CHAPTER 3

Coordinate invariance and manifolds

$\boxed{\boxed{\text{0.6P; Revised: 18-7-2007; Run: August 2, 2007}}}$

For the geometric applications we wish to make later (and of course many others) it is important to understand how the objects discussed above behave under coordinate transformations, so that they can be transferred to manifolds (and vector bundles). The basic principle is that the results above are independent of the choice of coordinates, which is to say diffeomorphisms of open sets.

1. Local diffeomorphisms

Let $\Omega_i \subset \mathbb{R}^n$ be open and $f : \Omega_1 \longrightarrow \Omega_2$ be a diffeomorphism, so it is a $\mathcal{C}^\infty$ map, which is equivalent to the condition

$$f^*u \in \mathcal{C}^\infty(\Omega_1)\ \forall\ u \in \mathcal{C}^\infty(\Omega_2),\ f^*u = u \circ f,\ f^*u(z) = u(f(z)), \tag{3.1}$$

and has a $\mathcal{C}^\infty$ inverse $f^{-1} : \Omega_2 \longrightarrow \Omega_1$. Such a map induces an isomorphisms $f^* : \mathcal{C}^\infty_c(\Omega_2) \longrightarrow \mathcal{C}^\infty_c(\Omega_1)$ and $f^* : \mathcal{C}^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_1)$ with inverse $(f^{-1})^* = (f^*)^{-1}$.

Recall also that, as a homeomorphism, f^* identifies the (Borel) measurable functions on Ω_2 with those on Ω_1. Since it is continuously differentiable it also identifies $L^1_{\text{loc}}(\Omega_2)$ with $L^1_{\text{loc}}(\Omega_1)$ and

$$u \in L^1_{\mathrm{c}}(\Omega_2) \Longrightarrow \int_{\Omega_1} f^*u(z)|J_f(z)|dz = \int_{\Omega_1} u(z')dz',\ J_f(z) = \det \frac{\partial f_i(z)}{\partial z_j}. \tag{3.2}$$

The absolute value appears because the definition of the Lebesgue integral is through the Lebesgue measure.

It follows that $f^* : L^2_{\text{loc}}(\Omega_2) \longrightarrow L^2_{\text{loc}}(\Omega_1)$ is also an isomorphism. If $u \in L^2(\Omega_2)$ has support in some compact subset $K \Subset \Omega_2$ then f^*u has support in the compact subset $f^{-1}(K) \Subset \Omega_1$ and

$$\|f^*u\|^2_{L^2} = \int_{\Omega_1} |f^*u|^2 dz \le C(K) \int_{\Omega_1} |f^*u|^2 |J_f(z)| dz = C(K)\|u\|^2_{L^2}. \tag{3.3}$$

Distributions are defined by duality, as the continuous linear functionals:-

$$u \in \mathcal{C}^{-\infty}(\Omega) \Longrightarrow u : \mathcal{C}^\infty_c(\Omega) \longrightarrow \mathbb{C}. \tag{3.4}$$

We always embed the smooth functions in the distributions using integration. This presents a small problem here, namely it is not consistent under pull-back. Indeed if $u \in \mathcal{C}^\infty(\Omega_2)$ and $\mu \in \mathcal{C}^\infty_c(\Omega_1)$ then

$$\int_{\Omega_1} f^*u(z)\mu(z)|J_f(z)|dz = \int_{\Omega_2} u(z')(f^{-1})^*\mu(z')dz' \text{ or} \tag{3.5}$$
$$\int_{\Omega_1} f^*u(z)\mu(z)dz = \int_{\Omega_2} u(z')(f^{-1})^*\mu(z')|J_{f^{-1}}(z')|dz',$$
$$\text{since } f^*J_{f^{-1}} = (J_f)^{-1}.$$

So, if we want distributions to be 'generalized functions', so that the identification of $u \in \mathcal{C}^\infty(\Omega_2)$ as an element of $\mathcal{C}^{-\infty}(\Omega_2)$ is consistent with the identification of $f^*u \in \mathcal{C}^\infty(\Omega_1)$ as an element of $\mathcal{C}^{-\infty}(\Omega_1)$ we need to use (3.5). Thus we *define*

$$f^* : \mathcal{C}^{-\infty}(\Omega_2) \longrightarrow \mathcal{C}^{-\infty}(\Omega_1) \text{ by } f^*u(\mu) = u((f^{-1})^*\mu|J_{f^{-1}}|). \tag{3.6}$$

There are better ways to think about this, namely in terms of densities, but let me not stop to do this at the moment. Of course one should check that f^* is a map as indicated and that it behaves correctly under composition, so $(f \circ g)^* = g^* \circ f^*$.

