18.156 – GRADUATE ANALYSIS
ELLIPTIC REGULARITY AND SCATTERING

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ABSTRACT.

BRIEF LECTURE NOTES

Lecture 6, 26 February, 2008

- Spectral/scattering theory for $\Delta + V(z)$ on $\mathbb{R}^n$. Initially, Δ will be the flat Laplacian, later it may get to have variable coefficients.
- Spectral theory for the Laplacian on a compact manifold. I will do some of this at various points later.
- Need to finish local elliptic regularity, but I will let you think about where we are for a while and do some lower-tech stuff.
- Today: Understand all tempered solutions of $(\Delta - \lambda)u = 0$ for $\lambda \in \mathbb{C}$.
- We already know this that for $\lambda \in \mathbb{C} \setminus [0, \infty)$ $u \equiv 0$ is the only solution.
- For $\lambda = 0$ we know that all tempered solutions are polynomials and so the null space is the (infinite-dimensional) space of harmonic polynomials.
- If $\lambda > 0$ set $\lambda = \tau^2$ with $\tau > 0$. Really we only need think about the case $\tau = 1$ since there is a scaling isomorphism

$$\{u \in \mathcal{S}'(\mathbb{R}^n); (\Delta - \tau^2)u = 0\} \ni u(z) \longmapsto u(\tau z) \in \{v \in \mathcal{S}'(\mathbb{R}^n); (\Delta - 1)v = 0\}. \tag{1}$$

  I will keep the τ anyway.
- Let's construct some solutions, which we can do using the Fourier transform since we want

$$(|\xi|^2 - \tau^2)\hat{u}(\xi) = 0. \tag{2}$$

  We know that this means $\hat{u}$ has support in $|\xi|^2 = \tau^2$. In fact the general tempered solution can be written down explicitly in terms of its pairing with $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$u(\overline{\phi}) = (2\pi)^{-n}(\hat{u})(\overline{\hat{\phi}}) = \int_{\mathbb{S}^{n-1}} F(\omega)\overline{\hat{\phi}}(\tau\omega)d\omega. \tag{3}$$

  I will look at this for $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ when it makes sense as an integral. In fact it makes sense as a distributional pairing for F a distribution on the sphere, but then I would have to talk about distributions on the sphere.
- Inserting the definition of the Fourier transform we see that

$$u(\phi) = \int_{\mathbb{S}^n \times \mathbb{R}^n} F(\omega) \exp(i\tau z \cdot \omega)\phi(z)dzd\omega. \tag{4}$$

This shows something we already know, that $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ and gives the integral formula

$$u(z) = \int_{\mathbb{S}^{n-1}} F(\omega) e^{i\tau z\cdot\omega} d\omega. \tag{5}$$

- What I want to talk about today, is the asymptotic behaviour of $u(z)$ as $|z| \to \infty$. Let's fix a direction and set $z = R\theta$, $\theta \in \mathbb{S}^{n-1}$ and look at

$$u(R\theta) = \int_{\mathbb{S}^{n-1}} F(\omega) e^{iR\tau\theta\cdot\omega} d\omega. \tag{6}$$

- In fact it is clear that if we rotate θ, sending it to $O\theta$ for an orthogonal transformation O then $O\theta \cdot \omega = \theta \cdot O^{-1}\theta$ so setting $\omega' = O^{-1}\omega$,

$$u(RO\theta) = \int_{\mathbb{S}^{n-1}} F(O\omega') e^{iR\tau\theta\cdot\omega'} d\omega' \tag{7}$$

we see that the effect is the same as rotating F. Since F was arbitrary anyway, we may as well set $\theta = (1, 0, \dots, 0)$ and worry about the general case afterwards. Thus

$$u(R, 0, \dots, 0) = \int_{\mathbb{S}^{n-1}} F(\omega) e^{iR\tau\omega_1} d\omega. \tag{8}$$

