CHAPTER 5

Suspended families and the resolvent

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For a compact manifold, M, the Sobolev spaces $H^s(M;E)$ (of sections of a vector bundle E) are defined above by reference to local coordinates and local trivializations of E. If M is not compact (but is paracompact, as is demanded by the definition of a manifold) the same sort of definition leads either to the spaces of sections with compact support, or the "local" spaces:

$$H^s_{\mathrm{c}}(M;E) \subset H^s_{\mathrm{loc}}(M;E),\ s \in \mathbb{R}. \tag{5.1}$$

Thus, if $F_a : \Omega_a \longrightarrow \Omega'_a$ is a covering of M, for $a \in A$, by coordinate patches over which E is trivial, $T_a : (F_a^{-1})^*E \cong \mathbb{C}^N$, and $\{\rho_a\}$ is a partition of unity subordinate to this cover then

$$\mu \in H^s_{\mathrm{loc}}(M;E) \Leftrightarrow T_a(F_a^{-1})^*(\rho_a\mu) \in H^s(\Omega'_a;\mathbb{C}^N)\ \forall\ a. \tag{5.2}$$

Practically, these spaces have serious limitations; for instance they are not Hilbert or even Banach spaaces. On the other hand they certainly have their uses and differential operators act on them in the usual way,

$$\begin{gathered} P \in \mathrm{Diff}^m(M;\mathbb{E}) \Rightarrow \\ P : H^{s+m}_{\mathrm{loc}}(M;E_+) \to H^s_{\mathrm{loc}}(M;E_-), \\ P : H^{s+m}_{\mathrm{c}}(M;E_+) \to H^s_{\mathrm{c}}(M;E_-). \end{gathered} \tag{5.3}$$

However, without some limitations on the growth of elements, as is the case in $H^s_{\mathrm{loc}}(M;E)$, it is not reasonable to expect the null space of the first realization of P above to be finite dimensional. Similarly in the second case it is not reasonable to expect the operator to be even close to surjective.

1. Product with a line

Some corrections from Fang Wang added, 25 July, 2007.

Thus, for non-compact manifolds, we need to find intermediate spaces which represent some growth constraints on functions or distributions. Of course this is precisely what we have done for $\mathbb{R}^n$ in defining the weighted Sobolev spaces,

$$H^{s,t}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); \langle z\rangle^{-t}u \in H^s(\mathbb{R}^n)\right\}. \tag{5.4}$$

However, it turns out that even these spaces are not always what we want.

To lead up to the discussion of other spaces I will start with the simplest sort of non-compact space, the real line. To make things more interesting (and useful) I will conisider

$$X = \mathbb{R} \times M \tag{5.5}$$

where M is a compact manifold. The new Sobolev spaces defined for this product will combine the features of $H^s(\mathbb{R})$ and $H^s(M)$. The Sobolev spaces on $\mathbb{R}^n$ are associated with the translation action of $\mathbb{R}^n$ on itself, in the sense that this fixes the "uniformity" at infinity through the Fourier transform. What happens on X is quite similar.

First we can define "tempered distributions" on X. The space of Schwartz functions of rapid decay on X can be fixed in terms of differential operators on M and differentiation on $\mathbb{R}$.

$$\mathcal{S}(\mathbb{R}\times M)=\left\{u:\mathbb{R}\times M\longrightarrow\mathbb{C};\ \sup_{\mathbb{R}\times M}\left|t^lD_t^kPu(t,\cdot)\right|<\infty\ \forall\ l,\ k,\ P\in\operatorname{Diff}^*(M)\right\}. \tag{5.6}$$

Exercise 1. Define the corresponding space for sections of a vector bundle E over M lifted to X and then put a topology on $\mathcal{S}(\mathbb{R}\times M;E)$ corresponding to these estimates and check that it is a complete metric space, just like $\mathcal{S}(\mathbb{R})$ in Chapter 1.

There are several different ways to look at

$$\mathcal{S}(\mathbb{R}\times M)\subset\mathcal{C}^\infty(\mathbb{R}\times M).$$

Namely we can think of either $\mathbb{R}$ or M as "coming first" and see that

$$\mathcal{S}(\mathbb{R}\times M)=\mathcal{C}^\infty(M;\mathcal{S}(\mathbb{R}))=\mathcal{S}(\mathbb{R};\mathcal{C}^\infty(M)). \tag{5.7}$$

The notion of a $\mathcal{C}^\infty$ function on M with values in a topological vector space is easy to define, since $\mathcal{C}^0(M;\mathcal{S}(\mathbb{R}))$ is defined using the metric space topology on $\mathcal{S}(\mathbb{R})$. In a coordinate patch on M higher derivatives are defined in the usual way, using difference quotients and these definitions are coordinate-invariant. Similarly, continuity and differentiability for a map $\mathbb{R}\longrightarrow\mathcal{C}^\infty(M)$ are easy to define and then

$$\mathcal{S}(\mathbb{R};\mathcal{C}^\infty(M))=\left\{u:\mathbb{R}\longrightarrow\mathcal{C}^\infty(M);\ \sup_t\left\|t^kD_t^pu\right\|_{\mathcal{C}^l(M)}<\infty,\ \forall\ k,\ p,\ l\right\}. \tag{5.8}$$

Using such an interpretation of $\mathcal{S}(\mathbb{R}\times M)$, or directly, it follows easily that the 1-dimensional Fourier transform gives an isomorphism $\mathcal{F}:\mathcal{S}(\mathbb{R}\times M)\longrightarrow\mathcal{S}(\mathbb{R}\times M)$ by

$$\mathcal{F}:u(t,\cdot)\longmapsto\hat{u}(\tau,\cdot)=\int_{\mathbb{R}}e^{-it\tau}u(t,\cdot)\,dt. \tag{5.9}$$

So, one might hope to use $\mathcal{F}$ to define Sobolev spaces on $\mathbb{R}\times M$ with uniform behavior as $t\to\infty$ in $\mathbb{R}$. However this is not so straightforward, although I will come back to it, since the 1-dimensional Fourier transform in (5.9) does *nothing* in the variables in M. Instead let us think about $L^2(\mathbb{R}\times M)$, the definition of which requires a choice of measure.

Of course there is an obvious class of product measures on $\mathbb{R}\times M$, namely $dt\cdot\nu_M$, where ν_M is a positive smooth density on M and dt is Lebesgue measure on $\mathbb{R}$. This corresponds to the functional

$$\int:\mathcal{C}^0_{\mathrm{c}}(\mathbb{R}\times M)\ni u\longmapsto\int u(t,\cdot)\,dt\cdot\nu\in\mathbb{C}. \tag{5.10}$$

The analogues of (5.7) correspond to Fubini's Theorem.

(5.11)
$$L^2_{\rm ti}(\mathbb{R}\times M)=\left\{u:\mathbb{R}\times M\to\mathbb{C}\text{ measurable; }\int|u(t,z)|^2\,dt\,\nu_z<\infty\right\}/\sim\ \text{a.e.}$$
$$L^2_{\rm ti}(\mathbb{R}\times M)=L^2(\mathbb{R};L^2(M))=L^2(M;L^2(\mathbb{R})).$$

Here the subscript "ti" is supposed to denote translation-invariance (of the measure and hence the space).