As already remarked, smooth functions pull back under a diffeomorphism (or any smooth map) to be smooth. Dually, vector fields push-forward. A vector field, in local coordinates, is just a first order differential operator without constant term

$$V = \sum_{j=1}^n v_j(z)D_{z_j},\ D_{z_j} = D_j = \frac{1}{i}\frac{\partial}{\partial z_j}. \tag{3.7}$$

For a diffeomorphism, the push-forward may be defined by

$$f^*(f_*(V)u) = Vf^*u \ \forall\ u \in \mathcal{C}^\infty(\Omega_2) \tag{3.8}$$

where we use the fact that f^* in (3.1) is an isomorphism of $\mathcal{C}^\infty(\Omega_2)$ onto $\mathcal{C}^\infty(\Omega_1)$. The chain rule is the computation of f_*V, namely

$$f_*V(f(z)) = \sum_{j,k=1}^n v_j(z)\frac{\partial f_j(z)}{\partial z_k}D_k. \tag{3.9}$$

As always this operation is natural under composition of diffeomorphism, and in particular $(f^{-1})_*(f_*)V = V$. Thus, under a diffeomorphism, vector fields push forward to vector fields and so, more generally, differential operators push-forward to differential operators.

Now, with these definitions we have

THEOREM 3. *For every $s \in \mathbb{R}$, any diffeomorphism $f : \Omega_1 \longrightarrow \Omega_2$ induces an isomorphism*

$$f^* : H^s_{\text{loc}}(\Omega_2) \longrightarrow H^s_{\text{loc}}(\Omega_1). \tag{3.10}$$

PROOF. We know this already for $s = 0$. To prove it for $0 < s < 1$ we use the norm on $H^s(\mathbb{R}^n)$ equivalent to the standard Fourier transform norm:-

$$\|u\|_s^2 = \|u\|_{L^2}^2 + \int_{\mathbb{R}^{2n}} \frac{|u(z) - u(\zeta)|^2}{|z-\zeta|^{2s+n}}dzd\zeta. \tag{3.11}$$

See Sect 7.9 of [**1**]. Then if $u \in H^s_{\text{c}}(\Omega_2)$ has support in $K \Subset \Omega_2$ with $0 < s < 1$, certainly $u \in L^2$ so $f^*u \in L^2$ and we can bound the second part of the norm in

(3.11) on f^*u :

$$\int_{\mathbb{R}^{2n}} \frac{|u(f(z)) - u(f(\zeta))|^2}{|z-\zeta|^{2s+n}} dz d\zeta \tag{3.12}$$
$$= \int_{\mathbb{R}^{2n}} \frac{|u(z') - u(\zeta')|^2}{|g(z') - g(\zeta')|^{2s+n}} |J_g(z')||J_g(\zeta')| dz' d\zeta'$$
$$\le C \int_{\mathbb{R}^{2n}} \frac{|u(z) - u(\zeta)|^2}{|z-\zeta|^{2s+n}} dz d\zeta$$

since $C|g(z') - g(\zeta')| \ge |z' - \zeta'|$ where $g = f^{-1}$.

For the spaces of order $m + s$, $0 \le s < 1$ and $m \in \mathbb{N}$ we know that

$$u \in H^{m+s}_{\text{loc}}(\Omega_2) \Longleftrightarrow Pu \in H^s_{\text{loc}}(\Omega_2) \ \forall\ P \in \text{Diff}^m(\Omega_2) \tag{3.13}$$

where $\text{Diff}^m(\Omega)$ is the space of differential operators of order at most m with smooth coefficients in Ω. As noted above, differential operators map to differential operators under a diffeomorphism, so from (3.13) it follows that $H^{m+s}_{\text{loc}}(\Omega_2)$ is mapped into $H^{m+s}_{\text{loc}}(\Omega_1)$ by f^*.

