18.156 – GRADUATE ANALYSIS
LECTURE NOTES FOR SPRING 2007

$\boxed{\text{0.4A; Revised: 26-1-2003; Run: February 20, 2007}}$

1. Lecture 1

[Last revised: 18 February, 2007. Includes corrections noted by Tim Nguyen]
What do I want to do:-

(1) Variable coefficient, elliptic PDE.
(2) Monopoles
(3) Something else if time permits

I have pretty much decided not to use one book but to take bits and pieces from several. For one thing there does not seem to be an approptriate treatment of monopoles. The elliptic part of the course is intended to serve as an introduction to many parts of geometic pde and differential geometry. The part on monopoles is basically an extended application of the first part.

First let me recall some things from 18.155 – or at least try to stir some memory of what it is about. Namely, using the Fourier transform we can say a lot about *constant coefficient linear* differential operators. In this course we want to develop methods to handle variable coefficient, non-linear and (because of gauge-invariance) non-elliptic operators. However in all three directions of generalization we will still ultimaltely use constant coefficient elliptic methods as a starting point.

A linear, constant coefficient differential operator can be thought of as a map

$$P : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ of the form } Pu(z) = \sum_{|\alpha|\le m} c_\alpha(z) D^\alpha u(z),$$
$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n},\ D_j = \frac{1}{i}\frac{\partial}{\partial z_j}, \tag{1.1}$$

but it also acts on various other spaces. So, really it is just a polynomial $P(\zeta)$ in n variables. This 'characteristic polynomial' has the property that

$$\mathcal{F}(P(D)u)(\zeta) = P(\zeta)\mathcal{F}u(\zeta) \tag{1.2}$$

and this is why the Fourier tranform is especially useful. Still it does not solve the important questions directly.

Question 1. $P(D)$ is always injective as a map (1.1) but usually not surjective. When is it surjective? If $\Omega \subset \mathbb{R}^n$ is a non-empty open set then

$$P(D) : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega) \tag{1.3}$$

is never injective, for which polynomials is it surjective?

The fist three points are relatively easy. As a map (1.1) $P(D)$ is injective since if $P(D)u = 0$ then by (1.2), $P(\zeta)\mathcal{F}u(\zeta) = 0$ on $\mathbb{R}^n$. However, a non-trivial polynomial cannot vanish on an open set, i.e. the set where it is non-zero is dense, so $\mathcal{F}u(\zeta) = 0$ on $\mathbb{R}^n$ (by continuity) and hence $u = 0$ by the invertibility of the Fourier transform. So (1.1) is injective (of course excepting the case that P is the zero polynomial). When is it surjective? That is, when can every $f \in \mathcal{S}(\mathbb{R}^n)$ be written as $P(D)u$ with $u \in \mathcal{S}(\mathbb{R}^n)$? Taking the Fourier transform again, this is the same as asking when every $g \in \mathcal{S}(\mathbb{R}^n)$ can be written in the form $P(\zeta)v(\zeta)$ with $v \in \mathcal{S}(\mathbb{R}^n)$. If $P(\zeta)$ has a zero in $\mathbb{R}^n$ then this is not possible, since $P(\zeta)v(\zeta)$ always vanishes at such a point. It is a little trickier to see the converse, that $P(\zeta) \neq 0$ on $\mathbb{R}^n$ implies that $P(D)$ in (1.1) is surjective. Why is this not obvious? Because we need to show that $v(\zeta) = g(\zeta)/P(\zeta) \in \mathcal{S}(\mathbb{R}^n)$ whenever $g \in \mathcal{S}(\mathbb{R}^n)$. Certainly, $v \in \mathcal{C}^\infty(\mathbb{R}^n)$ but we need to show that the derivatives decay rapidly at infinity. To do this we really need to show that

$$P(\zeta) \neq 0 \text{ on } \mathbb{R}^n \Longrightarrow C|P(\zeta)| \geq (1+|\zeta|)^a \tag{1.4}$$

for some C and $a \in \mathbb{R}$ (possibly negative) This is true! Thirdly the non-injectivity in (1.3) is obvious for the opposite reason. Namely for any polynomial there exists $\zeta \in \mathbb{C}^n$ such that $P(\zeta) = 0$. Since

$$P(D)e^{i\zeta\cdot z} = P(\zeta)e^{i\zeta\cdot z} \tag{1.5}$$

such a zero gives rise to a non-trivial element of the null space of (1.3). You can find an extensive discussion of the density of these sort of 'exponential' solutions (with polynomial factors) in all solutions in Hörmander's book [1].

What about the surjectivity of (1.3)? It is not always surjective unless Ω is *convex* but there are decent answers, to find them you should look under P-convexity in [1]. If $P(\zeta)$ is elliptic then (1.3) is surjective for every open Ω.