- We can use a partition of unity to decompose F into pieces with small support. First note that if $\operatorname{supp}(F) \cap \pm(1, 0, \dots, 0)$ then $u(R, 0, \dots, 0)$ is rapidly decreasing as $R \to \infty$. Change variables and use the Fourier transform.
- If F has support near either $\pm(1, 0, \dots, 0)$ then we can introduce $n-1$ local coordinates on the sphere such that

$$\omega_1 = \pm(1 \mp (y_1^2 + \dots + y_{n-1}^2)). \tag{9}$$

This allows us to evaluate the integral in the sense that (8) implies that

$$u(R, 0, \dots, 0) = e^{iR\tau} R^{-(n-1)/2} G_+(\frac{1}{R}) + e^{-iR\tau} R^{-(n-1)/2} G_-(\frac{1}{R}), \quad \text{as } R \to \infty \tag{10}$$

where $G_\pm(x) = \in \mathcal{C}^\infty([0, 1))$ and

$$G_\pm(0) = c_n^\pm F(\pm(1, 0, \dots, 0)), \ c_n^\pm = . \tag{11}$$

- Thus we can see what these solutions look like in general using our ability to rotate the angle θ :

$$\begin{gathered} u(R\theta) = e^{iR\tau} R^{-(n-1)/2} G_+(\frac{1}{R}, \theta) + e^{-iR\tau} R^{-(n-1)/2} G_-(\frac{1}{R}, \theta), \quad \text{as } R \to \infty, \\ G_\pm \in \mathcal{C}^\infty([0, 1) \times \mathbb{S}^{n-1}), \ G_\pm(0, \theta) = c_n^\pm F(\pm\theta). \end{gathered} \tag{12}$$

Lecture 4, 14 Feb Parametrices in open sets.

(a) For a constant coefficient elliptic operator $P(D)$ and $\Omega \subset \mathbb{R}^n$ we want to construct a parametrix, a continuous linear operator

$$Q_\Omega : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega) \text{ s.t. } P(D)Q_\Omega = \operatorname{Id} - R_\Omega \tag{13}$$

where R_Ω is a smoothing operator.

(b) Smoothing operators and proper supports.

(c) If $\hat{Q} = \frac{1-\chi(\xi)}{P(\xi)}$ where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\chi = 1$ near any zeros of P, then

$$Q_\Omega f(z) = \int \mu(z,z')Q(z-z')dz' \tag{14}$$

is a parametrix where $\mu \in \mathcal{C}^\infty(\Omega^2)$ has proper support and is equal to 1 in a neighbourhood of the diagonal.

Lecture 3, 12 Feb Local elliptic regularity (constant coefficients).

(a) Finish the proof of (17).

Chains of cutoffs. If $\Omega \subset \mathbb{R}^n$ is open and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ has support in Ω then for any k there is a sequence $\psi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ where with the notation $\psi = \psi_0$

$$\operatorname{supp}(\psi_{j-1}) \subset \{p \in \Omega; \psi_j(x) = 1 \text{ in } B(p,\epsilon) \text{ for } \epsilon > 0\} \subset \operatorname{supp}(\psi_j) \Subset \Omega. \tag{15}$$

Lecture 2, 7 Feb Sobolev spaces and ellipticity

- Sobolev spaces recalled – LN (=Lecture notes) 'Sobolev spaces' section of Chapter 1.
- Elliptic constant coefficient operators – LN Chapter 2, §2.
- I 'proved' that $P(D)$ of order m defines continuous linear map

$$P(D) : H^{s+m}_{\text{loc}}(\Omega) \longrightarrow H^s_{\text{loc}}(\Omega). \tag{16}$$

Started, but did not finish, the basic elliptic regularity result that

$$u \in \mathcal{C}^{-\infty}(\Omega),\ P(D)u \in H^s_{\text{loc}}(\Omega) \Longrightarrow u \in H^{s+m}_{\text{loc}}(\Omega). \tag{17}$$