For negative orders we may proceed in the same way. That is if $m \in \mathbb{N}$ and $0 \le s < 1$ then

$$u \in H^{s-m}_{\text{loc}}(\Omega_2) \Longleftrightarrow u = \sum_J P_J u_J,\ P_J \in \text{Diff}^m(\Omega_2),\ u_J \in H^s(\Omega_2) \tag{3.14}$$

where the sum over J is finite. A similar argument then applies to prove (3.10) for all real orders. □

Consider the issue of differential operators more carefully. If $P : \mathcal{C}^\infty(\Omega_1) \longrightarrow \mathcal{C}^\infty(\Omega_1)$ is a differential operator of order m with smooth coefficients then, as already noted, so is

$$P_f : \mathcal{C}^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_2),\ P_f v = (f^{-1})^*(P f^* v). \tag{3.15}$$

However, the formula for the coefficients, i.e. the explicit formula for P_f, is rather complicated:-

$$P = \sum_{|\alpha| \le m} \Longrightarrow P_f = \sum_{|\alpha| \le m} p_\alpha(g(z'))(J_f(z') D_{z'})^\alpha \tag{3.16}$$

since we have to do some serious differentiation to move all the Jacobian terms to the left.

Even though the formula (3.16) is complicated, the leading part of it is rather simple. Observe that we can compute the leading part of a differential operator by 'oscillatory testing'. Thus, on an open set Ω consider

$$P(z, D)(e^{it\psi} u) = e^{it\psi} \sum_{k=0}^{m} t^k P_k(z, D) u,\ u \in \mathcal{C}^\infty(\Omega),\ \psi \in \mathcal{C}^\infty(\Omega),\ t \in \mathbb{R}. \tag{3.17}$$

Here the $P_k(z, D)$ are differential operators of order $m-k$ acting on u (they involve derivatives of ψ of course). Note that the only way a factor of t can occur is from a derivative acting on $e^{it\psi}$ through

$$D_{z_j} e^{it\psi} = e^{it\psi} t \frac{\partial \psi}{\partial z_j}. \tag{3.18}$$

Thus, the coefficient of t^m involves no differentiation of u at all and is therefore multiplication by a smooth function which takes the simple form

$$\sigma_m(P)(\psi, z) = \sum_{|\alpha|=m} p_\alpha(z)(D\psi)^\alpha \in \mathcal{C}^\infty(\Omega). \tag{3.19}$$

In particular, the value of this function at any point $z \in \Omega$ is determined once we know $d\psi$, the differential of ψ at that point. Using this observation, we can easily compute the leading part of P_f given that of P in (3.15). Namely if $\psi \in \mathcal{C}^\infty(\Omega_2)$ and $(P_f)(z')$ is the leading part of P_f for

$$\begin{aligned} \sigma_m(P_f)(\psi', z')v &= \lim_{t\to\infty} t^{-m} e^{-it\psi} P_f(z', D_{z'})(e^{it\psi'}v) \\ &= \lim_{t\to\infty} t^{-m} e^{-it\psi} g^*(P(z, D_z)(e^{itf^*\psi'} f^*v) \\ &= g^*(\lim_{t\to\infty} t^{-m} e^{-itf^*\psi'} g^*(P(z, D_z)(e^{itf^*\psi'} f^*v) = g^* P_m(f^*\psi, z) f^*v. \end{aligned} \tag{3.20}$$

Thus

$$\sigma_m(P_f)(\psi', \zeta')) = g^*\sigma_m(P)(f^*\psi', z). \tag{3.21}$$

This allows us to 'geometrize' the transformation law for the leading part (called the principal symbol) of the differential operator P. To do so we think of $T^*\Omega$, for Ω and open subset of $\mathbb{R}^n$, as the union of the $T^*Z\Omega$, $z \in \Omega$, where $T^*_z\Omega$ is the linear space

$$\begin{aligned} T^*_z\Omega = \mathcal{C}^\infty(\Omega)/\sim_z,\ \psi \sim_z \psi' \Longleftrightarrow \\ \psi(Z) - \psi'(Z) - \psi(z) + \psi'(z) \text{ vanishes to second order at } Z = z. \end{aligned} \tag{3.22}$$

