CHAPTER 6

Manifolds with boundary

$\boxed{\text{0.6P; Revised: 18-7-2007; Run: August 2, 2007}}$

- Dirac operators – Photos-C5-16, C5-17.
- Homogeneity etc Photos-C5-18, C5-19, C5-20, C5-21, C5-23, C5-24.

1. Compactifications of $\mathbb{R}$.

As I will try to show by example later in the course, there are I believe considerable advantages to looking at compactifications of non-compact spaces. These advantages show up last in geometric and analytic considerations. Let me start with the simplest possible case, namely the real line. There are two standard compactifications which one can think of as 'exponential' and 'projective'. Since there is only one connected compact manifold with boundary compactification corresponds to the choice of a diffeomorphism onto the interior of $[0,1]$:

$$\begin{gathered}\gamma : \mathbb{R} \longrightarrow [0,1],\ \gamma(\mathbb{R}) = (0,1),\\ \gamma^{-1} : (0,1) \longrightarrow \mathbb{R},\ \gamma, \gamma^{-1}\mathcal{C}^\infty.\end{gathered} \tag{6.1}$$

In fact it is not particularly pleasant to have to think of the global maps γ, although we can. Rather we can think of separate maps

$$\begin{gathered}\gamma_+ : (T_+,\infty) \longrightarrow [0,1]\\ \gamma_- : (T_-,-\infty) \longrightarrow [0,1]\end{gathered} \tag{6.2}$$

which *both* have images $(0,x_\pm)$ and as diffeomorphism other than signs. In fact if we want the two ends to be the 'same' then we can take $\gamma_-(t) = \gamma_+(-t)$. I leave it as an exercise to show that γ then exists with

$$\begin{cases}\gamma(t) = \gamma_+(t) & t \gg 0\\ \gamma(t) = 1 - \gamma_-(t) & t \ll 0.\end{cases} \tag{6.3}$$

So, all we are really doing here is identifying a 'global coordinate' $\gamma_+^* x$ near ∞ and another near $-\infty$. Then two choices I refer to above are

$$\begin{aligned} x &= e^{-t} && \text{exponential compactification}\\ x &= 1/t && \text{projective compactification}.\end{aligned} \tag{CR.4}$$

Note that these are *alternatives*!

Rather than just consider $\mathbb{R}$, I want to consider $\mathbb{R} \times M$, with M compact, as discussed above.

Lemma 28. *If $R : H \longrightarrow H$ is a compact operator on a Hilbert space then* $\operatorname{Id} - R$ *is Fredholm.*

Proof. A compact operator is one which maps the unit ball (and hence any bounded subset) of H onto a precompact set, a set with compact closure. The unit ball in the null space of $\operatorname{Id} - R$ is

$$\{u \in H; \|u\| = 1\,,\ u = Ru\} \subset R\{u \in H; \|u\| = 1\}$$

and is therefore precompact. Since it is closed, it is compact and any Hilbert space with a compact unit ball is finite dimensional. Thus the null space of $(\operatorname{Id} - R)$ is finite dimensional.

Consider a sequence $u_n = v_n - Rv_n$ in the range of $\operatorname{Id} - R$ and suppose $u_n \to u$ in H. We may assume $u \neq 0$, since 0 is in the range, and by passing to a subsequence suppose that of γ on ?? fields. Clearly

$$\text{(CR.5)} \qquad \begin{array}{lll} \gamma(t) = e^{-t} & \Rightarrow & \gamma_*(\partial_t) = -x(\partial_x) \\ \tilde{\gamma}(t) = 1/t & \Rightarrow & \tilde{\gamma}_*(\partial_t) = -s^2\partial_s \end{array}$$

where I use 's' for the variable in the second case to try to reduce confusion, it is just a variable in $[0,1]$. Dually

$$\text{(CR.6)} \qquad \begin{aligned} \gamma^*\left(\frac{dx}{x}\right) &= -dt \\ \tilde{\gamma}^*\left(\frac{ds}{s^2}\right) &= -dt \end{aligned}$$

in the two cases. The minus signs just come from the fact that both γ's reverse orientation.