Lecture 1, 5 Feb

- What I plan to do in this course:-
  Local elliptic regularity:- If $P(z, D_z)$ is elliptic with smooth coefficients then $Pu = f$ with f smooth implies u is smooth.
  Spectral theory of the Laplacian $-\varDelta$ on a compact manifold.
  Scattering theory.
  Wave equation, maybe.
- Precursor:- The Laplacian on Euclidean space and tori.
- On $\mathbb{R}^n$, $\varDelta - \lambda$ is invertible on $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ for $\lambda \in \mathbb{C} \setminus [0,\infty)$. Recall the Fourier transform.
- For $\lambda > 0$ the only solutions have inverse Fourier transform $u(\omega)\delta(r - \lambda^{\frac{1}{2}})$.
- Eigenvalues of the Laplacian on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ by looking at periodic functions on $\mathbb{R}^n$. The general case of the Laplacian for a Riemann metric on a compact manifold is similar!

Problems following Lecture 6 – 26 February, 2008

Problem 1. Prove the Fourier transform formula in one dimensional space that

$$\mathcal{F}(\exp(\frac{i}{2}x^2) = \sqrt{\pi}\exp(\frac{i\pi}{8})\exp(-\frac{i}{2}\xi^2). \tag{18}$$

Hint. Observe that $\exp(\frac{i}{2}x^2)$ is bounded and continuous, so is an element of $\mathcal{S}'(\mathbb{R})$. Furthermore, for $t > 0$,

$$u_t = \exp(\frac{i-t}{2}x^2) \in \mathcal{S}(\mathbb{R}),\ \lim t \downarrow 0 u_t = \exp(\frac{i}{2}x^2) \text{ in } \mathcal{S}'(\mathbb{R}). \tag{19}$$

Thus it suffices to compute the Fourier transform of u_t. Now,

$$\begin{gathered}\widehat{u_t}(\xi)=\int_{\mathbb{R}} e^{-ix\xi}e^{\frac{i-t}{2}x^2}dx \text{ satisfies}\\ (\frac{d}{d\xi}+(i-t)^{-1}\xi)\widehat{u_t}(\xi)=0 \Longrightarrow \widehat{u_t}(\xi)=c(t)\exp((i-t)^{-1}\frac{\xi^2}{2})\end{gathered} \tag{20}$$

as follows by differentiating under the absolutely convergent integral. Thus it suffices to compute the constant, which is the value at $\xi=0$. In fact it is convenient to consider

$$f(z)=\int_{\mathbb{R}} e^{-z2x^2}dx \text{ for } \operatorname{Re}(z)>0 \tag{21}$$

in which region the integral converges absolutely, so $f(z)$ is holomorphic. For $z>0$ this is a Gaussian integral so

$$f(z)=\sqrt{\pi}z^{-\frac{1}{2}}, \tag{22}$$

for the main branch of the square root. Initially this is true for $z>0$ but holds in $\operatorname{Re}(z)>0$ by the uniqueness of analytic continuation. Thus in fact $c(t)$ can be computed as a limit, giving (18) (or something like it).

Comments preceeding and problems following Lecture 4 – 14 February, 2008

Problem 2. Recall at least what the Schwartz kernel theorem is about. The most general sort of operator we are likely to encounter is a continuous linear map

$$A:\mathcal{C}^\infty_c(\Omega)\longrightarrow\mathcal{C}^{-\infty}(\Omega') \tag{23}$$

where $\Omega\subset\mathbb{R}^n$ and $\Omega'\subset\mathbb{R}^{n'}$ are open sets. Make sure you understand that continuity of A is the statement that for each of compact set $K\Subset\Omega$, and each $\phi\in\mathcal{C}^\infty_c(\Omega')$ there exist constants C, k and k' such that

$$\psi\in\mathcal{C}^\infty_c(\Omega),\ \operatorname{supp}(\psi)\subset K\Longrightarrow \phi A(\psi)\in H^{k'}(\mathbb{R}^{n'}) \text{ and } \|\phi A(\psi)\|_{H^{k'}}\le C\|\psi\|_{H^k}. \tag{24}$$

Of course, k' may be very negative and k may be very large positive and both may go off to infinity, as may C, as K or $\operatorname{supp}(\phi)$ approach the boundaries of their respective sets.