Essentially by definition of the derivative, for any $\psi \in \mathcal{C}^\infty(\Omega)$,

$$\psi \sim_z \sum_{j=1}^n \frac{\partial \psi}{\partial z}(z)(Z_j - z_j). \tag{3.23}$$

This shows that there is an isomorphism, given by the use of coordinates

$$T^*\Omega \equiv \Omega \times \mathbb{R}^n,\ [z, \psi] \longmapsto (z, d\psi(z)). \tag{3.24}$$

The point of the complicated-looking definition (3.22) is that it shows easily (and I recommend you do it explicitly) that any smooth map $h : \Omega_1 \longrightarrow \Omega_2$ induces a smooth map

$$h^*T^*\Omega_2 \longrightarrow T^*\Omega_1,\ h([h(z), \psi]) = [z, h^*\psi] \tag{3.25}$$

which for a diffeomorphism is an isomorphism.

Lemma 12. *The transformation law* (3.21) *shows that for any element* $P \in \mathrm{Diff}^m(\Omega)$ *the principal symbol is well-defined as an element*

$$\sigma(P) \in \mathcal{C}^\infty(T^*\Omega) \tag{3.26}$$

which furthermore transforms as a function under the pull-back map (3.25) *induced by any diffeomorphism of open sets.*

Proof. The formula (3.19) is consistent with (3.23) and hence with (3.21) in showing that $\sigma_m(P)$ is a well-defined function on $T^*\Omega$. □

2. Manifolds

I will only give a rather cursory discussion of manifolds here. The main cases we are interested in are practical ones, the spheres $\mathbb{S}^n$ and the balls $\mathbb{B}^n$. Still, it is obviously worth thinking about the general case, since it is the standard setting for much of modern mathematics. There are in fact several different, but equivalent, definitions of a manifold.

2.1. Coordinate covers. Take a Hausdorff topological (in fact metrizable) space M. A *coordinate patch* on M is an open set and a homeomorphism

$$M \supset \Omega \xrightarrow{F} \Omega' \subset \mathbb{R}^n$$

onto an open subset of $\mathbb{R}^n$. An atlas on M is a covering by such coordinate patches (Ω_a, F_a),

$$M = \bigcup_{a \in A} \Omega_a.$$

Since each $F_{ab} : \Omega'_a \to \Omega'_a$ is, by assumption, a homeomorphism, the transition maps

$$\begin{aligned} F_{ab} &: \Omega'_{ab} \to \Omega'_{ba}\,, \\ \Omega'_{ab} &= F_b(\Omega_a \cap \Omega_b)\,, \\ (\Rightarrow \Omega'_{ba} &= F_a(\Omega_a \cap \Omega_b)) \\ F_{ab} &= F_a \circ F_b^{-1} \end{aligned}$$

are also homeomorphisms of open subsets of $\mathbb{R}^n$ (in particular n is constructed on components of M). The atlas is $\mathcal{C}^k$, $\mathcal{C}^\infty$, real analytic, etc.) if each F_{ab} is $\mathcal{C}^k$, $\mathcal{C}^\infty$ or real analytic. A $\mathcal{C}^\infty$ ($\mathcal{C}^k$ or whatever) structure on M is usually taken to be a *maximal* $\mathcal{C}^\infty$ atlas (meaning any coordinate patch compatible with all elements of the atlas is already in the atlas).

2.2. Smooth functions. A second possible definition is to take again a Hausdorff topological space and a subspace $\mathcal{F} \subset C(M)$ of the continuous real-valued function on M with the following two properties.

1) For each $p \in M \exists f_1, \dots, f_n \in \mathcal{F}$ and an open set $\Omega \ni p$ such that $F = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ is a homeomorphism onto an open set, $\Omega' \subset \mathbb{R}^n$ and $(F^{-1})^* g \in \mathcal{C}^\infty(\Omega')\, \forall g \in \mathcal{F}$.
2) $\mathcal{F}$ is maximal with this property.

