18.156 – Spring 2007 – Graduate Analysis
Elliptic regularity and Monopoles

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0.6P; Revised: 18-7-2007; Run: August 2, 2007

Contents

CHAPTER 1

Distributions

Summary of parts of 18.155 and a little beyond. With some corrections by Jacob Bernstein incorporated.

1. Fourier transform

The basic properties of the Fourier transform, tempered distributions and Sobolev spaces form the subject of the first half of this course. I will recall and slightly expand on such a standard treatment.

2. Schwartz space.

The space $\mathcal{S}(\mathbb{R}^n)$ of all complex-volumed functions with rapidly decreasing derivatives of all orders is a complete metric space with metric

$$\begin{aligned} d(u,v) &= \sum_{k=0}^{\infty} 2^{-k} \frac{\|u-v\|_{(k)}}{1+\|u-v\|_{(k)}} \text{ where} \\ \|u\|_{(k)} &= \sum_{|\alpha|+|\beta|\le k} \sup_{z\in\mathbb{R}^n} |z^\alpha D_z^\beta u(z)|. \end{aligned} \tag{1.1}$$

Here and below I will use the notation for derivatives

$$D_z^\alpha = D_{z_1}^{\alpha_1} \dots, D_{z_n}^{\alpha_n},\ D_{z_j} = \frac{1}{i} 1 \frac{\partial}{\partial z_j}.$$

These norms can be replaced by other equivalent ones, for instance by reordering the factors

$$\|u\|'_{(k)} = \sum_{|\alpha|+|\beta|\le k} \sup_{z\in\mathbb{R}^n} |D_z^\beta (z^\beta u)|.$$

In fact it is only the cumulative effect of the norms that matters, so one can use

$$\|u\|''_{(k)} = \sup_{z\in\mathbb{R}^n} |\langle z\rangle^{2k} (\Delta+1)^k u| \tag{1.2}$$

in (1.1) and the same topology results. Here

$$\langle z\rangle^2 = 1+|z|^2,\ \Delta = \sum_{j=1}^{n} D_j^2$$

(so the Laplacian is formally positive, the geometers' convention). It is not quite so trivial to see that inserting (1.2) in (1.1) gives an equivalent metric.

3. Tempered distributions.

The space of (metrically) continuous linear maps

$$f : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{1.3}$$

is the space of tempered distribution, denoted $\mathcal{S}'(\mathbb{R}^n)$ since it is the dual of $\mathcal{S}(\mathbb{R}^n)$. The continuity in (1.3) is equivalent to the estimates

$$\exists\ k,\ C_k > 0 \text{ s.t. } |f(\varphi)| \le C_k \|\varphi\|_{(k)}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{1.4}$$

There are several topologies which can be considered on $\mathcal{S}'(\mathbb{R}^n)$. Unless otherwise noted we consider the *uniform topology* on $\mathcal{S}'(\mathbb{R}^n)$; a subset $U \subset \mathcal{S}'(\mathbb{R}^n)$ is open in the uniform topology if for every $u \in U$ and every k sufficiently large there exists $\delta_k > 0$ (both k and δ_k depending on u) such that

$$v \in \mathcal{S}'(\mathbb{R}^n),\ |(u-u)(\varphi) \le \delta_k \|\varphi\|_{(k)} \Rightarrow v \in U.$$

For linear maps it is straightforward to work out continuity conditions. Namely

$$\begin{aligned} P &: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^m)\\ Q &: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m)\\ R &: \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^m)\\ S &: \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m) \end{aligned}$$

are, respectively, continuous for the metric and uniform topologies if

$$\begin{aligned} &\forall\ k\ \exists\ k',\ C \text{ s.t. } \|P\varphi\|_{(k)} \le C\|\varphi\|_{(k')}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\\ &\exists\ k,\ k',\ C \text{ s.t. } |Q\varphi(\psi)| \le C\|\varphi\|_{(k)}\|\psi\|_{(k')}\\ &\forall\ k,\ k'\ \exists\ C \text{ s.t. } |u(\varphi)| \le \|\varphi\|_{(k')}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \|Ru\|_{(k)} \le C\\ &\forall\ k'\ \exists\ k,\ C,\ C' \text{ s.t. } \|u(\varphi)\|_{(k)} \le \|\varphi\|_{(k)} \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow |Su(\psi)| \le C'\|\psi\|_{(k')}\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