We can now easily define the Sobolev spaces of positive integer order:

$$(5.12)\quad H^m_{\rm ti}(\mathbb{R}\times M)=\Big\{u\in L^2_{\rm ti}(\mathbb{R}\times M);\\ D^j_tP_ku\in L^2_{\rm ti}(\mathbb{R}\times M)\ \forall\, j\le m-k,\ 0\le k\le m,\ P_k\in\operatorname{Diff}^k(M)\Big\}.$$

In fact we can write them more succinctly by defining

(5.13)
$$\operatorname{Diff}^k_{\rm ti}(\mathbb{R}\times M)=\left\{Q\in\operatorname{Diff}^m(\mathbb{R}\times M);\ Q=\sum_{0\le j\le m}D^j_tP_j,\ P_j\in\operatorname{Diff}^{m-j}(M)\right\}.$$

This is the space of "t-translation-invariant" differential operators on $\mathbb{R}\times M$ and (5.12) reduces to

(5.14)
$$H^m_{\rm ti}(\mathbb{R}\times M)=\left\{u\in L^2_{\rm ti}(\mathbb{R}\times M);\ Pu\in L^2_{\rm ti}(\mathbb{R}\times M),\ \forall\ P\in\operatorname{Diff}^m_{\rm ti}(\mathbb{R}\times M)\right\}.$$

I will discuss such operators in some detail below, especially the elliptic case. First, we need to consider the Sobolev spaces of non-integral order, for completeness sake if nothing else. To do this, observe that on $\mathbb{R}$ itself (so for $M=\{\rm pt\}$), $L^2_{\rm ti}(\mathbb{R}\times\{\rm pt\})=L^2(\mathbb{R})$ in the usual sense. Let us consider a special partition of unity on $\mathbb{R}$ consisting of integral translates of *one* function.

Definition 5. An element $\mu\in\mathcal{C}^\infty_c(\mathbb{R})$ generates a "**ti-partition of unity**" (a non-standard description) on $\mathbb{R}$ if $0\le\mu\le1$ and $\sum_{k\in\mathbb{Z}}\mu(t-k)=1$.

It is easy to construct such a μ. Just take $\mu_1\in\mathcal{C}^\infty_c(\mathbb{R})$, $\mu_1\ge0$ with $\mu_1(t)=1$ in $|t|\le1/2$. Then let
$$F(t)=\sum_{k\in\mathbb{Z}}\mu_1(t-k)\in\mathcal{C}^\infty(\mathbb{R})$$
since the sum is finite on each bounded set. Moreover $F(t)\ge1$ and is itself invariant under translation by any integer; set $\mu(t)=\mu_1(t)/F(t)$. Then μ generates a ti-partition of unity.

Using such a function we can easily decompose $L^2(\mathbb{R})$. Thus, setting $\tau_k(t)=t-k$,

$$(5.15)\qquad f\in L^2(\mathbb{R})\iff(\tau_k^*f)\mu\in L^2_{\rm loc}(\mathbb{R})\ \forall\,k\in\mathbb{Z}\text{ and }\sum_{k\in\mathbb{Z}}\int|\tau_k^*f\mu|^2\,dt<\infty.$$

Of course, saying $(\tau_k^*f)\mu\in L^2_{\rm loc}(\mathbb{R})$ is the same as $(\tau_k^*f)\mu\in L^2_{\rm c}(\mathbb{R})$. Certainly, if $f\in L^2(\mathbb{R})$ then $(\tau_k^*f)\mu\in L^2(\mathbb{R})$ and since $0\le\mu\le1$ and $\operatorname{supp}(\mu)\subset[-R,R]$ for some R,
$$\sum_k\int|(\tau_k^*f)\mu|^2\le C\int|f|^2\,dt.$$

Conversely, since $\sum_{|k|\le T}\mu = 1$ on $[-1,1]$ for some T, it follows that

$$\int |f|^2\,dt \le C'\sum_k \int |(\tau_k^* f)\mu|^2\;dt.$$

Now, $D_t\tau_k^* f = \tau_k^*(D_t f)$, so we can use (5.15) to rewrite the definition of the spaces $H^k_{\text{ti}}(\mathbb{R}\times M)$ in a form that extends to *all* orders. Namely

$$u\in H^s_{\text{ti}}(\mathbb{R}\times M) \iff (\tau_k^*u)\mu \in H^s_{\text{c}}(\mathbb{R}\times M) \text{ and } \sum_k \|\tau_k^* u\|_{H^s}<\infty \tag{5.16}$$

provided we choose a fixed norm on $H^s_{\text{c}}(\mathbb{R}\times M)$ giving the usual topology for functions supported in a fixed compact set, for example by embedding $[-T,T]$ in a torus $\mathbb{T}$ and then taking the norm on $H^s(\mathbb{T}\times M)$.

LEMMA 19. *With* $\operatorname{Diff}^m_{ti}(\mathbb{R}\times M)$ *defined by* (5.13) *and the translation-invariant Sobolev spaces by* (5.16),

$$\begin{aligned} &P\in \operatorname{Diff}^m_{ti}(\mathbb{R}\times M)\Longrightarrow \\ &\qquad\qquad P : H^{s+m}_{ti}(\mathbb{R}\times M)\longrightarrow H^s_{ti}(\mathbb{R}\times M)\ \forall\, s\in\mathbb{R}. \end{aligned} \tag{5.17}$$

PROOF. This is basically an exercise. Really we also need to check a little more carefully that the two definitions of $H^(_{\text{ti}}\mathbb{R}\times M)$ for k a positive integer, are the same. In fact this is similar to the proof of (5.17) so is omitted. So, to prove (5.17) we will proceed by induction over m. For $m=0$ there is nothing to prove. Now observe that the translation-invariant of P means that $P\tau_k^*u=\tau_k^*(Pu)$ so

$$\begin{aligned} &u\in H^{s+m}_{\text{ti}}(\mathbb{R}\times M)\Longrightarrow \\ &\qquad P(\tau_k^* u\mu) = \tau_k^*(Pu) + \sum_{m'<m}\tau_k^*(P_{m'}u)D_t^{m-m'}\mu,\ P_{m'}\in\operatorname{Diff}^{m'}_{\text{ti}}(\mathbb{R}\times M). \end{aligned} \tag{5.18}$$

The left side is in $H^s_{\text{ti}}(\mathbb{R}\times M)$, with the sum over k of the squares of the norms bounded, by the regularity of u. The same is easily seen to be true for the sum on the right by the inductive hypothesis, and hence for the first term on the right. This proves the mapping property (5.17) and continuity follows by the same argument or the closed graph theorem. □

We can, and shall, extend this in various ways. If $\mathbb{E}=(E_1,E_2)$ is a pair of vector bundles over M then it lifts to a pair of vector bundles over $\mathbb{R}\times M$, which we can again denote by $\mathbb{E}$. It is then straightforward to define $\operatorname{Diff}^m_{\text{ti}}(\mathbb{R}\times M);\mathbb{E})$ and the Sobolev spaces $H^s_{\text{ti}}(\mathbb{R}\times M;E_i)$ and to check that (5.17) extends in the obvious way.

Then main question we want to understand is the *invertibility* of an operator such as P in (5.17). However, let me look first at these Sobolev spaces a little more carefully. As already noted we really have two definitions in the case of positive integral order. Thinking about these we can also make the following provisional definitions in terms of the 1-dimensional Fourier transform discussed above – where the '$\tilde H$' notation is only temporary since these will turn out to be the same as the spaces just considered.

For any compact manifold define

(5.19)
$$\begin{aligned}\tilde{H}^s_{\mathrm{ti}}(\mathbb{R}\times M) =&\big\{u\in L^2(\mathbb{R}\times M);\\ &\|u\|^2_s=\int_{\mathbb{R}}\left(\langle\tau\rangle^s|\hat{u}(\tau,\cdot)|^2_{L^2(M)}+\int_{\mathbb{R}}|\hat{u}(\tau,\cdot)|^2_{H^s(M)}\right)d\tau<\infty\big\},\ s\geq 0\end{aligned}$$

(5.20)
$$\begin{aligned}&\tilde{H}^s_{\mathrm{ti}}(\mathbb{R}\times M)=\big\{u\in\mathcal{S}'(\mathbb{R}\times M);u=u_1+u_2,\\ &\quad u_1\in L^2(\mathbb{R};H^s(M)),\ u_2\in L^2(M;H^s(\mathbb{R}))\big\},\ \|u\|^2_s=\inf\|u_1\|^2+\|u_2\|^2,\ s<0.\end{aligned}$$

The following interpolation result for Sobolev norms on M should be back in Chapter 3.