To discuss elliptic regularity, let me recall that Sobolev spaces were extensively discussed in 18.155. The basic global Sobolev spaces are defined in terms of the Fourier transform

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); (1+|\zeta|^2)^{-\frac{s}{2}}\mathcal{F}u(\zeta) \in L^2(\mathbb{R}^n)\}. \tag{1.6}$$

It turns out that these spaces are not really very useful as global spaces on $\mathbb{R}^,$ so we will later look at modified versions of them. Even this definition depends on knowing a few things. First that $\mathcal{F}$, the Fourier transform, acts as an isomorphism on $\mathcal{S}'(\mathbb{R}^n)$, the dual space of $\mathcal{S}(\mathbb{R}^n)$. Secondly that multiplication by $(1+|\zeta|^2)^{\frac{t}{2}}$ on $\mathcal{S}'(\mathbb{R}^n)$ is defined for any $t \in \mathbb{R}$ and thirdly that $L^2(\mathbb{R}^n)$ is embedded as a linear subspace of $\mathcal{S}'(\mathbb{R}^n)$. I invite you to remember these things – if they need clarification now would be a good time to ask. With these caveats, (1.6) is a good defintion of the Sobolev spaces and any constant coefficient differential operator of order m defines a continuous linear map

$$P(D) : H^{s+m}(\mathbb{R}^n) \longmapsto H^s(\mathbb{R}^n). \tag{1.7}$$

This map is *always* injective (of course I assume that P is not the zero polynomial). Why? It is seldom surjective. Recall that P is said to be elliptic (either as a polynomial or as a differential operator) if it is of order m and there is a constant $c > 0$ such that

$$|P(\zeta)| \geq c(1+|\zeta|)^m \text{ in } \{\zeta \in \mathbb{R}^n; |\zeta| > 1/c\}. \tag{1.8}$$

Exercise 1. Show that $P(D)$ is surjective as a map (1.7) if and only if P is elliptic and $P(\zeta) \neq 0$ on $\mathbb{R}^n$.

So, this is not frequently the case. In fact one of the questions I want to get to early on in this course – even though we are interested in variable coefficient operators – is improving (1.7) to get an isomorphism at least for homogeneous elliptic operators. One reason for this is that we need it for monopoles.

Let me recall *elliptic regularity* for constant coefficient operators. Since this is a local issue, I first want to define local versions of the Sobolev spaces. Maybe you already saw all this.

Definition 1. If $\Omega \subset \mathbb{R}^n$ is an open set then

$$H^s_{\text{loc}}(\Omega) = \left\{u \in \mathcal{C}^{-\infty}(\Omega); \phi u \in H^s(\mathbb{R}^n) \ \forall \ \phi \in \mathcal{C}^\infty_c(\Omega)\right\}. \tag{1.9}$$

Again you need to know what $\mathcal{C}^{-\infty}(\Omega)$ is (it is the dual of $\mathcal{C}^\infty_c(\Omega)$) and that multiplication by $\phi \in \mathcal{C}^\infty_c(\Omega)$ defines a linear continuous map from $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ to $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and gives a bounded operator on $H^m(\mathbb{R}^n)$ for all m.

Proposition 1. *If $P(D)$ is elliptic, $u \in \mathcal{C}^{-\infty}(\Omega)$ is a distribution on an open set and $P(D)u \in H^s_{\mathrm{loc}}(\Omega)$ then $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$.*

Let me discuss this in two slightly different ways. The first, older, approach is direct regularity estimates. The second is via the use of a parametrix.

First the regularity estimates. An easy case of Proposition 1 is if $u \in \mathcal{C}^{-\infty}_c(\Omega)$ has compact support to start with. Then $P(D)u$ also has compact support so in this case

$$u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n) \text{ and } P(D)u \in H^s(\mathbb{R}^n). \tag{1.10}$$

Then of course the Fourier transform works like a charm. Namely

$$(1+|\zeta|^2)^{\frac{s}{2}} P(\zeta)\hat{u}(\zeta) \in L^2 \Longrightarrow (1+|\zeta|^2)^{\frac{s+m}{2}} F(\zeta)\hat{u}(\zeta),\ F(\zeta) = (1+|\zeta|^2)^{-\frac{m}{2}} P(\zeta). \tag{1.11}$$

Ellipticity implies that $F(\zeta)$ is bounded above and below on $|\zeta| > 1/c$ and hence can be inverted there by a bounded function. Since $\hat{u}$ is smooth everywhere it follows from (1.11) that $(1+|\zeta|^2)^{\frac{s+m}{2}}\hat{u} \in L^2(\mathbb{R}^n)$ and hence that $u \in H^{s+m}(\mathbb{R}^n)$ as expected.

To do the general case of an open set we need to use cutoffs. We want to show that $\psi u \in H^{s+m}(\mathbb{R}^n)$ where $\psi \in \mathcal{C}^\infty_c(\Omega)$ is some fixed but arbitrary element. We can always choose some function $\phi \in \mathcal{C}^\infty_c(\Omega)$ which is equal to one in a neighbourhood of the support of ψ. Then $\phi u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ so $\phi u \in H^t(\mathbb{R}^n)$ for some (unknown) $t \in \mathbb{R}$. We will show that $\psi u \in H^T(\mathbb{R}^n)$ where T is the smaller of $s+m$ and $t+1$. To see this, compute

$$P(D)(\psi u) = \psi P(D)u + \sum_{|\beta|\le m-1, |\gamma|\ge 1} c_{\beta,\gamma} D^\gamma \psi D^\beta \phi u. \tag{1.12}$$