The particular case of R, for $m = 0$, where at least formally $\mathcal{S}(\mathbb{R}^0) = \mathbb{C}$, corresponds to the reflexivity of $\mathcal{S}(\mathbb{R}^n)$, that

$$\begin{gathered} R : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathbb{C} \text{ is cts. iff } \exists\ \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ s.t.}\\ Ru = u(\varphi) \text{ i.e. } (\mathcal{S}'(\mathbb{R}^n))' = \mathcal{S}(\mathbb{R}^n). \end{gathered}$$

In fact another extension of the middle two of these results corresponds to the Schwartz kernel theorem:

$$\begin{aligned} Q :& \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m) \text{ is linear and continuous}\\ &\text{iff } \exists\ Q \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n) \text{ s.t. } (Q(\varphi))(\psi) = Q(\psi \boxtimes \varphi)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^m)\ \psi \in \mathcal{S}(\mathbb{R}^n).\\ R :& \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ is linear and continuous}\\ &\text{iff } \exists\ R \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n) \text{ s.t. } (Ru)(z) = u(R(z, \cdot)). \end{aligned}$$

Schwartz test functions are dense in tempered distributions

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

where the *standard inclusion* is via Lebesgue measure

$$\mathcal{S}(\mathbb{R}^n) \ni \varphi \mapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n),\ u_\varphi(\psi) = \int_{\mathbb{R}^n} \varphi(z)\psi(z)dz. \tag{1.5}$$

The basic operators of differentiation and multiplication are transferred to $\mathcal{S}'(\mathbb{R}^n)$ by duality so that they remain consistent with the (1.5):

$$D_z u(\varphi) = u(-D_z\varphi)$$
$$fu(\varphi) = u(f\varphi)\ \forall\ f \in \mathcal{S}(\mathbb{R}^n)).$$

In fact multiplication extends to the space of function of polynomial growth:

$$\forall\ \alpha \in \mathbb{N}_0^n\ \exists\ k \text{ s.t. } |D_z^\alpha f(z)| \le C\langle z\rangle^k.$$

Thus such a function is a multiplier on $\mathcal{S}(\mathbb{R}^n)$ and hence by duality on $\mathcal{S}'(\mathbb{R}^n)$ as well.

4. Fourier transform

Many of the results just listed are best proved using the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$
$$\mathcal{F}\varphi(\zeta) = \hat{\varphi}(\zeta) = \int e^{-iz\zeta}\varphi(z)dz.$$

This map is an isomorphism that extends to an isomorphism of $\mathcal{S}'(\mathbb{R}^n)$

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$
$$\mathcal{F}\varphi(D_{z_j}u) = \zeta_j\mathcal{F}u,\ \mathcal{F}(z_j u) = -D_{\zeta_j}\mathcal{F}u$$

and also extends to an isomorphism of $L^2(\mathbb{R}^n)$ from the dense subset

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^2)\mathrm{d}ense,\ \|\mathcal{F}\varphi\|_{L^2}^2 = (2\pi)^n\|\varphi\|_{L^2}^2. \tag{1.6}$$

5. Sobolev spaces

Plancherel's theorem, (**??**), is the basis of the definition of the (standard, later there will be others) Sobolev spaces.

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n);\ (1+|\zeta|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)\}$$
$$\|u\|_s^2 = \int_{\mathbb{R}^n}(1+|\zeta|^2)^s|\hat{u}(\zeta)|d\zeta,$$

where we use the fact that $L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is a well-defined injection (regarded as an inclusion) by continuous extension from (1.5). Now,

$$D^\alpha : H^s(\mathbb{R}^n) \longrightarrow H^{s-|\alpha|}(\mathbb{R}^n)\ \forall\ s,\ \alpha. \tag{1.7}$$

The Sobolev spaces are Hilbert spaces, so their duals are (conjugate) isomorphic to themselves. However, in view of our inclusion $L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, we habitually identify

$$(H^s(\mathbb{R}^n))' = H^{-s}(\mathbb{R}^n),$$

with the 'extension of the L^2 paring'

$$(u,v) = \text{“}\int u(z)v(z)dz'' = (2\pi)^{-n}\int_{\mathbb{R}^n}\langle\zeta\rangle^s\hat{u}\cdot\langle\zeta\rangle^{-s}\hat{u}d\zeta.$$

Note that then (5) is a linear, not a conjugate-linear, isomorphism since (5) is a real pairing.