Lemma 20. *If M is a compact manifold or $\mathbb{R}^n$ then for any $m_1\geq m_2\geq m_3$ and any R, the Sobolev norms are related by*

$$\|u\|_{m_2}\leq C\left((1+R)^{m_2-m_1}\|u\|_{m_1}+(1+R)^{m_2-m_3}\|u\|_{m_3}\right). \tag{5.21}$$

Proof. On $\mathbb{R}^n$ this follows directly by dividing Fourier space in two pieces

(5.22)
$$\begin{aligned}\|u\|^2_{m_2}=&\int_{|\zeta|>R}\langle\zeta\rangle^{2m_2}|\hat{u}|d\zeta+\int_{|\zeta|\leq R}\langle\zeta\rangle^{2m_2}|\hat{u}|d\zeta\\ &\leq\langle R\rangle^{2(m_1-m_2)}\int_{|\zeta|>R}\langle\zeta\rangle^{2m_1}|\hat{u}|d\zeta+\langle R\rangle^{2(m_2-m_3)}\int_{|\zeta|\leq R}\langle\zeta\rangle^{2m_3}|\hat{u}|d\zeta\\ &\leq\langle R\rangle^{2(m_1-m_2)}\|u\|^2_{m_1}+\langle R\rangle^{2(m_2-m_3)}\|u\|^2_{m_3}.\end{aligned}$$

On a compact manifold we have defined the norms by using a partition ϕ_i of unity subordinate to a covering by coordinate patches $F_i:Y_i\longrightarrow U'_i$:

$$\|u\|^2_m=\sum_i\|(F_i)^*(\phi_iu)\|^2_m \tag{5.23}$$

where on the right we are using the Sobolev norms on $\mathbb{R}^n$. Thus, applying the estimates for Euclidean space to each term on the right we get the same estimate on any compact manifold. □

Corollary 1. *If $u\in\tilde{H}^s_{ti}(\mathbb{R}\times M)$, for $s>0$, then for any $0<t<s$*

$$\int_{\mathbb{R}}\langle\tau\rangle^{2t}\|\hat{u}(\tau,\cdot)\|^2_{H^{s-t}(M)}d\tau<\infty \tag{5.24}$$

which we can interpret as meaning '$u\in H^t(\mathbb{R};H^{s-t}(M))$ or $u\in H^{s-t}(M;H^s(\mathbb{R}))$.'

Proof. Apply the estimate to $\hat{u}(\tau,\cdot)\in H^s(M)$, with $R=|\tau|$, $m_1=s$ and $m_3=0$ and integrate over τ. □

Lemma 21. *The Sobolev spaces $\tilde{H}^s_{ti}(\mathbb{R}\times M)$ and $H^s_{ti}(\mathbb{R}\times M)$ are the same.*

Proof. □

Lemma 22. *For $0<s<1$ $u\in H^s_{ti}(\mathbb{R}\times M)$ if and only if $u\in L^2(\mathbb{R}\times M)$ and*

(5.25)
$$\begin{aligned}\int_{\mathbb{R}^2\times M}\frac{|u(t,z)-u(t',z)|^2}{|t-t'|^{2s+1}}dtdt'\nu+\int_{\mathbb{R}\times M^2}\frac{|u(t,z')-u(t,z)|^2}{\rho(z,z')^{s+\frac{n}{2}}}dt\nu(z)\nu(z')<\infty,\\ n=\dim M,\end{aligned}$$

where $0 \le \rho \in \mathcal{C}^\infty(M^2)$ *vanishes exactly quadratically at* $\operatorname{Diag} \subset M^2$.

Proof. This follows as in the cases of $\mathbb{R}^n$ and a compact manifold discussed earlier since the second term in (5.25) gives (with the L^2 norm) a norm on $L^2(\mathbb{R}; H^s(M))$ and the first term gives a norm on $L^2(M; H^s(\mathbb{R}))$. □

Using these results we can see directly that the Sobolev spaces in (5.19) have the following 'obvious' property as in the cases of $\mathbb{R}^n$ and M.

Lemma 23. *Schwartz space* $\mathcal{S}(\mathbb{R}\times M) = \mathcal{C}^\infty(M; \mathcal{S}(\mathbb{R}))$ *is dense in each* $H^s_{ti}(\mathbb{R}\times M)$ *and the* L^2 *pairing extends by continuity to a jointly continuous non-degenerate pairing*

$$H^s_{ti}(\mathbb{R}\times M) \times H^{-s}_{ti}(\mathbb{R}\times M) \longrightarrow \mathbb{C} \tag{5.26}$$

which identifies $H^{-s}_{ti}(\mathbb{R}\times M)$ *with the dual of* $H^s_{ti}(\mathbb{R}\times M)$ *for any* $s \in \mathbb{R}$.

Proof. I leave the density as an exercise – use convolution in $\mathbb{R}$ and the density of $\mathcal{C}^\infty(M)$ in $H^s(M)$ (explicity, using a partition of unity on M and convolution on $\mathbb{R}^n$ to get density in each coordinate patch).

Then the existence and continuity of the pairing follows from the definitions and the corresponding pairings on $\mathbb{R}$ and M. We can assume that $s > 0$ in (5.26) (otherwise reverse the factors). Then if $u \in H^s_{\mathrm{ti}}(\mathbb{R}\times M)$ and $v = v_1 + v_2 \in H^{-s}_{\mathrm{ti}}(\mathbb{R}\times M)$ as in (5.20),

$$(u, v) = \int_{\mathbb{R}} (u(t,\cdot), u_1(t,\cdot))\, dt + \int_M (u(\cdot, z), v_2(\cdot, z))\, \nu_z \tag{5.27}$$

where the first pairing is the extension of the L^2 pairing to $H^s(M) \times H^{-s}(M)$ and in the second case to $H^s(\mathbb{R}) \times H^{-s}(\mathbb{R})$. The continuity of the pairing follows directly from (5.27).

So, it remains only to show that the pairing is non-degenerate – so that

$$H^{-s}_{\mathrm{ti}}(\mathbb{R}\times M) \ni v \longmapsto \sup_{\|u\|_{H^s_{\mathrm{ti}}(\mathbb{R}\times M)}=1} |(u,v)| \tag{5.28}$$

is equivalent to the norm on $H^{-s}_{\mathrm{ti}}(\mathbb{R}\times M)$. We already know that this is bounded above by a multiple of the norm on H^{-s}_{ti} so we need the estimate the other way. To see this we just need to go back to Euclidean space. Take a partition of unity ψ_i with our usual ϕ_i on M subordinate to a coordinate cover and consider with $\phi_i = 1$ in a neighbourhood of the support of ψ_i. Then

$$(u, \psi_i v) = (\phi_i u, \psi_i v) \tag{5.29}$$

allows us to extend $\psi_i v$ to a continuous linear functional on $H^s(\mathbb{R}^n)$ by reference to the local coordinates and using the fact that for $s > 0$ $(F_i^{-1})^*(\phi_i u) \in H^s(\mathbb{R}^{n+1})$. This shows that the coordinate representative of $\psi_i v$ is a sum as desired and summing over i gives the desired bound. □

2. Translation-invariant Operators

Some corrections from Fang Wang added, 25 July, 2007.