With the final ϕu replaced by u this is just the effect of expanding out the derivatives on the product. Namely, the $\psi P(D)u$ term is when no derivative hits ψ and the other terms come from at least one derivative hitting ψ. Since $\phi = 1$ on the support of ψ we may then insert ϕ without changing the result. This the first term on the right in (1.12) is in $H^s(\mathbb{R}^n)$ and all terms in the sum are in $H^{t-m+1}(\mathbb{R}^n)$. Applying the simple case discussed above it follows that $\psi u \in H^r(\mathbb{R}^n)$ with r the minimum of $s+m$ and $t+1$. This result holds for any ψ with support in the interior of the set where $\phi = 1$. We can insert a chain of functions, of any finite length k, each contained in the interior of the set where the next is equal to 1, with ϕ having the largest support and ψ the smallest. The argument above then shows that r is the minimum of $t+k$ and $s+m$ for any k, proving the desired regularity.

The second method is actually quite similar, but we avoid the iteration technique, by doing it all at once. Namely, going back to the easy case of a function on $\mathbb{R}^n$ let's give the map a name:-

$$Q(D) : f \in \mathcal{S}'(\mathbb{R}^n) \longmapsto \mathcal{F}^{-1}\left(\frac{1-\chi(\zeta)}{P(\zeta)}\hat{f}(\zeta)\right) \in \mathcal{S}'(\mathbb{R}^n). \tag{1.13}$$

Here $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is chosen to be equal to one on the set $|\zeta| \le \frac{1}{c} + 1$ corresponding to the ellipticity estimate (1.8). Thus $\frac{1-\chi(\zeta)}{P(\zeta)} \in \mathcal{C}^\infty(\mathbb{R}^n)$ is bounded and in fact

$$|D^\alpha_\zeta \frac{1-\chi(\zeta)}{P(\zeta)}| \le C_\alpha(1+|\zeta|)^{-m-|\alpha|}\ \forall\ \alpha. \tag{1.14}$$

This has a straightforward proof by induction. Namely, these estimates are trivial on any compact set, where the function is smooth, so we need only consider the region where $\chi(\zeta) = 1$. The inductive statement is that for polynomials H_α,

$$D^\alpha_\zeta \frac{1}{P(\zeta)} = \frac{H_\alpha(\zeta)}{(P(\zeta))^{|\alpha|+1}},\ \deg(H_\alpha) \le (m-1)|\alpha|. \tag{1.15}$$

From this (1.14) follows.

So

$$Q(D) : H^s(\mathbb{R}^n) \longrightarrow H^{s+m}(\mathbb{R}^n) \tag{1.16}$$

is continuous for each s and it is also an essential inverse of $P(D)$ in the sense that as operators on $\mathcal{S}'(\mathbb{R}^n)$

$$Q(D)P(D) = P(D)Q(D) = \mathrm{Id} - E,\ E : H^s(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n)\ \forall\ s \in \mathbb{R}. \tag{1.17}$$

So, in the global case of $\mathbb{R}^n$, we get elliptic regularity by applying $Q(D)$ to both sides of the eqution $P(D)u = f$ to find

$$f \in H^s(\mathbb{R}^n) \Longrightarrow u = Eu + Qf \in H^{s+m}(\mathbb{R}^n). \tag{1.18}$$

The idea then, is to do the same thing for $P(D)$ acting on functions on the open set Ω and then for a variable coefficient operator. The problem of course is that $Q(D)$ does not act on functions (or distributions) defined just on Ω, then need to be defined on the whole of $\mathbb{R}^n$ and to be tempered. Now, $Q(D)$ is a convolution operator. Namely

$$Qf = q * f,\ q \in \mathcal{S}'(\mathbb{R}^n),\ \hat{q} = \frac{1-\chi(\zeta)}{P(\zeta)}. \tag{1.19}$$

This in fact is exactly what (1.13) means, since

$$\mathcal{F}(q * f) = \hat{q}\hat{f}. \tag{1.20}$$

We can write out convolution by a smooth function (which q is not, but let's not worry about that yet) as an integral

$$q * f(\zeta) = \int_{\mathbb{R}^n} q(z-z')f(z')dz'. \tag{1.21}$$

Restating the problem, (1.21) is an integral (really a distributional pairing) over the whole of $\mathbb{R}^n$ not the subset Ω. In essence the cutoff argument above inserts a cutoff ϕ in front of f (really of course in front of u but not to worry). So, let's think about inserting a cutoff into (1.21), replacing it by

$$Qf(\zeta) = \int_{\mathbb{R}^n} q(z-z')\psi(z,z')f(z')dz'. \tag{1.22}$$

Here we will take $\psi \in \mathcal{C}^\infty(\Omega^2)$. To get the integrand to have compact support in Ω for each $z \in \Omega$ we want to arrange that the projection onto the second variable, z'

$$\Omega \times \Omega \supset \mathrm{supp}(\psi) \longrightarrow \Omega \tag{1.23}$$

should be proper – the inverse image of a compact subset $K \subset \Omega$, namely $(\Omega \times K)\,\mathrm{supp}(\psi)$ should be compact in Ω^2.