2.3. Embedding. Alternatively one can simply say that a ($\mathcal{C}^\infty$) manifold is a subset $M \subset \mathbb{R}^N$ such that $\forall p \in M \exists$ an open set $U \ni p$, $U \subset \mathbb{R}^N$, and $h_1, \dots, h_{N-n} \in \mathcal{C}^\infty(U)$ s.t.

$$\begin{gathered} M \cap U = \{q \in U;\ h_i(q) = 0,\ i = 1, \dots, N-n\} \\ dh_i(p) \text{ are linearly independent.} \end{gathered}$$

I leave it to you to show that these definitions are equivalent in an *appropriate sense.* If we weaken the various notions of coordinates in each case, for instance in the first case, by requiring that $\Omega' \in \mathbb{R}^{n-k} \times [0, \infty)^k$ for some k, with a corresponding version of smoothness, we arrive at the notion of a manifold with cones.[1]

[1] I always demand in addition that the boundary faces of a manifold with cones be a *embedded* but others differ on this. I call the more general object a *tied manifold.*

So I will assume that you are reasonably familiar with the notion of a smooth ($\mathcal{C}^\infty$) manifold M, equipped with the space $\mathcal{C}^\infty(M)$ — this is just $\mathcal{F}$ in the second definition and in the first

$$\mathcal{C}^\infty(M) = \{u : M \to \mathbb{R};\ u \circ F^{-1} \in \mathcal{C}^\infty(\Omega') \,\forall \text{ coordinate patches}\}.$$

Typically I will not distinguish between complex and real-valued functions unless it seems necessary in this context.

Manifolds are always paracompact — so have countable covers by compact sets — and admit partitions of unity.

Proposition 7. *If $M = \bigcup_{a\in A} U_a$ is a cover of a manifold by open sets then there exist $\rho_a \in \mathcal{C}^\infty(M)$ s.t.* $\operatorname{supp}(\rho_a) \Subset U_a$ *(i.e., $\exists K_a \Subset U_a$ s.t. $\rho_a = 0$ on $M\backslash K_a$), these supports are locally finite, so if $K \Subset M$ then*

$$\{a \in A; \rho_a(m) \neq 0 \textit{ for some } m \in K\}$$

is finite, and finally

$$\sum_{a\in A} \rho_a(m) = 1,\ \forall\ m \in M.$$

It can also be arranged that

(1) $0 \le \rho_a(m) \le 1\ \forall\ a,\ \forall\ m \in M.$
(2) $\rho_a = \mu_a^2\,,\ \mu_a \in \mathcal{C}^\infty(M).$
(3) $\exists\ \varphi_a \in \mathcal{C}^\infty(M),\ 0 \le \varphi_a \le 1,\ \varphi = 1$ in a neighborhood of $\operatorname{supp}(\rho_a)$ and the sets $\operatorname{supp}(\varphi_a)$ are locally finite.

Proof. Up to you. □

Using a partition of unity subordinate to a covering by coordinate patches we may transfer definitions from $\mathbb{R}^n$ to M, provided they are coordinate-invariant in the first place and preserved by multiplication by smooth functions of compact support. For instance:

Definition 2. If $u : M \longrightarrow \mathbb{C}$ and $s \ge 0$ then $u \in H^s_{\text{loc}}(M)$ if for some partition of unity subordinate to a cover of M by coordinate patches

$$(F_a^{-1})^*(\rho_a u) \in H^s(\mathbb{R}^n) \quad \text{or } (F_a^{-1})^*(\rho_a u) \in H^s_{\text{loc}}(\Omega'_a). \tag{3.27}$$

Note that there are some abuses of notation here. In the first part of (3.27) we use the fact that $(F_a^{-1})^*(\rho_a u)$, defined really on Ω'_a (the image of the coordinate patch $F_a : \Omega_a \to \Omega'_a \in \mathbb{R}^n$), vanishes outside a compact subset and so can be unambiguously extended as zero outside Ω'_a to give a function on $\mathbb{R}^n$. The second form of (3.27) is better, but there is an equivalence relation, of equality off sets of measure zero, which is being ignored. The definition doesn't work well for $s < 0$ because u might then not be representable by a function so we don't know what u' is to start with.

The most sysetematic approach is to define distributions on M first, so we know what we are dealing with. However, there is a problem here too, because of the transformation law (3.5) that was forced on us by the local identification $\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^{-\infty}(\Omega)$. Namely, we really need *densities* on M before we can define distributions. I will discuss densities properly later; for the moment let me use a little ruse, sticking for simplicity to the compact case.