Next I will characterize those operators $P \in \operatorname{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M; \mathbb{E})$ which give invertible maps (5.17), or rather in the case of a pair of vector bundles $\mathbb{E} = (E_1, E_2)$ over M :

$$P : H^{s+m}_{\mathrm{ti}}(\mathbb{R}\times M; E_1) \longrightarrow H^s_{\mathrm{ti}}(\mathbb{R}\times M; E_2),\ P \in \operatorname{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M; \mathbb{E}). \tag{5.30}$$

This is a generalization of the 1-dimensional case, $M = \{\mathrm{pt}\}$ which we have already discussed. In fact it will become clear how to generalize some parts of the discussion below to products $\mathbb{R}^n \times M$ as well, but the case of a 1-dimensional Euclidean factor is both easier and more fundamental.

As with the constant coefficient case, there is a basic dichotomy here. A t-translation-invariant differential operator as in (5.30) is Fredholm if and only if it is invertible. To find necessary and sufficient conditons for invertibility we will we use the 1-dimensional Fourier transform as in (5.9).

If

$$P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M); \mathbb{E}) \Longleftrightarrow P = \sum_{i=0}^{m} D_t^i P_i,\ P_i \in \mathrm{Diff}^{m-i}(M; \mathbb{E}) \tag{5.31}$$

then

$$P : \mathcal{S}(\mathbb{R} \times M; E_1) \longrightarrow \mathcal{S}(\mathbb{R} \times M; E_2)$$

and

$$\widehat{Pu}(\tau, \cdot) = \sum_{i=0}^{m} \tau^i P_i \widehat{u}(\tau, \cdot) \tag{5.32}$$

where $\widehat{u}(\tau, \cdot)$ is the 1-dimensional Fourier transform from (5.9). So we clearly need to examine the "suspended" family of operators

$$P(\tau) = \sum_{i=0}^{m} \tau^i P_i \in \mathcal{C}^\infty\left(\mathbb{C}; \mathrm{Diff}^m(M; \mathbb{E})\right). \tag{5.33}$$

I use the term "suspended" to denote the addition of a parameter to $\mathrm{Diff}^m(M; \mathbb{E})$ to get such a family—in this case polynomial. They are sometimes called "operator pencils" for reasons that escape me. Anyway, the main result we want is

THEOREM 6. *If $P \in \mathrm{Diff}^m_{ti}(M; \mathbb{E})$ is elliptic then the suspended family $P(\tau)$ is invertible for all $\tau \in \mathbb{C} \setminus D$ with inverse*

$$P(\tau)^{-1} : H^s(M; E_2) \longrightarrow H^{s+m}(M; E_1) \tag{5.34}$$

where

$$D \subset \mathbb{C} \text{ is discrete and } D \subset \{\tau \in \mathbb{C};\ |\operatorname{Re}\tau| \le c|\operatorname{Im}\tau| + 1/c\} \tag{5.35}$$

for some $c > 0$ (see Fig. 1 – still not quite right).

In fact we need some more information on $P(\tau)^{-1}$ which we will pick up during the proof of this result. The translation-invariance of P can be written in operator form as

$$Pu(t + s, \cdot) = (Pu)(t + s, \cdot)\ \forall\ s \in \mathbb{R} \tag{5.36}$$

LEMMA 24. *If $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ is elliptic then it has a parametrix*

$$Q : \mathcal{S}(\mathbb{R} \times M; E_2) \longrightarrow \mathcal{S}(\mathbb{R} \times M; E_1) \tag{5.37}$$

which is translation-invariant in the sense of (5.36) *and preserves the compactness of supports in $\mathbb{R}$,*

$$Q : \mathcal{C}^\infty_c(\mathbb{R} \times M; E_2) \longrightarrow \mathcal{C}^\infty_c(\mathbb{R} \times M; E_1) \tag{5.38}$$

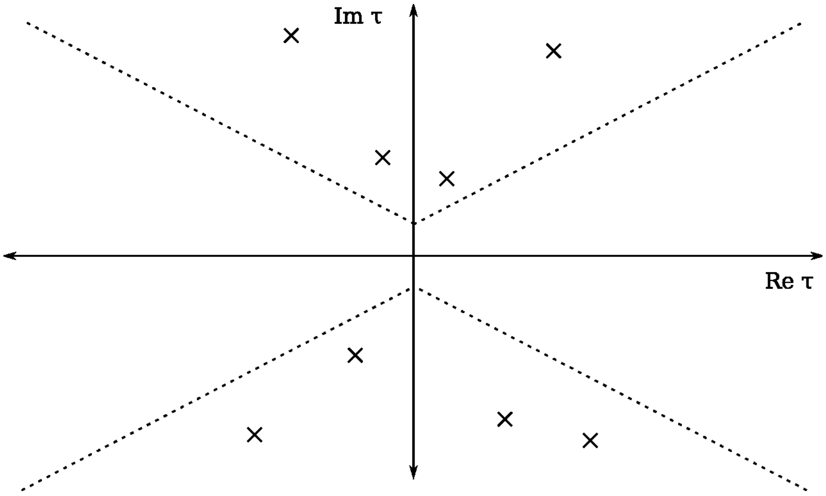

FIGURE 1. The region D.

PROOF. In the case of a compact manifold we contructed a global parametrix by patching local parametricies with a partition of unity. Here we do the same thing, treating the variable $t \in \mathbb{R}$ globally throughout. Thus if $F_a : \Omega_a \to \Omega'_a$ is a coordinate patch in M over which E_1 and (hence) E_2 are trivial, P becomes a square matrix of differential operators

$$P_a = \begin{bmatrix} P_{11}(z, D_t, D_z) & \cdots & P_{l1}(z, D_t, D_z) \\ \vdots & & \vdots \\ P_{1l}(z, D_t, D_z) & \cdots & P_{ll}(z, D_t, D_z) \end{bmatrix} \tag{5.39}$$

in which the coefficients do *not* depend on t. As discussed in Sections 2 and 3 above, we can construct a local parametrix in Ω'_a using a properly supported cutoff χ. In the t variable the parametrix is global anyway, so we use a fixed cutoff $\tilde{\chi} \in \mathcal{C}^\infty_c(\mathbb{R})$, $\tilde{\chi} = 1$ in $|t| < 1$, and so construct a parametrix

$$Q_a f(t, z) = \int_{\Omega'_a} q(t - t', z, z')\tilde{\chi}(t - t')\chi(z, z')f(t', z')\, dt'\, dz'. \tag{5.40}$$

This satisfies

$$P_a Q_a = \operatorname{Id} - R_a, \quad Q_a P_a = \operatorname{Id} - R'_a \tag{5.41}$$

where R_a and R'_a are smoothing operators on Ω'_a with kernels of the form

$$\begin{gathered} R_a f(t, z) = \int_{\Omega'_a} R_a(t - t', z, z')f(t', z')\, dt'\, dz' \\ R_a \in \mathcal{C}^\infty(\mathbb{R} \times \Omega'^2_a),\ R_a(t, z, z') = 0 \text{ if } |t| \geq 2 \end{gathered} \tag{5.42}$$

with the support proper in Ω'_a.