2. Lecture 2

8 February, 2007: Elliptic regularity, continued.

Photographs by Chris Kottke. Photo1, Photo2, Photo3, Photo4, Photo5, Photo6, Photo7, Photo8, Photo9, Photo10, Photo11, Photo12, Photo13, Photo14, Photo15.

Last time we showed that if $P(D)$ is elliptic of order m and $u \in \mathcal{C}^{-\infty}(\Omega)$, for $\Omega \subset \mathbb{R}^n$ open, satisfies $P(D)u \in H^s_{\mathrm{loc}}(\Omega)$ then $u \in H^{m+s}_{\mathrm{loc}}(\mathbb{R}^n)$ and for any ϕ, $\psi \in \mathcal{C}^\infty_c(\Omega)$ with $\phi = 1$ in a neighbourhood of $\mathrm{supp}(\phi)$,

$$\|\psi u\|_{s+m} \le C\|\psi P(D)u\|_s + C'\|\phi u\|_{s+m-1} \tag{2.1}$$

for any $M \in \mathbb{R}$, with C' depending only on ψ, ϕ, M and $P(D)$ and C only on $P(D)$ (so neither depends on u). We proceed to try to do the same thing in the variable coefficient case, so for

$$P(z, D) = \sum_{|\alpha|\le m} p_\alpha(z)D^\alpha,\ p_\alpha \in \mathcal{C}^\infty(\Omega). \tag{2.2}$$

We now assume ellipticity for the polynomial $P(z, \zeta)$ for each $z \in \Omega$. This is the same thing as ellipticity for the principal part, i.e. the condition for each compact subset of Ω

$$|\sum_{|\alpha|=m} p_\alpha(z)\zeta^\alpha| \ge C(K)|\zeta|^m,\ z \in K \Subset \Omega, C(K) > 0. \tag{2.3}$$

Now, we got the estimate (2.1) by iteration from the case $M = s + m - 1$ (by nesting cutoff functions). Pick a point $\bar z \in \Omega$. In a small ball around $\bar z$ the coefficients are almost constant. In fact by Taylor's theorem

$$P(z, \zeta) = P(\bar z, \zeta) + Q(z, \zeta),\ Q(z, \zeta) = \sum_j (z - \bar z)_j P_j(z, \bar z, \zeta) \tag{2.4}$$

where the P_j are also polynomials of degree m in ζ and smooth in z in the ball (and in $\bar z$.) We can apply the estimate (2.1) for $P(\bar z, D)$ and $s = 0$ to find

$$\|\psi u\|_m \le C\|\psi\left(P(z, D)u - Q(z, D)\right)u\|_0 + C'\|\phi u\|_{m-1}. \tag{2.5}$$

Because the coefficients are small we then find

$$\begin{aligned}\|\psi Q(z, D)u\|_0 &\le \sum_{j,|\alpha|\le m} \|(z - \bar z)_j r_{j,\alpha} D^\alpha \psi u\|_0 + C'\|\phi u\|_{m-1} \\ &\le \delta C\|\psi u\|_m + C'\|\phi u\|_{m-1}.\end{aligned} \tag{2.6}$$

What we would like to say next is that we can choose δ so small that $\delta C < \frac{1}{2}$ and so inserting (2.6) into (2.5) we would get

$$\begin{aligned}\|\psi u\|_m &\le C\|\psi P(z, D)u\|_0 + C\|\psi Q(z, D)u\|_0 + C'\|\phi u\|_{m-1} \\ &\le C\|\psi P(z, D)u\|_0 + \frac{1}{2}\|\psi u\|_m + C'\|\phi u\|_{m-1} \\ &\Longrightarrow \frac{1}{2}\|\psi u\|_m \le C\|\psi P(z, D)u\|_0 + C'\|\phi u\|_{m-1}.\end{aligned} \tag{2.7}$$

However, there is a problem here. Namely this is an *a priori* estimate – to move the norm term from right to left we need to know that it is *finite*. Really, that is what we are trying to prove! So more work is required. Nevertheless we will eventually get essentially the same estimate as in the constant coefficient case.

Theorem 1. *If $P(z,D)$ is an elliptic differential operator of order m with smooth coefficients in $\Omega \subset \mathbb{R}^n$ and $u \in \mathcal{C}^{-\infty}(\Omega)$ is such that $P(z,D)u \in H^s_{\mathrm{loc}}(\Omega)$ for some $s \in \mathbb{R}$ then $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$ and for any ϕ, $\psi \in \mathcal{C}^\infty_c(\Omega)$ with $\phi = 1$ in a neighbourhood of* $\operatorname{supp}(\psi)$ *and $M \in \mathbb{R}$, there exist constants C (depending only on P and ψ) and C' (independent of u) such that*

$$\|\psi u\|_{m+s} \le C\|\phi P(z,D)u\|_s + C'\|\phi u\|_M. \tag{2.8}$$

Let me add here to what I did in the lecture to observe how to get the *a priori* estimate first for general s, rather than $s = 0$ and then for general ψ (since up to this point it is only for ψ with sufficiently small support). In the estimates in (2.6) the L^2 norm of a product is estimated by the L^∞ norm of one factor and the L^2 norm of the other. For general Sobolev norms such an estimate does not hold, but something similar does.