Proposition 22. *Under exponential compactification the translation-invariant Sobolev spaces on $\mathbb{R} \times M$ are identified with*

$$\text{(6.4)} \qquad \begin{aligned} H^k_b([0,1] \times M) = \Big\{ u \in L^2\left([0,1] \times M; \frac{dx}{x} V_M\right)\ ;\ \forall\ \ell, p \le k \\ P_p \in \operatorname{Diff}^p(M)\,,\ (xD_x)^\ell P_p u \in L^2\big([0,1] \times M; \frac{dx}{x} V_M\big) \Big\} \end{aligned}$$

for k a positive integer, $\dim M = n$,

$$\text{(6.5)} \qquad \begin{aligned} H^s_b([0,1] \times M) = \big\{ u \in L^2\left([0,1] \times M; \frac{dx}{x} V_M\right); \\ \iint \frac{|u(x,z) - u(x',z')|^2}{\big(|\log \frac{x}{x'}|^2 + \rho(z,z')\big)^{\frac{n+s+1}{2}}} \frac{dx}{x} \frac{dx'}{x'} \nu\nu' < \infty \big\} 0 < s < 1 \end{aligned}$$

and for $s < 0$, $k \in \mathbb{N}$ s.t., $0 \le s + k < 1$,

$$\text{(6.6)} \qquad \begin{aligned} H^s_b([0,1] \times M) = \big\{ u = \sum_{0 \le j+p \le k} (X d^J_X) p_P u_{j,p}, \\ P_p \in \operatorname{Diff}^p(M)\,,\ u_{j,p} \in H^{s+k}_b([0,1] \times M) \big\}. \end{aligned}$$

Moreover the L^2 pairing with respect to the measure $\frac{dx}{x}\nu$ extends by continuity from the dense subspaces $\mathcal{C}^\infty_c((0,1) \times M)$ to a non-degenerate pairing

$$\text{(6.7)} \qquad H^s_b([0,1] \times M) \times H^{-s}_b([0,1] \times M) \ni (n,u) \longmapsto \int u \cdot v \frac{dx}{x} \nu \in \mathbb{C}.$$

□

PROOF. This is all just translation of the properties of the space $H^s_{\mathrm{ti}}(\mathbb{R} \times M)$ to the new coordinates. □

Note that there are other properties I have not translated into this new setting. There is one additional fact which it is easy to check. Namely $\mathcal{C}^\infty([0,1] \times M)$ acts as multipliers on all the spaces $H^s_{\mathrm{b}}([0,1] \times M)$. This follows directly from Proposition 22;

$$\mathcal{C}^\infty([0,1] \times M) \times H^s_{\mathrm{b}}([0,1] \times M) \ni (\varphi, u) \mapsto \varphi u \in H^s_{\mathrm{b}}([0,1] \times M)\,. \tag{CR.12}$$

What about the 'b' notation? Notice that $(1-x)x\partial_x$ and the smooth vector fields on M span, over $\mathcal{C}^\infty(X)$, for $X = [0,1] \times M$, *all* the vector fields tangent to $\{x = 0|u|x = 1\}$. Thus we can define the 'boundary differential operators' as

$$\mathrm{Diff}^m_{\mathrm{b}}([0,1] \times M_i)^E = \Bigg\{ P = \sum_{0 \le j+p \le m} a_{j,p}(x_j)((1-x)xD_x)^j P_p\ , \\ P_p \in \mathrm{Diff}^p(M_i)^E \Big\} \tag{CR.13}$$

and conclude from (CR.12) and the earlier properties that

$$P \in \mathrm{Diff}^m_{\mathrm{b}}(X; E) \Rightarrow \\ P : H^{s+m}_{\mathrm{b}}(X;E) \to H^s_{\mathrm{b}}(X;E) \,\forall s \in \mathbb{R}\,. \tag{CR.14}$$

THEOREM 8. *A differential operator as in (8) is Fredholm if and only if it is elliptic in the interior and the two "normal operators'*

$$I_\pm(P) = \sum_{0 \le j+p \le m} a_{j,p}(x_{\pm 1})(\pm D_k)^i P_p \quad x_+ = 0\,,\, x_- = 1 \tag{CR.16}$$

derived from (CR.13), are elliptic and invertible on the translation-invariant Sobolev spaces.

PROOF. As usual we are more interested in the sufficiency of these conditions than the necessity. To prove this result by using the present (slightly low-tech) methods requires going back to the beginning and redoing most of the proof of the Fredholm property for elliptic operators on a compact manifold.

The first step then is *a priori* bounds. What we want to show is that if the conditions of the theorem hold then for $u \in H^{s+m}_{\mathrm{b}}(X;E)$, $x = \mathbb{R} \times M$, $\exists C > 0$ s.t.

$$\|u\|_{m+s} \le C_s \|Pu\|_s + C_s \|x(1-x)u\|_{s-1+m}\,. \tag{CR.17}$$

Notice that the norm on the right has a factor, $x(1-x)$, which vanishes at the boundary. Of course this is supposed to come from the invertibility of $I_\pm(P)$ in $\mathbb{R}(0)$ and the ellipticity of P.