Now, we can sum these local parametricies, which are all t-translation-invariant to get a global parametrix with the same properties

$$Qf = \sum_a \chi_a (F_a^{-1})^* (T_a^{-1})^* Q_a T_a^* F_a^* f \tag{5.43}$$

where T_a denotes the trivialization of bundles E_1 and E_2. It follows that Q satisfies (5.38) and since it is translation-invariant, also (5.37). The global version of (5.41) becomes

$$\begin{aligned} PQ &= \operatorname{Id} - R_2, \quad QP = \operatorname{Id} - R_1, \\ R_i :&\mathcal{C}^\infty_c(\mathbb{R} \times M; E_i) \longrightarrow \mathcal{C}^\infty_c(\mathbb{R} \times M; E_i), \\ R_i f &= \int_{\mathbb{R}\times M} R_i(t - t', z, z') f(t', z')\, dt'\, \nu_{z'} \end{aligned} \tag{5.44}$$

where the kernels

$$R_i \in \mathcal{C}^\infty_c \left(\mathbb{R} \times M^2; \operatorname{Hom}(E_i)\right),\ i = 1, 2. \tag{5.45}$$

□

In fact we can deduce directly from (5.40) the boundedness of Q.

Lemma 25. *The properly-supported parametrix Q constructed above extends by continuity to a bounded operator*

$$\begin{aligned} Q :&H^s_{ti}(\mathbb{R} \times M; E_2) \longrightarrow H^{s+m}_{ti}(\mathbb{R} \times M; E_1)\ \forall\, s \in \mathbb{R} \\ Q :&\mathcal{S}(\mathbb{R} \times M; E_2) \longrightarrow \mathcal{S}(\mathbb{R} \times M; E_1). \end{aligned} \tag{5.46}$$

Proof. This follows directly from the earlier discussion of elliptic regularity for each term in (5.43) to show that

$$\begin{aligned} Q : \{f \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2;\ \operatorname{supp}(f) \subset [-2, 2] \times M\} \\ \longrightarrow \left\{u \in H^{s+m}_{\mathrm{ti}}(\mathbb{R} \times M; E_1;\ \operatorname{supp}(u) \subset [-2 - R, 2 + R] \times M\right\} \end{aligned} \tag{5.47}$$

for some R (which can in fact be taken to be small and positive). Indeed on compact sets the translation-invariant Sobolev spaces reduce to the usual ones. Then (5.46) follows from (5.47) and the translation-invariance of Q. Using a $\mu \in \mathcal{C}^\infty_c(\mathbb{R})$ generating a ti-paritition of unity on $\mathbb{R}$ we can decompose

$$H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2) \ni f = \sum_{k\in\mathbb{Z}} \tau^*_k(\mu\tau^*_{-k} f). \tag{5.48}$$

Then

$$Qf = \sum_{k\in\mathbb{Z}} \tau^*_k \left(Q(\mu\tau^*_{-k} f)\right). \tag{5.49}$$

The estimates corresponding to (5.47) give

$$\|Qf\|_{H^{s+m}_{\mathrm{ti}}} \le C\|f\|_{H^s_{\mathrm{ti}}}$$

if f has support in $[-2, 2] \times M$. The decomposition (5.48) then gives

$$\sum \|\mu\tau^*_{-k} f\|^2_{H^s} = \|f\|^2_{H_s} < \infty \implies \|Qf\|^2 \le C'\|f\|^2_{H^s}.$$

This proves Lemma 25. □

Going back to the remainder term in (5.44), we can apply the 1-dimensional Fourier transform and find the following uniform results.

Lemma 26. *If R is a compactly supported, t-translation-invariant smoothing operator as in* (5.44) *then*

$$\widehat{Rf}(\tau,\cdot)=\widehat{R}(\tau)\widehat{f}(\tau,\cdot) \tag{5.50}$$

where $\widehat{R}(\tau)\in\mathcal{C}^\infty\big(\mathbb{C}\times M^2;\operatorname{Hom}(E)\big)$ is entire in $\tau\in\mathbb{C}$ and satisfies the estimates

$$\forall\, k,p\ \exists\, C_{p,k} \text{ such that } \|\tau^k\widehat{R}(\tau)\|_{\mathcal{C}^p}\le C_{p,k}\exp(A|\operatorname{Im}\tau|). \tag{5.51}$$

Here A is a constant such that

$$\operatorname{supp} R(t,\cdot)\subset[-A,A]\times M^2. \tag{5.52}$$

Proof. This is a parameter-dependent version of the usual estimates for the Fourier-Laplace transform. That is,

$$\widehat{R}(\tau,\cdot)=\int e^{-i\tau t}R(t,\cdot)\,dt \tag{5.53}$$

from which all the statements follow just as in the standard case when $R\in\mathcal{C}^\infty_c(\mathbb{R})$ has support in $[-A,A]$. □

Proposition 18. *If R is as in Lemma 26 then there exists a discrete subset $D\subset\mathbb{C}$ such that $\Big(\operatorname{Id}-\widehat{R}(\tau)\Big)^{-1}$ exists for all $\tau\in\mathbb{C}\setminus D$ and*

$$\Big(\operatorname{Id}-\widehat{R}(\tau)\Big)^{-1}=\operatorname{Id}-\widehat{S}(\tau) \tag{5.54}$$

where $\widehat{S}:\mathbb{C}\longrightarrow\mathcal{C}^\infty(M^2;\operatorname{Hom}(E))$ is a family of smoothing operators which is meromorphic in the complex plane with poles of finite order and residues of finite rank at D. Furthermore,

$$D\subset\{\tau\in\mathbb{C};\ \log(|\operatorname{Re}\tau|)<c|\operatorname{Im}\tau|+1/c\} \tag{5.55}$$

for some $c>0$ and for any $C>0$, there exists C' such that

$$|\operatorname{Im}\tau|<C,\ |\operatorname{Re}\tau|>C' \implies \|\tau^k\widehat{S}(\tau)\|_{\mathcal{C}^p}\le C_{p,k}. \tag{5.56}$$

Proof. This is part of “Analytic Fredholm Theory” (although usually done with compact operators on a Hilbert space). The estimates (5.51) on $\widehat{R}(\tau)$ show that, in some region as on the right in (5.55),

$$\|\widehat{R}(\tau)\|_{L^2}\le 1/2. \tag{5.57}$$

Thus, by Neumann series,

$$\widehat{S}(\tau)=\sum_{k=1}^{\infty}\Big(\widehat{R}(\tau)\Big)^k \tag{5.58}$$

exists as a bounded operator on $L^2(M;E)$. In fact it follows that $\widehat{S}(\tau)$ is itself a family of smoothing operators in the region in which the Neumann series converges. Indeed, the series can be rewritten

$$\widehat{S}(\tau)=\widehat{R}(\tau)+\widehat{R}(\tau)^2+\widehat{R}(\tau)\widehat{S}(\tau)\widehat{R}(\tau) \tag{5.59}$$

The smoothing operators form a “corner” in the bounded operators in the sense that products like the third here are smoothing if the outer two factors are. This follows from the formula for the kernel of the product

$$\int_{M\times M}\widehat{R}_1(\tau;z,z')\widehat{S}(\tau;z',z'')\widehat{R}_2(\tau;z'',\tilde{z})\,\nu_{z'}\,\nu_{z''}.$$

Thus $\widehat{S}(\tau) \in \mathcal{C}^\infty(M^2;\operatorname{Hom}(E))$ exists in a region as on the right in (5.55). To see that it extends to be meromorphic in $\mathbb{C} \setminus D$ for a discrete divisor D we can use a finite-dimensional approximation to $\widehat{R}(\tau)$.

Recall — if neccessary from local coordinates — that given any $p \in \mathbb{N}, R > 0, q > 0$ there are finitely many sections $f_i^{(\tau)} \in \mathcal{C}^\infty(M;E'), g_i^{(\tau)} \in \mathcal{C}^\infty(M;E)$ and such that

$$\|\widehat{R}(\tau) - \sum_i g_i(\tau,z)\cdot f_i(\tau,z')\|_{\mathcal{C}^p} < \epsilon, \quad |\tau| < R. \tag{5.60}$$

Writing this difference as $M(\tau)$,

$$\operatorname{Id} - \widehat{R}(\tau) = \operatorname{Id} - M(\tau) + F(\tau)$$

where $F(\tau)$ is a finite rank operator. In view of (5.60), $\operatorname{Id} - M(\tau)$ is invertible and, as seen above, of the form $\operatorname{Id} - \widehat{M}(\tau)$ where $\widehat{M}(\tau)$ is holomorphic in $|\tau| < R$ as a smoothing operator.