Now, what the Schwartz kernel theorem says, is that any such continuous linear operator corresponds to a unique 'kernel' $K_A\in\mathcal{C}^{-\infty}(\Omega'\times\Omega)$, that is a distribution on this open set in $\mathbb{R}^{n+n'}$ in the sense that

$$(Au(\psi))(\phi)=K(\phi\otimes\psi)\text{``}='' \int_{\Omega\times\Omega'}K(z',z)\phi(z')\psi(z)dzdz' \tag{25}$$

and conversely each such kernel corresponds to an operator A.

This is important, at least psychologically, since it means that linear operators are again just distributions, so we do not need to have a separate field of operator theory (haha, of course there is one and it is BIG). The Schwartz kernel theorem is not very important practically, since for the operators we are interested in, the kernel tends to be rather obvious. However, it is certainly the case that we use the easy direction, and the uniqueness, to construct operators.

The 'easy' direction is to show that a kernel K does define an operator A through (25) and I suggest you see if you can write this down. The uniqueness is also pretty easy, that this operator A determines K, meaning that two K's cannot give the same operator. The existence of K, given that A is continuous as in (24) involves a bit more work so it is a separate problem.

Problem 3. Suppose that A is a continuous linear operator as in (23) and (24). Fix $\psi \in \mathcal{C}_c^\infty(\Omega)$ and $\phi \in \mathcal{C}_c^\infty(\Omega')$ and consider the cut-off operator $A'u = \phi A(\psi u)$. Let $\langle D\rangle^s$ be the operator (on $\mathcal{S}'(\mathbb{R}^n)$) defined as multiplication of the Fourier transform by $\langle\xi\rangle^s = (1+|\xi|^2)^{s/2}$. Show that if s is be a large negative integer then

$$A''(u) = \langle D\rangle^k A' \langle D\rangle^k : L^2(\mathbb{R}^n) \longrightarrow \mathcal{C}^0(\mathbb{R}^n) \tag{26}$$

(meaning it maps L^2 into bounded continuous functions and is continuous as an operator from Hilbert to Banach space, the latter having supremum norm). You might want to use Sobolev embedding to do this! Then use the Riesz representation theorem (for L^2) to show that there is a kernel

$$K''(z', z) \in \mathcal{C}^0(\mathbb{R}^{n'}; L^2(\mathbb{R}^n)) \subset \mathcal{S}'(\mathbb{R}^{n'} \times \mathbb{R}^n)$$

for A''. Go back and show that this gives a kernel for A'. Finally try to show that the uniqueness of the kernels shows that the original A has a kernel.

The following result is used in the construction of a parametrix.

Problem 4. A 'covered' partition of unity. Given an open $\Omega \subset \mathbb{R}^n$ we can always find a partition of unity. That is, a countable collection of functions $\phi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that

$$\begin{gathered} 0 \le \phi_j(z) \le 1 \ \forall\ j,\ z \in \Omega \\ \text{If } K \Subset \Omega \text{ then } \{j; \operatorname{supp}\phi_j \cap K \neq \emptyset\} \text{ is finite} \\ \sum_j \phi_j(z) = 1 \ \forall\ z \in \Omega. \end{gathered} \tag{27}$$

Now, show that this can be improved in that we can choose these ϕ_j so that in addition there exist $\phi'_j \in \mathcal{C}_c^\infty(\Omega)$ which also have 'locally finite supports' in the sense of the second condition above, and also satisfy

$$\phi'_j = 1 \text{ in a neighbourhood of } \operatorname{supp}(\phi_j). \tag{28}$$