The Sobolev spaces decrease with increasing s,

$$H^s(\mathbb{R}^n) \subset H^{s'}(\mathbb{R}^n)\ \forall\ s \ge s'.$$

One essential property is the relationship between the 'L^2 derivatives' involved in the definition of Sobolev spaces and standard derivatives. Namely, the Sobolev embedding theorem:

$$
\begin{aligned}
s > \frac{n}{2} &\Longrightarrow H^s(\mathbb{R}^n) \subset \mathcal{C}^0_\infty(\mathbb{R}^n) \\
&\qquad = \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ its continuous and bounded}\}. \\
s > \frac{n}{2} + k,\ k \in \mathbb{N} &\Longrightarrow H^s(\mathbb{R}^n) \subset \mathcal{C}^k_\infty(\mathbb{R}^n) \\
&\qquad \overset{\text{def}}{=} \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ s.t. } D^\alpha u \in \mathcal{C}^0_\infty(\mathbb{R}^n)\ \forall\ |\alpha| \le k\}.
\end{aligned}
$$

For positive integral s the Sobolev norms are easily written in terms of the functions, without Fourier transform:

$$
\begin{gathered}
u \in H^k(\mathbb{R}^n) \Leftrightarrow D^\alpha u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| \le k \\
\|u\|_k^2 = \sum_{|\alpha| \le k} \int_{\mathbb{R}^n} |D^\alpha u|^2 dz.
\end{gathered}
$$

For negative integral orders there is a similar characterization by duality, namely

$$
\begin{aligned}
H^{-k}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \text{ s.t. }, \exists\ u_\alpha \in L^2(\mathbb{R}^n),\ |\alpha| \ge k \\
u = \sum_{|\alpha| \le k} D^\alpha u_\alpha\}.
\end{aligned}
$$

In fact there are similar "Hölder" characterizations in general. For $0 < s < 1$, $u \in H^s(\mathbb{R}^n) \Longrightarrow u \in L^2(\mathbb{R}^n)$ and

$$
\int_{\mathbb{R}^{2n}} \frac{|u(z) - u(z')|^2}{|z - z'|^{n+2s}} dz dz' < \infty. \tag{1.8}
$$

Then for $k < s < k+1$, $k \in \mathbb{N}$ $u \in H^s(\mathbb{R}^2)$ is equivalent to $D^\alpha \in H^{s-k}(\mathbb{R}^n)$ for all $|\alpha| \in k$, with corresponding (Hilbert) norm. Similar realizations of the norms exist for $s < 0$.

One simple consequence of this is that

$$
\mathcal{C}^\infty_\infty(\mathbb{R}^n) = \bigcap_k \mathcal{C}^k_\infty(\mathbb{R}^n) = \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ s.t. } |D^\alpha u| \text{ is bounded } \forall\ \alpha\}
$$

is a multiplier on *all* Sobolev spaces

$$
\mathcal{C}^\infty_\infty(\mathbb{R}^n) \cdot H^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)\ \forall\ s \in \mathbb{R}.
$$

6. Weighted Sobolev spaces.

It follows from the Sobolev embedding theorem that

$$
\bigcap_s H^s(\mathbb{R}^n) \subset \mathcal{C}^\infty_\infty(\mathbb{R}^n); \tag{1.9}
$$

in fact the intersection here is quite a lot smaller, but nowhere near as small as $\mathcal{S}(\mathbb{R}^n)$. To discuss decay at infinity, as will definitely want to do, we may use weighted Sobolev spaces.