By comparison $I_\pm(P) : H^{s+m}_{\hbar}(\mathbb{R} \times M) \to H^s_{\hbar}(\mathbb{R} \times M)$ are isomorphisms — necessary and sufficient conditions for this are given in Theorem ???. We can use the compactifying map γ to convert this to a statement as in (CR.17) for the operators

$$P_\pm \in \mathrm{Diff}^m_{\mathrm{b}}(X)\,,\, P_\pm = I_\pm(P)(\gamma_* D_t, \cdot)\,. \tag{CR.18}$$

Namely

$$\|u\|_{m+s} \le C_s \|P_\pm u\|_s \tag{CR.19}$$

where these norms, as in (CR.17) are in the H^s_{b} spaces. Note that near $x = 0$ or $x = 1$, $P_\pm$ are obtained by substituting $D_t \mapsto xD_x$ or $(1-x)D_x$ in (CR.17). Thus

$$P - P_\pm \in (x - x_\pm)\,\mathrm{Diff}^m_{\mathrm{b}}(X)\,, \quad x_\pm = 0, 1 \tag{CR.20}$$

have coefficients which *vanish* at the appropriate boundary. This is precisely how (CR.16) is derived from (CR.13). Now choose $\varphi \in \mathcal{C}^\infty, (0,1) \times M$ which is equal to 1 on a sufficiently large set (and has $0 \le \varphi \le 1$) so that

$$1 - \varphi = \varphi_+ + \varphi_-\,,\ \varphi_\pm \in \mathcal{C}^\infty([0,1] \times M) \tag{CR.21}$$

have $\operatorname{supp}(\varphi_\pm) \subset \{|x - x_\pm| \le \epsilon)$, $0 \le \varphi_+ 1$.

By the interim elliptic estimate,

$$\|\varphi u\|_{s+m} \le C_s \|\varphi P u\|_s + C'_s \|\psi u\|_{s-1+m} \tag{CR.22}$$

where $\psi \in \mathcal{C}^\infty_c((0,1) \times M)$. On the other hand, because of (CR.20)

$$\begin{aligned} \|\varphi_\pm u\|_{m+s} &\le C_s \|\varphi_\pm P_\pm u\|_s + C_s \|[\varphi_\pm, P_\pm u]\|_s \\ &\le C_s \|\varphi_\pm P u\|_s + C_s \varphi_\pm (P - P_\pm) u\|_s + C_s \|[\varphi_\pm, P_\pm] u\|_s\,. \end{aligned} \tag{CR.23}$$

Now, if we can choose the support at $\varphi_\pm$ small enough — recalling that C_s truly depends on $I_\pm(P_t)$ and s — then the second term on the right in (CR.23) is bounded by $\frac{1}{4}\|u\|_{m+s}$, since *all* the coefficients of $P - P_\pm$ are small on the support off $\varphi_\pm$. Then (CR.24) ensures that the final term in (CR.17), since the coefficients vanish at $x = x_\pm$.

The last term in (CR.22) has a similar bound since ψ has compact support in the interim. This combining (CR.2) and (CR.23) gives the desired bound (CR.17).

To complete the proof that P is Fredholm, we need another property of these Sobolev spaces.

Lemma 29. *The map*

$$Xx(1-x) : H^s_b(X) \longrightarrow H^{s-1}_b(X) \tag{6.8}$$

is compact.

Proof. Follow it back to $\mathbb{R} \times M$!

□

Now, it follows from the *a priori* estimate (CR.17) that, as a map (CR.14), P has finite dimensional null space and closed range. This is really the proof of Proposition ?? again. Moreover the adjoint of P with respect to $\frac{dx}{x}V, P^*$, is again elliptic and satisfies the condition of the theorem, so it too has finite-dimensional null space. Thus the range of P has finite codimension so it is Fredholm.

□

A corresponding theorem, with similar proof follows for the cusp compactification. I will formulate it later.

2. Basic properties

A discussion of manifolds with boundary goes here.

3. Boundary Sobolev spaces

Generalize results of Section 1 to arbitrary compact manifolds with boundary.