Problems following Lecture 3– 12 February, 2008

Problem 5. Show that the statment for a differential operator with constant coefficients that for any open set Ω

$$u \in \mathcal{C}^{-\infty}(\Omega),\ P(D)u \in H^s_{\text{loc}}(\Omega) \Longrightarrow u \in H^{s+m}_{\text{loc}}(\Omega) \tag{29}$$

actually implies the estimates we used to prove it. Namely if $\psi \in \mathcal{C}_c^\infty(\Omega)$ and $\phi \in \mathcal{C}_c^\infty(\Omega)$ satisfies $\phi = 1$ in a neighbourhood of $\operatorname{supp}(\psi)$ then for each t there exists constants C, C' such that

$$\|\psi u\|_{H^{s+m}} \le C\|\psi P(D)u\|_{H^s} + C'\|\phi u\|_{H^t}. \tag{30}$$

Hints. One good way is to show that (29) implies that $P(D)$ is elliptic (if $\Omega \neq \emptyset$!) Use the fact that if $P(D)$ is not elliptic then the principal part vanishes on a radial line. Try to construct a function which is not in $H^m(\mathbb{R}^n)$ but such that $P(D)u$ is in L^2 and work from there. This is not so easy.

Make sure you recall some of the basic properties of convolution. In particualar that convolution $u * v$ is always defined for distributions if one of u and v has compact support, that $u * V = v * u$ and that

$$P(D)(u * v) = (P(D)u) * v,\ u * f \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ if } u \in \mathcal{C}^\infty(\mathbb{R}^n),\ f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n). \tag{31}$$

Problems following Lecture 2 – 7 February, 2008

Problems following Lecture 1 – 5 February, 2008

These problems are intended to help you recall the treatment of the Fourier transform and Sobolev spaces in 18.155.

Problem 6. Recall and explain the definition of Sobolev spaces on $\mathbb{R}^n$:

$$H^s(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); \hat{u} \in L^1_{\text{loc}}(\mathbb{R}^n); (1 + |\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)\right\},\ s \in \mathbb{R}. \tag{32}$$

Here I mean that you should explain why the definition as stated makes sense and that each $H^s(\mathbb{R}^n)$ is a Hilbert space.

Problem 7. Show that $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ is dense and that the bilinear map

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (u, v) \longmapsto \int_{\mathbb{R}^n} u(z)v(z)dz \in \mathbb{C} \tag{33}$$

extends to a non-degenerate pairing for any $s \in \mathbb{R}$, i.e. a continuous bilinear map

$$H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{34}$$

which allows $H^{-s}(\mathbb{R}^n)$ to be identified with the dual of $H^s(\mathbb{R}^n)$.

Problem 8. Show that if $\lambda \in \mathbb{C} \setminus [0, \infty)$ then $\Delta - \lambda$ defines an isomorphism

$$\Delta - \lambda : H^{s+2}(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n)\ \forall\ s \in \mathbb{R}. \tag{35}$$

Problem 9. Show, following the idea of a similar proof in class on 5 Feb, that

$$|D_z^\alpha \langle z\rangle| \le C_\alpha \langle z\rangle^{1-|\alpha|},\ \langle z\rangle = (1 + |z|^2)^{\frac{1}{2}}. \tag{36}$$

Problem 10. Show that on $\mathbb{R}^3$, for a certain non-zero constant c,

$$\Delta_z |z|^{-1} = (D_1^2 + D_2^2 + D_3^2)|z|^{-1} = c\delta(z) \text{ in } \mathcal{S}'(\mathbb{R}^3). \tag{37}$$

Problem 11. Using convolution show that if $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^3)$ is a distribution with compact support (if necessary, remind yourself as to what this means) then

$$u = c^{-1}(|z|^{-1}) * f \in \mathcal{S}(\mathbb{R}^3) \text{ satisfies } \Delta u = f. \tag{38}$$

How is this consistent with the fact, from class, that Δ is not an isomorphism on $\mathcal{S}'(\mathbb{R}^3)$?

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