The ordinary Sobolev spaces do not effectively define decay (or growth) at infinity. We will therefore also set

$$H^{m,l}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n);\ \langle z\rangle^{\ell} u \in H^m(\mathbb{R}^n)\},\ m, \ell \in \mathbb{R},$$
$$= \langle z\rangle^{-\ell} H^m(\mathbb{R}^n)\,,$$

where the second notation is supported to indicate that $u \in H^{m,l}(\mathbb{R}^n)$ may be written as a product $\langle z\rangle^{-\ell} v$ with $v \in H^m(\mathbb{R}^n)$. Thus

$$H^{m,\ell}(\mathbb{R}^n) \subset H^{m',\ell'}(\mathbb{R}^n) \text{ if } m \geq m' \text{ and } \ell \geq \ell',$$

so the spaces are decreasing in each index. As consequences of the *Schwartz structure theorem*

$$\begin{aligned} \mathcal{S}'(\mathbb{R}^n) &= \bigcup_{m,\ell} H^{m,\ell}(\mathbb{R}^n) \\ \mathcal{S}(\mathbb{R}^n) &= \bigcap_{m,\ell} H^{m,\ell}(\mathbb{R}^n). \end{aligned} \tag{1.10}$$

This is also true 'topologically' meaning that the first is an 'inductive limit' and the second a 'projective limit'.

Similarly, using some commutation arguments

$$D_{z_j} : H^{m,\ell}(\mathbb{R}^n) \longrightarrow H^{m-1,\ell}(\mathbb{R}^n),\ \forall\ m,\ elll$$
$$\times z_j : H^{m,\ell}(\mathbb{R}^n) \longrightarrow H^{m,\ell-1}(\mathbb{R}^n).$$

Moreover there is symmetry under the Fourier transform

$$\mathcal{F} : H^{m,\ell}(\mathbb{R}^n) \longrightarrow H^{\ell,m}(\mathbb{R}^n) \text{ is an isomorphism } \forall\ m,\ \ell.$$

As with the usual Sobolev spaces, $\mathcal{S}(\mathbb{R}^n)$ is dense in all the $H^{m,\ell}(\mathbb{R}^n)$ spaces and the continuous extension of the L^2 paring gives an identification

$$H^{m,\ell}(\mathbb{R}^n) \cong (H^{-m,-\ell}(\mathbb{R}^n))' \text{ fron}$$
$$H^{m,\ell}(\mathbb{R}^n) \times H^{-m,-\ell}(\mathbb{R}^n) \ni u, v \mapsto$$
$$(u, v) = \text{``}\int u(z)v(z)dz''.$$

Let R_s be the operator defined by Fourier multiplication by $\langle \zeta\rangle^s$:

$$R_s : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\ \widehat{R_s f}(\zeta) = \langle\zeta\rangle^s \hat{f}(\zeta). \tag{1.11}$$

LEMMA 1. *If $\psi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$M_s = [\psi, R_s *] : H^t(\mathbb{R}^n) \longrightarrow H^{t-s+1}(\mathbb{R}^n) \tag{1.12}$$

is bounded for each t.

PROOF. Since the Sobolev spaces are defined in terms of the Fourier transform, first conjugate and observe that (1.12) is equivalent to the boundeness of the integral operator with kernel

$$K_{s,t}(\zeta, \zeta') = (1+|\zeta|^2)^{\frac{t-s+1}{2}} \hat{\psi}(\zeta-\zeta') \left((1 + |\zeta'|^2)^{\frac{s}{2}} - (1 + |\zeta|^2)^{\frac{s}{2}}\right) (1+|\zeta'|^2)^{-\frac{t}{2}} \tag{1.13}$$

on $L^2(\mathbb{R}^n)$. If we insert the characteristic function for the region near the diagonal

$$|\zeta - \zeta'| \leq \frac{1}{4}(|\zeta| + |\zeta'|) \Longrightarrow |\zeta| \leq 2|\zeta'|,\ |\zeta'| \leq 2|\zeta| \tag{1.14}$$

then $|\zeta|$ and $|\zeta'|$ are of comparable size. Using Taylor's formula

(1.15)
$$(1+|\zeta'|^2)^{\frac{s}{2}}-(1+|\zeta|^2)^{\frac{s}{2}} = s(\zeta-\zeta')\cdot\int_0^1 (t\zeta+(1-t\zeta')\left(1+|t\zeta+(1-t)\zeta'|^2\right)^{\frac{s}{2}-1}dt$$
$$\Longrightarrow \left|(1+|\zeta'|^2)^{\frac{s}{2}}-(1+|\zeta|^2)^{\frac{s}{2}}\right| \le C_s|\zeta-\zeta'|(1+|\zeta|)^{s-1}.$$