4. Dirac operators

Euclidean and then general Dirac operators

5. Homogeneous translation-invariant operators

One application of the results of Section 3 is to homogeneous constant-coefficient operators on $\mathbb{R}^n$, including the Euclidean Dirac operators introduced in Section 4. Recall from Chapter 2 that an elliptic constant-coefficient operator is Fredholm, on the standard Sobolev spaces, if and only if its characteristic polynomial has no real zeros. If P is homogeneous

$$P_{ij}(t\zeta) = t^m P_{ij}(\zeta) \ \forall\ \zeta \in \mathbb{C}^n\,,\ t \in \mathbb{R}\,, \tag{6.9}$$

and elliptic, then the only real zero (of the determinant) is at $\zeta = 0$. We will proceed to discuss the radial compactification of Euclidean space to a ball, or more conveniently a half-sphere

$$\gamma_R : \mathbb{R}^n \hookrightarrow \mathbb{S}^{n,1} = \{Z \in \mathbb{R}^{n+1}\,;\ |Z| = 1\,,\ Z_0 \geq 0\}\,. \tag{6.10}$$

Transferring P to $\mathbb{S}^{n,1}$ gives

$$P_R \in Z_0^m \operatorname{Diff}^m_{\mathrm{b}}(\mathbb{S}^{n,1}\,;\,\mathbb{C}^N) \tag{6.11}$$

which is elliptic and to which the discussion in Section 3 applies.

In the 1-dimensional case, the map (6.10) reduces to the second 'projective' compactification of $\mathbb{R}$ discussed above. It can be realized globally by

$$\gamma_R(z) = \left(\frac{1}{\sqrt{1+|z|^2}}\,,\ \frac{z}{\sqrt{1+|z|^2}}\right) \in \mathbb{S}^{n,1}\,. \tag{6.12}$$

Geometrically this corresponds to a form of stereographic projection. Namely, if $\mathbb{R}^n \ni z \mapsto (1, z) \in \mathbb{R}^{n+1}$ is embedded as a 'horizontal plane' which is then projected radially onto the sphere (of radius one around the origin) one arrives at (6.12). It follows easily that γ_R is a diffeomorphism onto the open half-sphere with inverse

$$z = Z'/Z_0\,,\ Z' = (Z_1, \dots, Z_n)\,. \tag{6.13}$$

Whilst (6.12) is concise it is not a convenient form of the compactification as far as computation is concerned. Observe that

$$x \mapsto \frac{x}{\sqrt{1+x^2}}$$

is a diffeomorphism of neighborhoods of $0 \in \mathbb{R}$. It follows that Z_0, the first variable in (6.12) can be replaced, near $Z_0 = 0$, by $1/|z| = x$. That is, there is a diffeomorphism

$$\{0 \leq Z_0 \leq \epsilon\} \cap \mathbb{S}^{n,1} \leftrightarrow [0, \delta]_x \times \mathbb{S}^{n-1}_\theta \tag{6.14}$$

which composed with (6.12) gives $x = 1/|z|$ and $\theta = z/|z|$. In other words the compactification (6.12) is equivalent to the introduction of polar coordinates near infinity on $\mathbb{R}^n$ followed by inversion of the radial variable.

Lemma 30. *If $P = (P_{ij}(D_z))$ is an $N \times N$ matrix of constant coefficient operators in $\mathbb{R}^n$ which is homogeneous of degree $-m$ then* (6.11) *holds after radial compactification. If P is elliptic then P_R is elliptic.*

PROOF. This is a bit tedious if one tries to do it by direct computation. However, it is really only the homogeneity that is involved. Thus if we use the coordinates $x = 1/|z|$ and $\theta = z/|z|$ valid near the boundary of the compactification (i.e., near ∞ on $\mathbb{R}^n$) then

$$P_{ij} = \sum_{0\le \ell\le m} D_x^\ell P_{\ell,i,j}(x,\theta,D_\theta)\,,\ P_{\ell,i,j} \in \mathcal{C}^\infty(0,\delta)_x\,;\,\mathrm{Diff}^{m-\ell}(\mathbb{S}^{n-1}). \tag{6.15}$$

Notice that we *do* know that the coefficients are smooth in $0 < x < \delta$, since we are applying a diffeomorphism there. Moreover, the operators $P_{\ell,i,j}$ are uniquely determined by (6.15).