Lemma 1. *If $u \in H^s(\mathbb{R}^n)$ and $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ then*

$$\|\psi u\|_s \le \|\psi\|_{L^\infty}\|u\|_s + C\|u\|_{s-1} \tag{2.9}$$

where the constant depends on s and ψ but not u.

Proposition 2. *Under the hypotheses of Theorem 1 if in addition $u \in \mathcal{C}^\infty(\Omega)$ then* (2.8) *follows.*

See the addenda for proofs.

At this point let me return to the discussion for the constant coefficient case. I will construct an operator Q_Ω which will turn out to be an inverse, modulo smoothing errors, for $P(D)$ acting functions on Ω. The idea is to set

$$Q_\Omega f(z) = \int_\Omega q(z-z')\chi(z,z')f(z')dz' \tag{2.10}$$

where q is from (1.19). For the moment let us not worry about the precise meaning of the integral, only that it should not be ruled out by support difficulties. For this we want $\chi \in \mathcal{C}^\infty(\Omega^2)$ to have proper support in the following sense:

$$\text{If } K \subset \Omega \text{ then } \pi_R\left((\Omega \times K) \cap \operatorname{supp}(\chi)\right) \cup \pi_L\left((L \times \Omega) \cap \operatorname{supp}(\chi)\right) \Subset \Omega. \tag{2.11}$$

Here π_L, $\pi_R : \Omega^2 \longrightarrow \Omega$ are the two projections, onto left and right factors. This condition means that if we multiply the integral in (2.10) on the left by $\phi(z)$, $\phi \in \mathcal{C}^\infty_c(\Omega)$ then the integrand has compact support in z' as well – and so should exist at least as a distributional pairing. The second property we want of χ is that it should not change the properties of q as a convolution operator too much. This reduces to

$$\chi = 1 \text{ in a neighbourhood of } \operatorname{Diag} = \{(z,z); z \in \Omega\} \subset \Omega^2. \tag{2.12}$$

Before discussing why these conditions help us, let me just check that it is possible to find such a ψ. This follows easily from the existence of a partition of unity in Ω as follows. I claim that it is possible to find functions $\phi_i \in \mathcal{C}^\infty_c(\Omega)$, $i \in \mathbb{N}$, which have locally finite supports (i.e. any compact subset of Ω only meets the supports of a finite number of the ϕ_i,) that $\sum\limits_i \phi_i(z) = 1$ in Ω and also that there exist functions $\phi_i' \in \mathcal{C}^\infty_c(\Omega)$, also with locally finite supports in the same sense and such that $\phi_i' = 1$ on a neighborhood of the support of ϕ_i. I leave the existence of such functions as an exercise.

Accepting that such functions exists, consider

$$\chi(z,z') = \sum_i \phi_i(z)\phi_i'(z'). \tag{2.13}$$

Any compact subset of Ω^2 is contained in a compact set of the form $K \times K$ and hence meets the supports of only a finite number of terms in (2.13). Thus the sum is locally finite and hence $\chi \in \mathcal{C}^\infty(\Omega^2)$. Moreover, its support has the property (2.11). Clearly, by the assumption that $\phi_i' = 1$ on the support of ϕ_i and that the latter form a partition of unity, $\psi(z,z) = 1$. In fact $\chi(z,z') = 1$ in a neighborhood of the diagonal since each z has a neighborhood N such that $z' \in N$, $\psi_i(z) \neq 0$ implies $\psi_i'(z') = 1$. Thus we have shown that such a cutoff function ψ exists.

Now, why do we want (2.12)? This is important because

$$\operatorname{sing\,supp}(q) \subset \{0\} \tag{2.14}$$

as follows from (1.14). Indeed these estimates on the Fourier transform show that

$$z^\alpha q(z) \in C^N(\mathbb{R}^n) \text{ if } |\alpha| > n + N \tag{2.15}$$

since this is enough to show that the Fourier transform is L^1. At every point of $\mathbb{R}^n$ other than 0 one of the z_j is non-zero and so, taking $z^\alpha = z_j^k$, (2.15) shows that $q(z)$ is in C^N in $\mathbb{R}^n \setminus \{0\}$ for all N, i.e. (2.14) holds.

Thus the distribution $q(z - z')$ in (2.10) is only singular at the diagonal. It follows that different choices of χ with the properties listed above lead to kernels in (2.10) which differ by smooth functions in Ω^2 with proper supports.

Definition 2. An properly supported smoothing operator, which is by defnινition given by an integral operator

$$Ef(z) = \int_\Omega E(z,z')f(z')dz' \tag{2.16}$$

where $E \in \mathcal{C}^\infty(\Omega^2)$ has proper support (so both maps

$$\pi_L,\ \pi_R : \operatorname{supp}(E) \longrightarrow \Omega \tag{2.17}$$

are proper) gives continuous operators

$$E : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega),\ CmIc(\Omega) \longrightarrow \mathcal{C}_c^\infty(\Omega) \tag{2.18}$$

and has an adjoint of the same type.