Thus

$$\operatorname{Id} - \widehat{R}(\tau) = (\operatorname{Id} - M(\tau))(\operatorname{Id} + F(\tau) - \widehat{M}(\tau)F(\tau))$$

is invertible if and only if the finite rank perturbation of the identity by $(\operatorname{Id} - \widehat{M}(\tau))F(\tau)$ is invertible. For R large, by the previous result, this finite rank perturbation must be invertible in an open set in $\{|\tau| < R\}$. Then, by standard results for finite dimensional matrices, it has a meromorphic inverse with finite rank (generalized) residues. The same is therefore true of $\operatorname{Id} - \widehat{R}(\tau)$ itself.

Since $R > 0$ is arbitrary this proves the result. □

Proof. Proof of Theorem 6 We have proved (5.44) and the corresponding form for the Fourier transformed kernels follows:

$$\widehat{P}(\tau)\widehat{Q}'(\tau) = \operatorname{Id} - \widehat{R}_2(\tau),\ \widehat{Q}'(\tau)\widehat{P}(\tau) = \operatorname{Id} - \widehat{R}_1(\tau) \tag{5.61}$$

where $\widehat{R}_1(\tau), \widehat{R}_2(\tau)$ are families of smoothing operators as in Proposition 18. Applying that result to the first equation gives a new meromorphic right inverse

$$\widehat{Q}(\tau) = \widehat{Q}'(\tau)(\operatorname{Id} - \widehat{R}_2(\tau))^{-1} = \widehat{Q}'(\tau) - \widehat{Q}'(\tau)M(\tau)$$

where the first term is entire and the second is a meromorphic family of smoothing operators with finite rank residues. The same argument on the second term gives a left inverse, but his shows that $\widehat{Q}(\tau)$ must be a two-sided inverse.

This we have proved everything except the locations of the poles of $\widehat{Q}(\tau)$ — which are only constrained by (5.55) instead of (5.35). However, we can apply the same argument to $P_\theta(z, D_t, D_z) = P(z, e^{i\theta}D_t, D_z)$ for $|\theta| < \delta$, $\delta > 0$ small, since P_θ stays elliptic. This shows that the poles of $\widehat{Q}(\tau)$ lie in a set of the form (5.35). □

3. Invertibility

We are now in a position to characterize those t-translation-invariant differential operators which give isomorphisms on the translation-invariant Sobolev spaces.

Theorem 7. *An element $P \in \operatorname{Diff}^m_{ti}(\mathbb{R} \times M;E)$ gives an isomorphism* (5.30) *(or equivalently is Fredholm) if and only if it is elliptic and $D \cap \mathbb{R} = \emptyset$, i.e. $\hat{P}(\tau)$ is invertible for all $\tau \in \mathbb{R}$.*

PROOF. We have already done most of the work for the important direction for applications, which is that the ellipticity of P and the invertibility at $\hat{P}(\tau)$ for all $\tau \in \mathbb{R}$ together imply that (5.30) is an isomorphism for any $s \in \mathbb{R}$.

Recall that the ellipticity of P leads to a parameterix Q which is translation-invariant and has the mapping property we want, namely (5.46).

To prove the same estimate for the true inverse (and its existence) consider the difference

$$\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{\mathbb{R}}(\tau),\ \tau \in \mathbb{R}. \tag{5.62}$$

Since $\hat{P}(\tau) \in \mathrm{Diff}^m(M;\mathbb{E})$ depends smoothly on $\tau \in \mathbb{R}$ and $\hat{Q}(\tau)$ is a paramaterix for it, we know that

$$\hat{R}(\tau) \in \mathcal{C}^\infty(\mathbb{R};\Psi^{-\infty}(M;\mathbb{E})) \tag{5.63}$$

is a smoothing operator on M which depends smoothly on $\tau \in \mathbb{R}$ as a parameter. On the other hand, from (5.61) we also know that for large real τ,

$$\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{Q}(\tau)M(\tau)$$

where $M(\tau)$ satisfies the estimates (5.56). It follows that $\hat{Q}(\tau)M(\tau)$ also satisfies these estimates and (5.63) can be strengthened to

$$\sup_{\tau\in\mathbb{R}} \|\tau^k \hat{R}(\tau,\cdot,\cdot)\|_{\mathcal{C}^p} < \infty\ \forall\ p,k. \tag{5.64}$$

That is, the kernel $\hat{R}(\tau) \in \mathcal{S}(\mathbb{R};\mathcal{C}^\infty(M^2;\mathrm{Hom}(\mathbb{E})))$. So if we define the t-translation-invariant operator

$$Rf(t,z) = (2\pi)^{-1}\int e^{it\tau}\hat{R}(\tau)\hat{f}(\tau,\cdot)d\tau \tag{5.65}$$

by inverse Fourier transform then

$$R : H^s_{\mathrm{ti}}(\mathbb{R}\times M;E_2) \longrightarrow H^\infty_{\mathrm{ti}}(\mathbb{R}\times M;E_1)\ \forall\ s \in \mathbb{R}. \tag{5.66}$$

It certainly suffices to show this for $s < 0$ and then we know that the Fourier transform gives a map

$$\mathcal{F} : H^s_{\mathrm{ti}}(\mathbb{R}\times M;E_2) \longrightarrow \langle\tau\rangle^{|s|}L^2(\mathbb{R};H^{-|s|}(M;E_2)). \tag{5.67}$$

Since the kernel $\hat{R}(\tau)$ is rapidly decreasing in τ, as well as being smooth, for every $N > 0$,

$$\hat{R}(\tau) : \langle\tau\rangle^{|s|}L^2(\mathbb{R};H^{-|s|}M;E_2) \longrightarrow \langle\tau\rangle^{-N}L^2(\mathbb{R};H^N(M;E_2)) \tag{5.68}$$

and inverse Fourier transform maps

$$\mathcal{F}^{-1} : \langle\tau\rangle^{-N}H^N(M;E_2) \longrightarrow H^N_{\mathrm{ti}}(\mathbb{R}\times M;E_2)$$

which gives (5.66).

Thus $Q+R$ has the same property as Q in (5.46). So it only remains to check that $Q+R$ is the two-sided version of P and it is enough to do this on $\mathcal{S}(\mathbb{R}\times M;E_i)$ since these subspaces are dense in the Sobolev spaces. This in turn follows from (5.62) by taking the Fourier transform. Thus we have shown that the invertibility of P follows from its ellipticity and the invertibility of $\hat{P}(\tau)$ for $\tau \in \mathbb{R}$.