It follows that in the region (1.14) the kernel in (1.13) is bounded by

$$C|\zeta-\zeta'||\hat{\psi}(\zeta-\zeta')|. \tag{1.16}$$

In the complement to (1.14) the kernel is rapidly decreasing in ζ and ζ' in view of the rapid decrease of $\hat{\psi}$. Both terms give bounded operators on L^2, in the first case using the same estimates that show convolution by an element of $\mathcal{S}$ to be bounded. □

LEMMA 2. *If $u \in H^s(\mathbb{R}^n)$ and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then*

$$\|\psi u\|_s \le \|\psi\|_{L^\infty}\|u\|_s + C\|u\|_{s-1} \tag{1.17}$$

where the constant depends on s and ψ but not u.

PROOF. This is really a standard estimate for Sobolev spaces. Recall that the Sobolev norm is related to the L^2 norm by

$$\|u\|_s = \|\langle D\rangle^s u\|_{L^2}. \tag{1.18}$$

Here $\langle D\rangle^s$ is the convolution operator with kernel defined by its Fourier transform

$$\langle D\rangle^s u = R_s * u,\ \widehat{R_s}(\zeta) = (1+|\zeta|^2)^{\frac{s}{2}}. \tag{1.19}$$

To get (1.17) use Lemma 1.

From (1.12), (writing 0 for the L^2 norm)

(1.20)
$$\|\psi u\|_s = \|R_s*(\psi u)\|_0 \le \|\psi(R_s*u)\|_0 + \|M_s u\|_0$$
$$\le \|\psi\|_{L^\infty}\|R_s u\|_0 + C\|u\|_{s-1} \le \|\psi\|_{L^\infty}\|u\|_s + C\|u\|_{s-1}.$$

This completes the proof of (1.17) and so of Lemma 2. □

7. Multiplicativity

Of primary importance later in our treatment of non-linear problems is some version of the multliplicative property

$$A^s(\mathbb{R}^n) = \begin{cases} H^s(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n) & s \le \frac{n}{2} \\ H^s(\mathbb{R}^n) & s > \frac{n}{2}\end{cases} \text{ is a } \mathcal{C}^\infty \text{ algebra.} \tag{1.21}$$

Here, a $\mathcal{C}^\infty$ algebra is an algebra with an additional closure property. Namely if $F : \mathbb{R}^N \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ function vanishing at the origin and $u_1, \dots, u_N \in A^s$ are *real-valued* then

$$F(u_1,\dots,u_n) \in A^s.$$

I will only consider the case of real interest here, where s is an integer and $s > \frac{n}{2}$. The obvious place to start is

LEMMA 3. *If $s > \frac{n}{2}$ then*

$$u,\ v \in H^s(\mathbb{R}^n) \Longrightarrow uv \in H^s(\mathbb{R}^n). \tag{1.22}$$

Proof. We will prove this directly in terms of convolution. Thus, in terms of weighted Sobolev spaces $u \in H^s(\mathbb{R}^n) = H^{s,0}(\mathbb{R}^n)$ is equivalent to $\hat{u} \in H^{0,s}(\mathbb{R}^n)$. So (1.22) is equivalent to

$$u,\ v \in H^{0,s}(\mathbb{R}^n) \Longrightarrow u * v \in H^{0,s}(\mathbb{R}^n). \tag{1.23}$$

Using the density of $\mathcal{S}(\mathbb{R}^n)$ it suffices to prove the estimate

$$\|u * v\|_{H^{0,s}} \le C_s \|u\|_{H^{0,s}} \|v\|_{H^{0,s}} \text{ for } s > \frac{n}{2}. \tag{1.24}$$

Now, we can write $u(\zeta) = \langle\zeta\rangle^{-s} u'$ etc and convert (1.24) to an estimate on the L^2 norm of

$$\langle\zeta\rangle^{-s} \int \langle\xi\rangle^{-s} u'(\xi) \langle\zeta - \xi\rangle^{-s} v'(\zeta - \xi) d\xi \tag{1.25}$$

in terms of the L^2 norms of u' and $v' \in \mathcal{S}(\mathbb{R}^n)$.