Proposition 3. *If $P(D)$ is an elliptic operator with constant coefficients then the kernel in* (2.10) *defines an operator $Q_\Omega : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega)$ which maps $H^s_{\text{loc}}(\Omega)$ to $H^{s+m}_{\text{loc}}(\Omega)$ for each $s \in \mathbb{R}$ and gives a 2-sided parametrix for $P(D)$ in Ω :*

$$P(D)Q_\Omega = \operatorname{Id} - R,\ Q_\Omega P(D) = \operatorname{Id} - R' \tag{2.19}$$

where R and R' are smoothing operators.

Proof. Since we have already seen that changing χ in (2.10) changes Q_Ω by a smoothing operator – and that such a change will just change R and R' in (2.19), we can use the explicit choice for χ made above in terms of a partition unity. Thus, multiplying on the left by some $\mu \in \mathcal{C}_c^\infty(\Omega)$ the sum becomes finite and

$$\mu Q_\Omega f = \sum_j \mu\psi_j q * (\psi_j' f). \tag{2.20}$$

It follows that Q_Ω acts on $\mathcal{C}^{-\infty}(\Omega)$. To check (2.19) we may apply $P(D)$ to (2.20) and consider a region where $\mu = 1$. Since $P(D)q* = \operatorname{Id} - \tilde{R}*$ where $\tilde{R} \in \mathcal{S}(\mathbb{R}^n)$, $P(D)Q_\Omega f = \operatorname{Id} - R$ where additional 'error terms' arise from any differentiation of ψ_j, but all such terms have smooth kernels (since $\psi_j' = 1$ on the support of ψ_j and $q(z - z')$ is smooth outside the diagonal. The second identity in (2.19) comes from the same computation for the adjoint of $P(D)$ and E_Ω. $\square$

Addenda to Lecture 2

Proof of Lemma 1. This is really a standard estimate for Sobolev spaces. Recall that the Sobolev norm is related to the L^2 norm by

$$\|u\|_s = \|\langle D\rangle^s u\|_{L^2}. \tag{2.21}$$

Here $\langle D\rangle^s$ is the convolution operator with kernel defined by its Fourier transform

$$\langle D\rangle^s u = R_s * u,\ \widehat{R_s}(\zeta) = (1 + |\zeta)^2)^{\frac{s}{2}}. \tag{2.22}$$

To get (2.9) use the following standard bound on the commutator.

Lemma 2. *If $\psi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$M_s = [\psi, R_s*] : H^t(\mathbb{R}^n) \longrightarrow H^{t-s+1}(\mathbb{R}^n) \tag{2.23}$$

is bounded for each t.

Proof. Since the Sobolev spaces are defined in terms of the Fourier transform, first conjugate and observe that (2.23) is equivalent to the boundeness of the integral operator with kernel

$$K_{s,t}(\zeta,\zeta') = (1+|\zeta|^2)^{\frac{t-s+1}{2}}\hat{\psi}(\zeta-\zeta')\left((1+|\zeta'|^2)^{\frac{s}{2}} - (1+|\zeta|^2)^{\frac{s}{2}}\right)(1+|\zeta'|^2)^{-\frac{t}{2}} \tag{2.24}$$

on $L^2(\mathbb{R}^n)$. If we insert the characteristic function for the region near the diagonal

$$|\zeta - \zeta'| \le \frac{1}{4}(|\zeta| + |\zeta'|) \Longrightarrow |\zeta| \le 2|\zeta'|,\ |\zeta'| \le 2|\zeta| \tag{2.25}$$

then $|\zeta|$ and $|\zeta'|$ are of comparable size. Using Taylor's formula

$$\begin{gathered}(1+|\zeta'|^2)^{\frac{s}{2}} - (1+|\zeta|^2)^{\frac{s}{2}} = s(\zeta-\zeta')\cdot\int_0^1 (t\zeta + (1-t\zeta')\left(1 + |t\zeta + (1-t)\zeta'|^2\right)^{\frac{s}{2}-1} dt \\ \Longrightarrow \left|(1+|\zeta'|^2)^{\frac{s}{2}} - (1+|\zeta|^2)^{\frac{s}{2}}\right| \le C_s|\zeta-\zeta'|(1+|\zeta|)^{s-1}.\end{gathered} \tag{2.26}$$

It follows that in the region (2.25) the the kernel in (2.24) is bounded by

$$C|\zeta - \zeta'||\hat{\psi}(\zeta - \zeta')|. \tag{2.27}$$

In the complement to (2.25) the kernel is rapidly decreasing in ζ and ζ' in view of the rapid decrease of $\hat{\psi}$. Both terms give bounded operators on L^2, in the first case using the same estimates that show convolution by an element of $\mathcal{S}$ to be bounded. $\square$

From (2.23), (writing 0 for the L^2 norm)

$$\begin{aligned}\|\psi u\|_s = \|R_s * (\psi u)\|_0 &\le \|\psi(R_s * u)\|_0 + \|M_s u\|_0 \\ &\le \|\psi\|_{L^\infty}\|R_s u\|_0 + \|u\|_{s-1} \le \|\psi\|_{L^\infty}\|u\|_s + \|u\|_{s-1}.\end{aligned} \tag{2.28}$$