So we can exploit the assumed homogeneity of P_{ij}. This means that for any $t > 0$, the transformation $z \mapsto tz$ gives

$$P_{ij} f(tz) = t^m (P_{ij} f)(tz)\,. \tag{6.16}$$

Since $|tz| = t|z|$, this means that the transformed operator must satisfy

$$\sum_\ell D_x^\ell P_{\ell,i,j}(x,\theta,D_\theta) f(x/t,\theta) = t^m (\sum_\ell D^\ell P_{\ell,i,j}(\cdot,\theta,D_\theta) f(\cdot,\theta))(x/t)\,. \tag{6.17}$$

Expanding this out we conclude that

$$x^{-m-\ell} P_{\ell,i,j}(x,\theta,D_\theta) = P_{\ell,i,j}(\theta,D_\theta) \tag{6.18}$$

is independent of x. Thus in fact (6.15) becomes

$$P_{ij} = x^m \sum_{0\le j\le \ell} x^\ell D_x^\ell P_{\ell,j,i}(\theta,D_\theta)\,. \tag{6.19}$$

Since we can rewrite

$$x^\ell D_x = \sum_{0\le j\le \ell} C_{\ell,j} (xD_x)^j \tag{6.20}$$

(with explicit coefficients if you want) this gives (6.11). Ellipticity in this sense, meaning that

$$x^{-m} P_R \in \mathrm{Diff}^m_{\mathrm{b}}(\mathbb{S}^{n,1};\mathbb{C}^N) \tag{6.21}$$

(6.19) and the original ellipticity at P. Namely, when expressed in terms of xD_x the coefficients of 6.21 are independent of x (this of course just reflects the homogeneity), ellipticity in $x > 0$ follows by the coordinate independence of ellipticity, and hence extends down to $x = 0$. □

Now the coefficient function Z_0^{w+m} in (6.11) always gives an isomorphism

$$\times Z_0^m : Z_0^w H^s_{\mathrm{b}}(\mathbb{S}^{n,1}) \longrightarrow Z_0^{w+m} H^s_{\mathrm{b}}(\mathbb{S}^{n,1})\,. \tag{6.22}$$

Combining this with the results of Section 3 we find most of

THEOREM 9. *If P is an $N \times N$ matrix of constant coefficient differential operators on $\mathbb{R}^n$ which is elliptic and homogeneous of degree $-m$ then there is a discrete set $-\operatorname{Im}(D(P)) \subset \mathbb{R}$ such that*

$$P : Z_0^w H_b^{m+s}(\mathbb{S}^{n,1}) \longrightarrow Z_0^{w+m} H_b^s(\mathbb{S}^{n,1}) \text{ is Fredholm } \forall\ w \notin -\operatorname{Im}(D(P)) \tag{6.23}$$

where (6.12) is used to pull these spaces back to $\mathbb{R}^n$. Moreover,

$$\begin{aligned} &P \text{ is injective for } w \in [0,\infty) \text{ and}\\ &P \text{ is surjective for } w \in (-\infty, n-m] \cap (-\operatorname{Im}(D)(P))\,. \end{aligned} \tag{6.24}$$

PROOF. The conclusion (6.23) is exactly what we get by applying Theorem X knowing (6.11).

To see the specific restriction (6.24) on the null space and range, observe that the domain spaces in (6.23) are tempered. Thus the null space is contained in the null space on $\mathcal{S}'(\mathbb{R}^n)$. Fourier transform shows that $P(\zeta)\hat{u}(\zeta) = 0$. From the assumed ellipticity of P and homogeneity it follows that $\operatorname{supp}(\hat{u}(\zeta)) \subset \{0\}$ and hence $\hat{u}$ is a sum of derivatives of delta functions and finally that u itself is a polynomial. If $w \geq 0$ the domain in (6.23) contains no polynomials and the first part of (6.24) follows.

The second part of (6.24) follows by a duality argument. Namely, the adjoint of P with respect to $L^2(\mathbb{R}^n)$, the usual Lebesgue space, is P^* which is another elliptic homogeneous differential operator with constant coefficients. Thus the first part of (6.24) applies to P^*. Using the homogeneity of Lebesgue measure,

$$|dz| = \frac{dx}{x^{n+1}} \cdot \nu_\theta \text{ near } \infty \tag{6.25}$$

and the shift in weight in (6.23), the second part of (6.24) follows. □

One important consequence of this is a result going back to Nirenberg and Walker (although expressed in different language).

COROLLARY 2. *If P is an elliptic $N \times N$ matrix constant coefficient differential operator which is homogeneous of degree m, with $n > m$, the the map (6.23) is an isomorphism for $w \in (0, n-m)$.*

In particular this applies to the Laplacian in dimensions $n > 2$ and to the constant coefficient Dirac operators discussed above in dimensions $n > 1$. In these cases it is also straightforward to compute the index and to identify the surjective set. Namely, for a constant coefficient Dirac operator

$$D(P) = i\mathbb{N}_0 \cup i(n - m + \mathbb{N}_0)\,. \tag{6.26}$$

Figure goes here.