Writing out the L^2 norm as in the proof of Lemma 1 above, we need to estimate the absolute value of

$$\int\int\int d\zeta d\xi d\eta \langle\zeta\rangle^{2s} \langle\xi\rangle^{-s} u_1(\xi) \langle\zeta-\xi\rangle^{-s} v_1(\zeta-\xi) \langle\eta\rangle^{-s} u_2(\eta) \langle\zeta-\eta\rangle^{-s} v_2(\zeta-\eta) \tag{1.26}$$

in terms of the L^2 norms of the u_i and v_i. To do so divide the integral into the four regions,

$$\begin{gathered}
|\zeta - \xi| \le \frac{1}{4}(|\zeta| + |\xi|),\ |\zeta - \eta| \le \frac{1}{4}(|\zeta| + |\eta|) \\
|\zeta - \xi| \le \frac{1}{4}(|\zeta| + |\xi|),\ |\zeta - \eta| \ge \frac{1}{4}(|\zeta| + |\eta|) \\
|\zeta - \xi| \ge \frac{1}{4}(|\zeta| + |\xi|),\ |\zeta - \eta| \le \frac{1}{4}(|\zeta| + |\eta|) \\
|\zeta - \xi| \ge \frac{1}{4}(|\zeta| + |\xi|),\ |\zeta - \eta| \ge \frac{1}{4}(|\zeta| + |\eta|).
\end{gathered} \tag{1.27}$$

Using (1.14) the integrand in (1.26) may be correspondingly bounded by

$$\begin{gathered}
C\langle\zeta - \eta\rangle^{-s} |u_1(\xi)||v_1(\zeta - \xi)| \cdot \langle\zeta - \xi\rangle^{-s} |u_2(\eta)||v_2(\zeta - \eta)| \\
C\langle\eta\rangle^{-s} |u_1(\xi)||v_1(\zeta - \xi)| \cdot \langle\zeta - \xi\rangle^{-s} |u_2(\eta)||v_2(\zeta - \eta)| \\
C\langle\zeta - \eta\rangle^{-s} |u_1(\xi)||v_1(\zeta - \xi)| \cdot \langle\xi\rangle^{-s} |u_2(\eta)||v_2(\zeta - \eta)| \\
C\langle\eta\rangle^{-s} |u_1(\xi)|v_1(\zeta - \xi)| \cdot \langle\xi\rangle^{-s} |u_2(\eta)||v_2(\zeta - \eta)|.
\end{gathered} \tag{1.28}$$

Now applying Cauchy-Schwarz inequality, with the factors as indicated, and changing variables appropriately gives the desired estimate. □

Next, we extend this argument to (many) more factors to get the following result which is close to the Gagliardo-Nirenberg estimates (since I am concentrating here on L^2 methods I will not actually discuss the latter).

Lemma 4. *If $s > \frac{n}{2}$, $N \ge 1$ and $\alpha_i \in \mathbb{N}_0^k$ for $i = 1, \dots, N$ are such that*

$$\sum_{i=1}^{N} |\alpha_i| = T \le s$$

then

$$u_i \in H^s(\mathbb{R}^n) \Longrightarrow U = \prod_{i=1}^{N} D^{\alpha_i} u_i \in H^{s-T}(\mathbb{R}^n),\ \|U\|_{H^{s-T}} \le C_N \prod_{i=1}^{N} \|u_i\|_{H^s}. \tag{1.29}$$

PROOF. We proceed as in the proof of Lemma 3 using the Fourier transform to replace the product by the convolution. Thus it suffices to show that

$$u_1 * u_2 * u_3 * \cdots * u_N \in H^{0,s-T} \text{ if } u_i \in H^{0,s-\alpha_i}. \tag{1.30}$$

Writing out the convolution symmetrically in all variables,

$$u_1 * u_2 * u_3 * \cdots * u_N(\zeta) = \int_{\zeta=\sum\limits_i \xi_i} u_1(\xi_1)\cdots u_N(\xi_N) \tag{1.31}$$

it follows that we need to estimate the L^2 norm in ζ of

$$\langle\zeta\rangle^{s-T}\int_{\zeta=\sum\limits_i \xi_i} \langle\xi_1\rangle^{-s+a_1} v_1(\xi_1)\cdots\langle\xi_N\rangle^{-s+a_N} v_N(\xi_N) \tag{1.32}$$

for N factors v_i which are in L^2 with the $a_i = |\alpha|_i$ non-negative integers summing to $T \le s$. Again writing the square as the product with the complex conjuage it is enough to estimate integrals of the type