The converse statement is less important but certainly worth knowing! If P is an isomorphism as in (5.30), even for one value of s, then it must be elliptic — this follows as in the compact case since it is everywhere a local statement. Then if $\hat{P}(\tau)$ is not invertible for some $\tau \in \mathbb{R}$ we know, by ellipticity, that it is Fredholm and, by the stability of the index, of index zero (since $\hat{P}(\tau)$ *is* invertible for a dense set of $\tau \in \mathbb{C}$). There is therefore some $\tau_0 \in \mathbb{R}$ and $f_0 \in \mathcal{C}^\infty(M; E_2)$, $f_0 \neq 0$, such that

$$\hat{P}(\tau_0)^* f_0 = 0. \tag{5.69}$$

It follows that f_0 is *not* in the range of $\hat{P}(\tau_0)$. Then, choose a cut off function, $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\rho(\tau_0) = 1$ (and supported sufficiently close to τ_0) and define $f \in \mathcal{S}(\mathbb{R} \times M; E_2)$ by

$$\hat{f}(\tau, \cdot) = \rho(\tau) f_0(\cdot). \tag{5.70}$$

Then $f \notin P \cdot H^s_{\text{ti}}(\mathbb{R} \times M; E_1)$ for any $s \in \mathbb{R}$. To see this, suppose $u \in H^s_{\text{ti}}(\mathbb{R} \times M; E_1)$ has

$$Pu = f \Rightarrow \hat{P}(\tau)\hat{u}(\tau) = \hat{f}(\tau) \tag{5.71}$$

where $\hat{u}(\tau) \in \langle\tau\rangle^{|s|} L^2(\mathbb{R}; H^{-|s|}(M; E_1))$. The invertibility of $P(\tau)$ for $\tau \neq \tau_0$ on $\operatorname{supp}(\rho)$ (chosen with support close enough to τ_0) shows that

$$\hat{u}(\tau) = \hat{P}(\tau)^{-1} \hat{f}(\tau) \in \mathcal{C}^\infty((\mathbb{R} \backslash \{\tau_0\}) \times M; E_1).$$

Since we know that $\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{R}(\tau)$ is a meromorphic family of smoothing operators it actually follows that $\hat{u}(\iota)$ is meromorphic in τ near τ_0 in the sense that

$$\hat{u}(\tau) = \sum_{j=1}^{k} (\tau - \tau_0)^{-j} u_j + v(\tau) \tag{5.72}$$

where the $u_j \in \mathcal{C}^\infty(M; E_1)$ and $v \in \mathcal{C}^\infty((\tau - \epsilon, \tau + \epsilon) \times M; E_1)$. Now, one of the u_j is not identically zero, since otherwise $\hat{P}(\tau_0) v(\tau_0) = f_0$, contradicting the choice of f_0. However, a function such as (5.72) is *not* locally in L^2 with values in any Sobolev space on M, which contradicts the existence of $u \in H^s_{\text{ti}}(\mathbb{R} \times M; E_1)$.

This completes the proof for invertibility of P. To get the Fredholm version it suffices to prove that if P is Fredholm then it is invertible. Since the arguments above easily show that the null space of P is empty on any of the $H^s_{\text{ti}}(\mathbb{R} \times M; E_1)$ spaces and the same applies to the adjoint, we easily conclude that P is an isomorphism if it is Fredholm. □

This result allows us to deduce similar invertibility conditions on exponentially-weighted Sobolev spaces. Set

$$e^{at} H^s_{\text{ti}}(\mathbb{R} \times M; E) = \{u \in H^s_{\text{loc}}(\mathbb{R} \times M; E);\ e^{-at} u \in H^s_{\text{ti}}(\mathbb{R} \times M; E)\} \tag{5.73}$$

for any $\mathcal{C}^\infty$ vector bundle E over M. The translation-invariant differential operators also act on these spaces.

LEMMA 27. *For any $a \in \mathbb{R}$, $P \in \operatorname{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ defines a continuous linear operator*

$$P : e^{at} H^{s+m}_{ti}(\mathbb{R} \times M; E_1) \longrightarrow e^{at} H^{s+m}_{ti}(\mathbb{R} \times M; E_2). \tag{5.74}$$

PROOF. We already know this for $a = 0$. To reduce the general case to this one, observe that (5.74) just means that

$$P \cdot e^{at} u \in e^{at} H^s_{\text{ti}}(\mathbb{R} \times M; E_2)\ \forall\ u \in H^s_{\text{ti}}(\mathbb{R} \times M; E_1) \tag{5.75}$$

with continuity meaning just continuous dependence on u. However, (5.75) in turn means that the conjugate operator

$$P_a = e^{-at} \cdot P \cdot e^{at} : H^{s+m}_{\mathrm{ti}}(\mathbb{R} \times M; E_1) \longrightarrow H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2). \tag{5.76}$$

Conjugation by an exponential is actually an isomorphism

$$\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M; \mathbb{E}) \ni P \longmapsto e^{-at} P e^{at} \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M; \mathbb{E}). \tag{5.77}$$

To see this, note that elements of $\mathrm{Diff}^j(M; \mathbb{E})$ commute with multiplication by e^{at} and

$$e^{-at} D_t e^{at} = D_t - ia \tag{5.78}$$

which gives (5.77)).

The result now follows. □

Proposition 19. *If $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ is elliptic then as a map (5.74) it is invertible precisely for*

$$a \notin -\mathrm{Im}(D),\ D = D(P) \subset \mathbb{C}, \tag{5.79}$$

that is, a is not the negative of the imaginary part of an element of D.

Note that the set $-\mathrm{Im}(D) \subset \mathbb{R}$, for which invertibility fails, is discrete. This follows from the discreteness of D and the estimate (5.35). Thus in Fig 1 invertibility on the space with weight e^{at} correspond exactly to the horizonatal line with $\mathrm{Im}\,\tau = -a$ missing D.

Proof. This is direct consequence of (**??**) and the discussion around (5.76). Namely, P is invertible as a map (5.74) if and only if P_a is invertible as a map (5.30) so, by Theorem 7, if

and only if

$$D(P_a) \cap \mathbb{R} = \emptyset.$$

From (5.78), $D(P_a) = D(P) + ia$ so this condition is just $D(P) \cap (\mathbb{R} - ia) = \emptyset$ as claimed. □

Although this is a characterization of the Fredholm properties on the standard Sobolev spaces, it is not the end of the story, as we shall see below.

One important thing to note is that $\mathbb{R}$ has *two* ends. The exponential weight e^{at} treats these differently – since if it is big at one end it is small at the other – and in fact we (or rather you) can easily define doubly-exponentially weighted spaces and get similar results for those. Since this is rather an informative extended exercise, I will offer some guidance.

Definition 6. Set

$$\begin{aligned} H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R} \times M; E) = \{&u \in H^s_{\mathrm{loc}}(\mathbb{R} \times M; E);\\ &\chi(t) e^{-at} u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E)(1 - \chi(t)) e^{bt} u \in H^s_{ti}(\mathbb{R} \times M; E)\} \end{aligned} \tag{5.80}$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R})$, $\chi = 1$ in $t > 1$, $\chi = 0$ in $t < -1$.

Exercises.

(1) Show that the spaces in (5.80) are independent of the choice of χ, are all Hilbertable (are complete with respect to a Hilbert norm) and show that if $a+b\ge 0$

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E)=e^{at}H^{s}_{\mathrm{ti}}(\mathbb{R}\times M;E)+e^{-bt}H^{s}_{\mathrm{ti}}(\mathbb{R}\times M;E) \tag{5.81}$$

whereas if $a+b\le 0$ then

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E)=e^{at}H^{s}_{\mathrm{ti}}(\mathbb{R}\times M;E)\cap e^{-bt}H^{s}_{\mathrm{ti}}(\mathbb{R}\times M;E). \tag{5.82}$$

(2) Show that any $P\in\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M;\mathbb{E})$ defines a continuous linear map for any s, a, $b\in\mathbb{R}$

$$P:H^{s+m,a,b}_{\mathrm{ti\text{-}exp}}(\mathbb{R}\times M;E_1)\longrightarrow H^{s,a,b}_{\mathrm{ti\text{-}exp}}(\mathbb{R}\times M;E_2). \tag{5.83}$$

(3) Show that the standard L^2 pairing, with respect to dt, a smooth positive density on M and an inner product on E extends to a non-degenerate bilinear pairing

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E)\times H^{-s,-a,-b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E)\longrightarrow\mathbb{C} \tag{5.84}$$

for any s, a and b. Show that the adjoint of P with respect to this pairing is P^* on the 'negative' spaces – you can use this to halve the work below.