6. Scattering structure

Let me briefly review how the main result of Section 5 was arrived at. To deal with a constant coefficient Dirac operator we first radially compactified $\mathbb{R}^n$ to a ball, then peeled off a multiplicative factor Z_0 from the operator showed that the remaining operator was Fredholm by identifing a neighbourhood of the boundary with part of $\mathbb{R} \times \mathbb{S}^{n-1}$ using the exponential map to exploit the results of Section 1 near infinity. Here we will use a similar, but different, procedure to treat a different class of operators which are Fredholm on the *standard* Sobolev spaces.

Although we will only apply this in the case of a ball, coming from $\mathbb{R}^n$, I cannot resist carrying out the discussed for a general compact manifolds — since I think the generality clarifies what is going on. Starting from a compact manifold with boundary, M, the first step is essentially the reverse of the radial compactification of $\mathbb{R}^n$.

Near any point on the boundary, $p \in \partial M$, we can introduce 'admissible' coordinates, $x, y_1, \dots, y_{n-1}$ where $\{x = 0\|$ is the local form of the boundary and $y_1, \dots, y_{n-1}$ are tangential coordinates; we normalize $y_1 = \cdots = y_{n-1} = 0$ at p. By

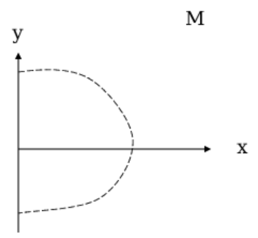

FIGURE 1. Boundary coordinate patch

reversing the radial compactification of $\mathbb{R}^n$ I mean we can introduce a diffeomorphism of a neighbourhood of p to a conic set in $\mathbb{R}^n$:

$$z_n = 1/x\,,\; z_j = y_j/x\,,\; j = 1, \dots, n-1\,. \tag{6.27}$$

Clearly the 'square' $|y| < \epsilon$, $0 < x < \epsilon$ is mapped onto the truncated conic set

$$z_n \geq 1/\epsilon\,,\; |z'| < \epsilon|z_n|\,, z' = (z_1, \dots, z_{n-1})\,. \tag{6.28}$$

DEFINITION 7. We define spaces $H^s_{\mathrm{sc}}(M)$ for any compact manifold with boundary M by the requirements

$$u \in H^s_{\mathrm{sc}}(M) \Longleftrightarrow u \in H^s_{\mathrm{loc}}(M \setminus \partial M) \text{ and } R^*_j(\varphi_j u) \in H^s(\mathbb{R}^n) \tag{6.29}$$

for $\varphi_j \in \mathcal{C}^\infty(M)$, $0 \leq \varphi_i \leq 1$, $\sum \varphi_i = 1$ in a neighbourhood of the boundary and where each φ_j is supported in a coordinate patch (6), (6.28) with R given by (6.27).

Of course such a definition would not make much sense if it depended on the choice of the partition of unity near the boundary $\{\varphi_i\|$ or the choice of coordinate. So really (7) should be preceded by such an invariance statement. The key to this is the following observation.

PROPOSITION 23. *If we set $\mathcal{V}_{\mathrm{sc}}(M) = x\mathcal{V}_{\mathrm{b}}(M)$ for any compact manifold with boundary then for any $\psi \in \mathcal{C}^\infty(M)$ supported in a coordinate patch* (6)*, and any $\mathcal{C}^\infty$ vector field V on M*

$$\psi V \in \mathcal{V}_{\mathrm{sc}}(M) \Longleftrightarrow \psi V = \sum_{j=1}^{n} \mu_j (R^{-1})_*(D_{z_j})\,,\ \mu_j \in \mathcal{C}^\infty(M)\,. \tag{6.30}$$

PROOF. The main step is to compute the form of D_{z_j} in terms of the coordinate obtained by inverting (6.27). Clearly

$$D_{z_n} = x^2 D_x\,,\ D_{z_j} = xD_{y_j} - y_i x^2 D_x\,,\ j < n\,. \tag{6.31}$$

Now, as discussed in Section 3, xD_x and D_{y_j} locally span $\mathcal{V}_{\mathrm{b}}(M)$, so x^2D_x, xD_{y_j} locally span $\mathcal{V}_{\mathrm{sc}}(M)$. Thus (6.31) shows that in the singular coordinates (6.27), $\mathcal{V}_{\mathrm{sc}}(M)$ is spanned by the D_{z_ℓ}, which is exactly what (6.30) claims. □

Next let's check what happens to Euclidean measure under R, actually we did this before:

$$|dz| = \frac{|dx|}{x^{n+1}}\nu_y\,. \tag{SS.9}$$

Thus we can first identify what (6.29) means in the case of $s = 0$.

