny equations

(1) Compact operators, spectral theorem
(2) Families of Fredholm operators(*)
(3) Non-compact self-adjoint operators, spectral theorem
(4) Spectral theory of the Laplacian on a compact manifold
(5) Pseudodifferential operators(*)
(6) Invertibility of the Laplacian on Euclidean space
(7) Lie groups(‡), bundles and gauge invariance
(8) Bogomolny equations on $\mathbb{R}^3$
(9) Gauge fixing
(10) Charge and monopoles
(11) Monopole moduli spaces

* I will drop these if it looks as though time will become an issue.

†,‡ I will provide a brief and elementary discussion of manifolds and Lie groups if that is found to be necessary.