(4) Show that if P is elliptic then (5.83) is Fredholm precisely when

$$a\notin-\operatorname{Im}(D)\text{ and }b\notin\operatorname{Im}(D). \tag{5.85}$$

Hint:- Assume for instance that $a+b\ge 0$ and use (5.81). Given (5.85) a parametrix for P can be constructed by combining the inverses on the single exponential spaces

$$Q_{a,b}=\chi' P_a^{-1}\chi+(1-\chi'')P_{-b}^{-1}(1-\chi) \tag{5.86}$$

where χ is as in (5.80) and χ' and χ'' are similar but such that $\chi'\chi=1$, $(1-\chi'')(1-\chi)=1-\chi$.

(5) Show that P is an isomorphism if and only if

$$a+b\le 0\text{ and }[a,-b]\cap-\operatorname{Im}(D)=\emptyset\text{ or }a+b\ge 0\text{ and }[-b,a]\cap-\operatorname{Im}(D)=\emptyset.$$

(6) Show that if $a+b\le 0$ and (5.85) holds then

$$\operatorname{ind}(P)=\dim\operatorname{null}(P)=\sum_{\tau_i\in D\cap(\mathbb{R}\times[b,-a])}\operatorname{Mult}(P,\tau_i)$$

where $\operatorname{Mult}(P,\tau_i)$ is the *algebraic* multiplicity of τ as a 'zero' of $\hat{P}(\tau)$, namely the dimension of the generalized null space

$$\operatorname{Mult}(P,\tau_i)=\dim\left\{u=\sum_{p=0}^{N}u_p(z)D^p_\tau\delta(\tau-\tau_i);\,P(\tau)u(\tau)\equiv 0\right\}.$$

(7) Characterize these multiplicities in a more algebraic way. Namely, if τ' is a zero of $P(\tau)$ set $E_0=\operatorname{null}P(\tau')$ and $F_0=\mathcal{C}^\infty(M;E_2)/P(\tau')\mathcal{C}^\infty(M;E_1)$. Since $P(\tau)$ is Fredholm of index zero, these are finite dimensional vector spaces of the same dimension. Let the derivatives of P be $T_i=\partial^i P/\partial\tau^i$ at $\tau=\tau'$ Then define $R_1:E_0\longrightarrow F_0$ as T_1 restricted to E_0 and projected to F_0. Let E_1 be the null space of R_1 and $F_1=F_0/R_1E_0$. Now proceed inductively and define for each i the space E_i as the null space of R_i, $F_i=F_{i-1}/R_iE_{i-1}$ and $R_{i+1}:E_i\longrightarrow F_i$ as T_i restricted to E_i and projected to F_i. Clearly E_i and F_i have the same, finite, dimension which

is non-increasing as i increases. The properties of $P(\tau)$ can be used to show that for large enough i, $E_i = F_i = \{0\}$ and

$$\operatorname{Mult}(P,\tau') = \sum_{i=0}^{\infty} \dim(E_i) \tag{5.87}$$

where the sum is in fact finite.

(8) Derive, by duality, a similar formula for the index of P when $a+b \geq 0$ and (5.85) holds, showing in particular that it is injective.

4. Resolvent operator

Addenda to Chapter 5

More?

- Why – manifold with boundary later for Euclidean space, but also resolvent (Photo-C5-01)
- Hölder type estimates – Photo-C5-03. Gives interpolation.

As already noted even a result such as Proposition 19 and the results in the exercises above by no means exhausts the possibile realizations of an element $P \in \operatorname{Diff}^m_{\text{ti}}(\mathbb{R}\times M;\mathbb{E})$ as a Fredholm operator. Necessarily these other realization cannot simply be between spaces like those in (5.80). To see what else one can do, suppose that the condition in Theorem 7 is violated, so

$$D(P) \cap \mathbb{R} = \{\tau_1, \dots, \tau_N\} \neq \emptyset. \tag{5.88}$$

To get a Fredholm operator we need to change either the domain or the range space. Suppose we want the range to be $L^2(\mathbb{R}\times M; E_2)$. Now, the condition (5.85) guarantees that P is Fredholm as an operator (5.83). So in particular

$$P : H^{m,\epsilon,\epsilon}_{\text{ti}-\exp}(\mathbb{R}\times M; E_1) \longrightarrow H^{0,\epsilon,\epsilon}_{\text{ti-}\exp}(\mathbb{R}\times M; E_2) \tag{5.89}$$

is Fredholm for all $\epsilon > 0$ sufficiently small (becuase D is discrete). The image space (which is necessarily the range in this case) just consists of the sections of the form $\exp(a|t|)f$ with f in L^2. So, in this case the range certainly contains L^2 so we can define

$$\operatorname{Dom}_{\text{AS}}(P) = \{u \in H^{m,\epsilon,\epsilon}_{\text{ti-}\exp}(\mathbb{R}\times M; E_1); Pu \in L^2(\mathbb{R}\times M; E_2)\},\ \epsilon > 0 \text{ sufficiently small.} \tag{5.90}$$

This space is independent of $\epsilon > 0$ if it is taken smalle enough, so the same space arises by taking the intersection over $\epsilon > 0$.

Proposition 20. *For any elliptic element $P \in \operatorname{Diff}^m_{ti}(\mathbb{R}\times M;\mathbb{E})$ the space in (5.90) is Hilbertable space and*

$$P : \operatorname{Dom}_{AS}(P) \longrightarrow L^2(\mathbb{R}\times M; E_2) \textit{ is Fredholm.} \tag{5.91}$$

I have not made the assumption (5.88) since it is relatively easy to see that if $D \cap \mathbb{R} = \emptyset$ then the domain in (5.90) reduces again to $H^m_{\text{ti}}(\mathbb{R}\times M; E_1)$ and (5.91) is just the standard realization. Conversely of course under the assumption (5.88) the domain in (5.91) is strictly larger than the standard Sobolev space. To see what it actually is requires a bit of work but if you did the exercises above you are in a position to work this out! Here is the result when there is only one pole of $\hat{P}(\tau)$ on the real line and it has order one.

Proposition 21. *Suppose* $P \in \operatorname{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ *is elliptic,* $\hat{P}(\tau)$ *is invertible for* $\tau \in \mathbb{R} \setminus \{0\}$ *and in addition* $\tau \hat{P}(\tau)^{-1}$ *is holomorphic near* 0. *Then the Atiyah-Singer domain in* (5.91) *is*

$$\begin{gathered} \operatorname{Dom}_{AS}(P) = \big\{u = u_1 + u_2; u_1 \in H^m_{ti}(\mathbb{R} \times M; E_1), \\ u_2 = f(t)v,\ v \in \mathcal{C}^\infty(M; E_1),\ \hat{P}(0)v = 0,\ f(t) = \int_0^t g(t)dt,\ g \in H^{m-1}(\mathbb{R})\big\}. \end{gathered} \tag{5.92}$$

Notice that the 'anomalous' term here, u_2, need not be square-integrable. In fact for any $\delta > 0$ the power $\langle t \rangle^{\frac{1}{2}-\delta} v \in \langle t \rangle^{1-\delta} L^2(\mathbb{R} \times M; E_1)$ is included and conversely

$$f \in \bigcap_{\delta > 0} \langle t \rangle^{1+\delta} H^{m-1}(\mathbb{R}). \tag{5.93}$$

One can say a lot more about the growth of f if desired but it is generally quite close to $\langle t \rangle L^2(\mathbb{R})$.

Domains of this sort are sometimes called 'extended L^2 domains' – see if you can work out what happens more generally.