LEMMA 31. *For $s = 0$, Definition* (7) *unambiguously defines*

$$H^0_{sc}(M) = \left\{u \in L^2_{\mathrm{loc}}(M)\,;\ \int |u|^2 \frac{\nu_M}{x^{n+1}} < \infty\right\} \tag{6.32}$$

where ν_M is a positive smooth density on M (smooth up to the boundary of course) and $x \in \mathcal{C}^\infty(M)$ is a boundary defining function.

PROOF. This is just what (6.29) and (SS.9) mean. □

Combining this with Proposition 23 we can see directly what (6.29) means for $kin\mathbb{N}$.

LEMMA 32. *If* (6.29) *holds for $s = k \in \mathbb{N}$ for any one such partition of unity then $u \in H^0_{sc}(M)$ in the sense of* (6.32) *and*

$$V_1 \dots V_j u \in H^0_{sc}(M)\ \forall\ V_i \in \mathcal{V}_{\mathrm{sc}}(M)\ \text{if}\ j \le k\,, \tag{6.33}$$

and conversely.

PROOF. For clarity we can proceed by induction on k and replace (6.33) by the statements that $u \in H^{k-1}_{\mathrm{sc}}(M)$ and $Vu \in H^{k-1}_{\mathrm{sc}}(M)$ $\forall V \in \mathcal{V}_{\mathrm{sc}}(M)$. In the interior this is clear and follows immediately from Proposition 23 provided we carry along the inductive statement that

$$\mathcal{C}^\infty(M)\ \text{acts by multiplication on}\ H^k_{\mathrm{sc}}(M)\,. \tag{6.34}$$

□

As usual we can pass to general $s \in \mathbb{R}$ by treating the cases $0 < s < 1$ first and then using the action of the vector fields.

PROPOSITION 24. *For* $0 < s < 1$ *the condition* (6.29) *(for any one partition of unity) is equivalent to requiring* $u \in H^0_{sc}(M)$ *and*

$$\iint_{M\times M} \frac{|u(p) - u(p')|^2}{\rho_{sc}^{n+2s}} \frac{\nu_M}{x^{n+1}} \frac{\nu'_M}{(x')^{n+1}} < \infty \tag{6.35}$$

where $\rho_{sc}(p,p') = \chi\chi' p(p,p') + \sum_j \varphi_j\varphi'_j\langle z - z'\rangle$.

PROOF. Use local coordinates. □

Then for $s \geq 1$ if k is the integral part of s, so $0 \leq s - k < 1$, $k \in \mathbb{N}$,

$$u \in H^s_{\text{sc}}(M) \Longleftrightarrow V_1, \dots, V_j u \in H^{s-k}_{\text{sc}}(M)\,, V_i \in \mathcal{V}_{\text{sc}}(M)\,,\ j \leq k \tag{6.36}$$

and for $s < 0$ if $k \in \mathbb{N}$ is chosen so that $0 \leq k + s < 1$, then

$$\begin{gathered} u \in H^s_{\text{sc}}(M) \Leftrightarrow \exists\ V_j \in H^{s+k}_{\text{sc}}(M)\,,\ j = 1, \dots, \mathbb{N}\,, \\ u_j \in H^{s-k}_{\text{sc}}(M)\,, V_{j,i}(M)\,,\ 1 \leq i \leq \ell_j \leq k \text{ s.t.} \\ u = u_0 + \sum_{j=1}^{N} V_{j,i} \cdots V_{j,\ell_j} u_j. \end{gathered} \tag{6.37}$$

All this complexity is just because we are preceding in such a 'low-tech' fashion. The important point is that these Sobolev spaces are determined by the choice of 'structure vector fields', $V \in \mathcal{V}_{\text{sc}}(M)$. I leave it as an important exercise to check that