This completes the proof of (2.9) and so of Lemma 1. $\square$

Proof of Proposition 2. First we can generalize (2.5), now using Lemma 1. Thus, if ψ has support near the point $\bar{z}$

$$\begin{aligned}\|\psi u\|_{s+m} &\le C\|\psi P(\bar{z},D)u\|_s + \|\phi Q(z,D)\psi u\|_s + C'\|\phi u\|_{s+m-1} \\ &\le C\|\psi P(\bar{z},D)u\|_s + \delta C\|\psi uu\|_{s+m} + C'\|\phi u\|_{s+m-1}.\end{aligned} \tag{2.29}$$

This gives the extension of (2.7) to general s (where now we are assuming that u is indeed smooth:

$$\|\psi u\|_{s+m} \le C_s\|\psi P(z,D)u\|_s + C'\|\phi u\|_{s+m-1}. \tag{2.30}$$

Now, given a general element $\psi \in \mathcal{C}_c^\infty(\Omega)$ and $\phi \in \mathcal{C}_c^\infty(\Omega)$ with $\phi = 1$ in a neighbourhood of $\operatorname{supp}(\psi)$ we may choose a partition of unity psi_j for $\operatorname{supp}(\psi)$ for each element of which (2.30) holds for some $\phi_j \in \mathcal{C}_c^\infty(\Omega)$ where in addition $\phi = 1$ in a neighbourhood of $\operatorname{supp}(\phi_j)$. Then, with various constants

$$\begin{aligned}\|\psi u\|_{s+m} \le \sum_j \|\psi_j\|_{s+m} &\le C_s \sum_j \|\psi_j \phi P(z,D)u\|_s + C'\sum_j \|\phi_j \phi u\|_{s+m-1} \\ &\le C_s(K)\|\phi P(z,D)u\|_s + C''\|\phi u\|_{s+m-1},\end{aligned} \tag{2.31}$$

where K is the support of ψ and Lemma 1 has been used again. This removes the restriction on supports.

Now, to get the full (a priori) estimate (2.8), where the error term on the right has been replaced by one with arbitrarily negative Sobolev order, it is only necessary to iterate (2.31) on a nested sequence of cutoff functions as we did earlier in the constant coefficient case.

This completes the proof of Proposition 2. $\square$

3. Lecture 3

13 February, 2007. Parametrices for elliptic operators.

Photo1, Photo2, Photo3, Photo4, Photo5, Photo6, Photo7, Photo8, Photo9

Today I will extend to a variable coefficient elliptic operator the results we had last time in the constant coefficient case. If you recall, last time I worked out *a priori* estimates for solutions of the elliptic operator in terms of Sobolev norms. To use these we need to show the regularity of solutions and I will do this by constructing parameterices just like last time in the constant coefficient case.

Theorem 2. *If $P(z, D)$ is an elliptic differential operator of order m with smooth coefficients in $\Omega \subset \mathbb{R}^n$ then there is a continuous linear operator*

$$Q : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega) \tag{3.1}$$

such that

$$P(z, D)Q = \mathrm{Id} - R_R,\ QP(z, D) = \mathrm{Id} - R_L \tag{3.2}$$

where R_R, R_L are properly-supported smoothing operators.

That is, both R_R and R_L have kernels in $\mathcal{C}^\infty(\Omega^2)$ with proper supports. We will in fact conclude that

$$Q : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^{s+m}_{\mathrm{loc}}(\Omega),\ \forall\ s \in \mathbb{R} \tag{3.3}$$

using the *a priori* estimates.

To construct at least a first approximation to Q we will use essentially the same formula as in the constant coefficient case. Thus consider

$$Q_0 f(z) = \int_\Omega q(z, z - z')\chi(z, z')f(z')dz'. \tag{3.4}$$

Here q is defined as last time, except it now depends on both variables, rather than just the difference, and is defined by inverse Fourier transform

$$q_0(z, Z) = \mathcal{F}^{-1}_{\zeta \longmapsto Z}\hat{q}_0(z, \zeta),\ \hat{q}_0 = \frac{1 - \chi(z, \zeta)}{P(z, \zeta} \tag{3.5}$$

where $\chi \in \mathcal{C}^\infty(\Omega \times \mathbb{R})$ is chosen to have compact support in the second variable, so $\mathrm{supp}(\chi) \cap (K \times \mathbb{R}^n)$ is compact for each $K \Subset \Omega$, and to be equal to 1 on such a large set that $P(z, \zeta) \neq 0$ on the support of $1 - \chi(z, \zeta)$. Thus the right side makes sense and the inverse Fourier transform exists.