Definition 3. If M is a compact $\mathcal{C}^\infty$ manifold then $\mathcal{C}^0(M)$ is a Banach space with the supremum norm and a continuous linear functional

$$\mu : \mathcal{C}^0(M) \longrightarrow \mathbb{R} \tag{3.28}$$

is said to be a positive smooth measure if for every coordinate patch on M, $F : \Omega \longrightarrow \Omega'$ there exists $\mu_F \in \mathcal{C}^\infty(\Omega')$, $\mu_F > 0$, such that

$$\mu(f) = \int_{\Omega'} (F^{-1})^* f \mu_F dz \ \forall\ f \in \mathcal{C}^0(M) \text{ with } \operatorname{supp}(f) \subset \Omega. \tag{3.29}$$

Now if μ, $\mu' : \mathcal{C}^0(M) \longrightarrow \mathbb{R}$ is two such smooth measures then $\mu'_F = v_F \mu_F$ with $v_F \in \mathcal{C}^\infty(\Omega')$. In fact $\exists\ v \in \mathcal{C}^\infty(M)$, $v > 0$, such that $F^*_{v_F} = v$ on Ω. That is, the v's patch to a well-defined function globally on M. To see this, notice that every $g \in \mathcal{C}^0_{\mathrm{c}}(\Omega')$ is of the form $(F^{-1})^* g$ for some $g \in \mathcal{C}^0(M)$ (with support in Ω) so (3.29) certainly *determines* μ_F on Ω'. Thus, assuming we have two smooth measures, v_F is determined on Ω' for every coordinate patch. Choose a partition of unity ρ_a and *define*

$$v = \sum_a \rho_a F^*_a v_{F_a} \in \mathcal{C}^\infty(M).$$

Exercise. Show (using the transformation of integrals under diffeomorphisms) that

$$\mu'(f) = \mu(vf) \ \forall\ f \in \mathcal{C}^\infty(M). \tag{3.30}$$

Thus we have 'proved' half of

Proposition 8. *Any (compact) manifold admits a positive smooth density and any two positive smooth densities are related by* (3.30) *for some (uniquely determined)* $v \in \mathcal{C}^\infty(M)$, $v > 0$.

Proof. I have already unloaded the hard part on you. The extension is similar. Namely, chose a covering of M by coordinate patches and a corresponding partition of unity as above. Then simply *define*

$$\mu(f) = \sum_a \int_{\Omega'_a} (F_a^{-1})^* (\rho_a f) dz$$

using Lebesgue measure in each Ω'_a. The fact that this satisfies (3.29) is similar to the exercise above. □

Now, for a compact manifold, we can define a *smooth positive density* $\mu' \in \mathcal{C}^\infty(M; \Omega)$ as a continuous linear functional of the form

$$\mu' : \mathcal{C}^0(M) \longrightarrow \mathbb{C},\ \mu'(f) = \mu(\varphi f) \text{ for some } \varphi \in \mathcal{C}^\infty(M) \tag{3.31}$$

where φ is allowed to be complex-valued. For the moment the notation, $\mathcal{C}^\infty(M; \Omega)$, is not explained. However, the *choice* of a fixed positive $\mathcal{C}^\infty$ measure allows us to identify

$$\mathcal{C}^\infty(M; \Omega) \ni \mu' \longrightarrow \varphi \in \mathcal{C}^\infty(M),$$

meaning that this map is an isomorphism.

Lemma 13. *For a compact manifold, M, $\mathcal{C}^\infty(M;\Omega)$ is a complete metric space with the norms and distance function*

$$\|\mu'\|_{(k)} = \sup_{|\alpha|\le k} |V_1^{\alpha_1}\cdots V_p^{\alpha_1}\varphi|$$
$$d(\mu_1',\mu_2') = \sum_{k=0}^\infty 2^{-k}\frac{\|\mu'\|_{(k)}}{1+\|\mu'\|_{(k)}}$$

where $\{V_1,\dots,V_p\}$ is a collection of vector fields spanning the tangent space at each point of M.

This is really a result of about $\mathcal{C}^\infty(M)$ itself. I have put it this way because of the current relevance of $\mathcal{C}^\infty(M;\Omega)$.