LEMMA 33. *For the ball, or half-sphere,*

$$\gamma^*_R H^s_{sc}(\mathbb{S}^{n,1}) = H^s(\mathbb{R}^n)\,.$$

Thus on Euclidean space we have done *nothing*. However, my claim is that we understand things better by doing this! The idea is that we should Fourier analysis on $\mathbb{R}^n$ to analyse differential operators which are made up out of $\mathcal{V}_{\text{sc}}(M)$ on any compact manifold with boundary M, and this includes $\mathbb{S}^{n,1}$ as the radial compactification of $\mathbb{R}^n$. Thus set

$$\begin{aligned} \text{Diff}^m_{\text{sc}}(M) = \big\{P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M);\ \exists\ f \in \mathcal{C}^\infty(M) \text{ and} \\ V_{i,j} \in \mathcal{V}_{\text{sc}}(M) \text{ s.t. } P = f + \sum_{i,1\leq j\leq m} V_{i,1} \dots V_{i,j}\big\}. \end{aligned} \tag{6.38}$$

In local coordinates this is just a differential operator and it is smooth up to the boundary. Since only scattering vector fields are allowed in the definition such an operator is quite degenerate at the boundary. It always looks like

$$P = \sum_{k+|\alpha|\leq m} a_{k,\alpha}(x,y)(x^2 D_x)^k (xD_y)^\alpha, \tag{6.39}$$

with smooth coefficients in terms of local coordinates (6).

Now, if we *freeze* the coefficients at a point, p, on the boundary of M we get a polynomial

$$\sigma_{\text{sc}}(P)(p) = \sum_{k+|\alpha|\leq m} a_{k,\alpha}(p)\tau^k\eta^\alpha. \tag{6.40}$$

Note that this is *not* in general homogeneous since the lower order terms are retained. Despite this one gets essentially the same polynomial at each point, independent of the admissible coordinates chosen, as will be shown below. Let's just assume this for the moment so that the condition in the following result makes sense.

THEOREM 10. *If* $P \in \mathrm{Diff}^m_{sc}(M;\mathbb{E})$ *acts between vector bundles over* M*, is elliptic in the interior and each of the polynomials (matrices)* (6.40) *is elliptic and has no real zeros then*

$$P : H^{s+m}_{sc}(M, E_1) \longrightarrow H^s_{sc}(M; E_2) \text{ is Fredholm} \tag{6.41}$$

for each $s \in \mathbb{R}$ *and conversely.*

7. Manifolds with corners

8. Blow up

Last time at the end I gave the following definition and theorem.

DEFINITION 8. We define weighted (non-standard) Sobolev spaces for $(m, w) \in \mathbb{R}^2$ on $\mathbb{R}^n$ by

$$\tilde{H}^{m,w}(\mathbb{R}^n) = \{u \in M^m_{\mathrm{loc}}(\mathbb{R}^n); F^*\left((1-\chi)r^{-w}u\right) \in H^m_{\mathrm{ti}}(\mathbb{R} \times \mathbb{S}^{n-1})\} \tag{6.42}$$

where $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\chi(y) = 1$ in $|y| < 1$ and

$$F : \mathbb{R} \times \mathbb{S}^{n-1} \ni (t, \theta) \longrightarrow (e^t, e^t\theta) \in \mathbb{R}^n \setminus \{0\}. \tag{6.43}$$

THEOREM 11. *If* $P = \sum\limits_{i=1}^n \Gamma_i D_i$, $\Gamma_i \in M(N, \mathbb{C})$*, is an elliptic, constant coefficient, homogeneous differential operator of first order then*

$$P : \tilde{H}^{m,w}(\mathbb{R}^n) \longrightarrow \tilde{H}^{m-1,w+1}(\mathbb{R}^n) \ \forall \ (m, w) \in \mathbb{R}^2 \tag{6.44}$$

is continuous and is Fredholm for $w \in \mathbb{R} \setminus \tilde{D}$ *where* $\tilde{D}$ *is discrete.*

If P *is a Dirac operators, which is to say explicitly here that the coefficients are 'Pauli matrices' in the sense that*

$$\Gamma_i^* = \Gamma_i,\ \Gamma_i^2 = \mathrm{Id}_{N\times N},\ \forall\ i,\ \Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 0,\ i \neq j, \tag{6.45}$$

then

$$\tilde{D} = -\mathbb{N}_0 \cup (n - 2 + \mathbb{N}_0) \tag{6.46}$$

and if $n > 2$ *then for* $w \in (0, n-2)$ *the operator* P *in* (6.44) *is an isomorphism.*

I also proved the following result from which this is derived

LEMMA 34. *In polar coordinates on* $\mathbb{R}^n$ *in which* $\mathbb{R}^n \setminus \{0\} \simeq (0, \infty) \times \mathbb{S}^{n-1}$, $y = r\theta$,

$$D_{y_j} = \tag{6.47}$$