$$\begin{aligned} &\int_{\{(\xi,\eta)\in\mathbb{R}^{2N};\sum\limits_i \xi_i=\sum\limits_i \eta_i\}} \langle\sum_i \xi\rangle^{2s-2T}\langle\xi_1\rangle^{-s+a_1} \\ &\qquad v_1(\xi_1)\cdots\langle\xi_N\rangle^{-s+a_N} v_N(\xi_N)\langle\eta_1\rangle^{-s+a_1}\bar{v}_1(\eta_1)\cdots\langle\eta_N\rangle^{-s+a_N}\bar{v}_N(\eta_N). \end{aligned} \tag{1.33}$$

This is really an integral over $\mathbb{R}^{2N-1}$ with respect to Lebesgue measure. Applying Cauchy-Schwarz inequality the absolute value is estimated by

$$\int_{\{(\xi,\eta)\in\mathbb{R}^{2N};\sum\limits_i \xi_i=\sum\limits_i \eta_i\}} \prod_{i=1}^N |v_i(\xi_i)|^2 \langle\sum_l \eta_l\rangle^{2s-2T}\prod_{i=1}^N\langle\eta_i\rangle^{-2s+2a_i} \tag{1.34}$$

The domain of integration, given by $\sum\limits_i \eta_i = \sum\limits_i \xi_i$, is covered by the finite number of subsets Γ_j on which in addition $|\eta_j| \ge |\eta_i|$, for all i. On this set we may take the variables of integration to be η_i for $i \ne j$ and the ξ_l. Then $|\eta_i| \ge |\sum\limits_l \eta_l|/N$ so the second part of the integrand in (1.34) is estimated by

$$\langle\eta_j\rangle^{-2s+2a_j}\langle\sum_l \eta_l\rangle^{2s-2T}\prod_{i\ne j}\langle\eta_i\rangle^{-2s+2a_i} \le C_N\langle\eta_j\rangle^{-2T+2a_j}\prod_{i\ne j}\langle\eta_i\rangle^{-2s+2a_i} \le C_N'\prod_{i\ne j}\langle\eta_i\rangle^{-2s} \tag{1.35}$$

Thus the integral in (1.34) is finite and the desired estimate follows. □

PROPOSITION 1. *If $F \in \mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R})$ and $u \in H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}$ an integer then*

$$F(z,u(z)) \in H^s_{\text{loc}}(\mathbb{R}^n). \tag{1.36}$$

PROOF. Since the result is local on $\mathbb{R}^n$ we may multiply by a compactly supported function of z. In fact since $u \in H^s(\mathbb{R}^n)$ is bounded we also multiply by a compactly supported function in $\mathbb{R}$ without changing the result. Thus it suffices to show that

$$F \in \mathcal{C}^\infty_c(\mathbb{R}^n\times\mathbb{R}) \Longrightarrow F(z,u(z)) \in H^s(\mathbb{R}^n). \tag{1.37}$$

Now, Lemma 4 can be applied to show that $F(z, u(z)) \in H^s(\mathbb{R}^n)$. Certainly $F(z, u(z)) \in L^2(\mathbb{R}^n)$ since it is continuous and has compact support. Moreover, differentiating s times and applying the chain rule gives

$$D^\alpha F(z, u(z)) = \sum F_{\alpha_1,\dots,\alpha_N}(z, u(z)) D^{\alpha_1} u \cdots D^{\alpha_N} u \tag{1.38}$$

where the sum is over all (finitely many) decomposition with $\sum_{i=1}^{N} \alpha_i \le \alpha$ and the $F_{\cdot}(z, u)$ are smooth with compact support, being various derivitives of $F(z, u)$. Thus it follows from Lemma 4 that all terms on the right are in $L^2(\mathbb{R}^n)$ for $|\alpha| \le s$. $\square$

Note that slightly more sophisticated versions of these arguments give the full result (1.21) but Proposition 1 suffices for our purposes below.

8. Some bounded operators

LEMMA 5. *If $J \in \mathcal{C}^k(\Omega^2)$ is properly supported then the operator with kernel J (also denoted J) is a map*

$$J : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^k_{\mathrm{loc}}(\Omega) \ \forall\ s \ge -k. \tag{1.39}$$