Proof. First notice that there are indeed such vector fields on a compact manifold. Simply take a covering by coordinate patches and associated partitions of unity, φ_a, supported in the coordinate patch Ω_a. Then if $\Psi_a \in \mathcal{C}^\infty(M)$ has support in Ω_a and $\Psi_a \equiv 1$ in a neighborhood of $\operatorname{supp}(\varphi_a)$ consider

$$V_{a\ell} = \Psi_a(F_a^{-1})_*(\partial_{z_\ell}),\ \ell = 1,\dots,n,$$

just the coordinate vector fields cut off in Ω_a. Clearly, taken together, these span the tangent space at each point of M, i.e., the local coordinate vector fields are really linear combinations of the V_i given by renumbering the $V_{a\ell}$. It follows that

$$\|\mu'\|_{(k)} = \sup_{|\alpha\le k} |V_1^{\alpha_1}\cdots V_p^{\alpha_p}\varphi| \in M$$

is a norm on $\mathcal{C}^\infty(M;\Omega)$ locally equivalent to the $\mathcal{C}^k$ norm on φ_f on compact subsets of coordinate patches. It follows that (3.32) gives a distance function on $\mathcal{C}^\infty(M;\Omega)$ with respect to what is complete — just as for $\mathcal{S}(\mathbb{R}^n)$. □

Thus we can define the space of distributions on M as the space of continuous linear functionals $u \in \mathcal{C}^{-\infty}(M)$

$$u : \mathcal{C}^\infty(M;\Omega) \longrightarrow \mathbb{C},\ |u(\mu)| \le C_k\|\mu\|_{(k)}. \tag{3.32}$$

As in the Euclidean case smooth, and even locally integrable, functions embed in $\mathcal{C}^{-\infty}(M)$ by integration

$$L^1(M) \hookrightarrow \mathcal{C}^{-\infty}(M),\ f \mapsto f(\mu) = \int_M f\mu \tag{3.33}$$

where the integral is defined unambiguously using a partition of unity subordinate to a coordinate cover:

$$\int_M f\mu = \sum_a \int_{\Omega_a'} (F_a^{-1})^*(\varphi_a f\mu_a)dz$$

since $\mu = \mu_a dz$ in local coordinates.

Definition 4. The Sobolev spaces on a compact manifold are defined by reference to a coordinate case, namely if $u \in \mathcal{C}^{-\infty}(M)$ then

(3.34)
$$u \in H^s(M) \Leftrightarrow u(\psi\mu) = u_a((F_a^{-1})^*\psi\mu_a),\ \forall\ \psi \in \mathcal{C}_c^\infty(\Omega_a) \text{ with } u_a \in H^s_{\text{loc}}(\Omega_a').$$

Here the condition can be the requirement *for all* coordinate systems or for a covering by coordinate systems in view of the coordinate independence of the local Sobolev spaces on $\mathbb{R}^n$, that is the weaker condition implies the stronger.

Now we can transfer the properties of Sobolev for $\mathbb{R}^n$ to a compact manifold; in fact the compactness simplifies the properties

$$H^m(M) \subset H^{m'}(M),\ \forall\ m \geq m' \tag{3.35}$$

$$H^m(M) \hookrightarrow \mathcal{C}^k(M),\ \forall\ m > k + \frac{1}{2}\dim M \tag{3.36}$$

$$\bigcap_m H^m(M) = \mathcal{C}^\infty(M) \tag{3.37}$$

$$\bigcup_m H^m(M) = \mathcal{C}^{-\infty}(M). \tag{3.38}$$

These are indeed Hilbert(able) spaces — meaning they do not have a *natural* choice of Hilbert space structure, but they do have one. For instance

$$\langle u, v\rangle_s = \sum_a \langle (F_a^{-1})^* \varphi_a u, (F_a^{-1})^* \varphi_a v\rangle_{H^s(\mathbb{R}^n)}$$

where φ_a is a *square* partition of unity subordinate to coordinate covers.

3. Vector bundles

Although it is *not* really the subject of this course, it is important to get used to the coordinate-free language of vector bundles, etc. So I will insert here at least a minimum treatment of bundles, connections and differential operators on manifolds.

Addenda to Chapter 3