Last time I showed how to bound the ζ derivatives of such a quotient, using the ellipticity of P. The same argument works for derivatives with respect to z except no decay results. That is, for any compact set $K \Subset \Omega$

$$|D^\beta_z D^\alpha_\zeta \hat{q}_0(z, \zeta)| \leq C_{\alpha,\beta}(K)(1 + |\zeta|)^{-m-|\alpha|}. \tag{3.6}$$

Now the argument concerning the singularities of q_0 works with z derivatives as well. It shows that

$$(z_j - z'_j)^{N+k} q_0(z, z - z') \in C^N(\Omega \times \mathbb{R}^n) \text{ if } k + m > n/2. \tag{3.7}$$

So as before

$$\mathrm{sing\,supp}\, q_0 \subset \mathrm{Diag} = \{(z, z) \in \Omega^2\}. \tag{3.8}$$

So, as in the earlier case, changing the cutoff function in (3.4) changes Q_0 by a properly supported smoothing operator and this will not affect the validity of (3.2) one way or the other! For the moment not worrying too much about how to make sense of (3.4) consider (formally)

$$P(z,D)Q_0f = \int_\Omega P(z,D_Z)q_0(z,Z)_{Z=z-z'}\chi(z,z')f(z')dz' + E_1 + R_1. \tag{3.9}$$

To apply $P(z,D)$ we just need to apply D^α to E_0f, multiply the result by $p_\alpha(z)$ and add. Applying D_z^α (formally) under the integral sign in (3.4) each derivative may fall on either the 'parameter' z in $q_0(z,z-z')$, the variable $Z=z-z'$ or else on the cutoff $\chi(z,z')$. Now, if χ is every differentiated the result vanishes near the diagonal and as a consequence of (3.8) this gives a smooth kernel. So any such term is included in R_1 in (3.9) which is a smoothing operator and we only have to consider derivatives falling on the first or second variables of q_0. The first term in (3.9) corresponds to *all* derivatives falling on the second variable. Thus

$$E_1f = \int_\Omega P(z,D_Z)e_1(z,z-z')\chi(z,z')f(z')dz' \tag{3.10}$$

is the new term which arises from at least one derivative of the coefficients of q_0 (which is to say ultimately the coefficient of $P(z,\zeta)$. We need to examine this in detail. First however notice that we may rewrite (3.9) as

$$P(z,D)Q_0f = \mathrm{Id} + E_1 + R_1' \tag{3.11}$$

where E_1 is unchanged and R_1' is a new properly supported smoothing operator which comes from the fact that

$$P(z,\zeta)q_0(z,\zeta) = 1-\zeta(z,\zeta) \Longrightarrow P(z,D_Z)q_0(z,Z) = \delta(Z)+r(z,Z),\ r\in\mathcal{C}^\infty(\Omega\times\mathbb{R}^n) \tag{3.12}$$

from the choice of q_0. This part is just as in the constant coefficient case.

So, it is the new error term, E_1 which we must examine more carefully. This arises, as already noted, directly from the fact that the coefficients of $P(z,D)$ are not assumed to be constant, hence $q_0(z,Z)$ depends parameterically on z and this is differentiated in (3.9). So, using Leibniz' formula to get an explicit representation of e_1 in (3.10) we see that

$$e_1(z,Z) = \sum_{|\alpha|\le m,\ |\gamma|<m} p_\alpha(z)\binom{\alpha}{\gamma}D_z^{\alpha-\gamma}D_Z^\gamma q_0(z,Z). \tag{3.13}$$

The precise form of this expansion is not really significant. What *is* important is that at most $m-1$ derivatives are acting on the second variable of $q_0(z,Z)$ since all the terms where all m act here have already been treated. Taking the Fourier transform in the second variable, as before, we find that

$$\hat e_1(z,\zeta) = \sum_{|\alpha|\le m,\ |\gamma|<m} p_\alpha(z)\binom{\alpha}{\gamma}D_z^{\alpha-\gamma}\zeta^\gamma\hat q_0(z,\zeta)\in\mathcal{C}^\infty(\Omega\times\mathbb{R}^n). \tag{3.14}$$

So $\hat e_1$ is the product of $q_0(z,\zeta)$ and a polynomial in ζ of degree at most $m-1$ (with smooth dependence on z. We may therefore transfer the estimates (3.6) to e_1 and conclude that

$$|D_z^\beta D_\zeta^\alpha \hat e_1(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{-1-|\alpha|}. \tag{3.15}$$

Let us denote by $S^m(\Omega\times\mathbb{R}^n)\subset\mathcal{C}^\infty(\Omega\times\mathbb{R}^n)$ the linear space of functions satisfyign (3.6) when $-m$ is replaced by m, i.e.

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{m-|\alpha|} \Longleftrightarrow a\in S^m(\Omega\times\mathbb{R}^n). \tag{3.16}$$

This allows (3.15) to be written succinctly as $\hat{e}_1 \in S^{-1}(\Omega \times \mathbb{R}^n)$.

To summarize so far, we have chosen $\hat{q}_0 \in S^{-m}(\Omega \times \mathbb{R}^n)$ such that with Q_0 given by (3.4),

$$P(z,D)Q_0 = \mathrm{Id} + E_1 + R_1' \tag{3.17}$$

where E_1 is given by the same formula (3.4), as (3.10), where now $\hat{e}_1 \in S^{-1}(\Omega \times \mathbb{R}^n)$. In fact we can easily generalize this discussion, to do so let me use the notation

$$\mathrm{Op}(a)f(z) = \int_\Omega A(z, z-z')\chi(z,z')f(z')dz', \text{ if } \hat{A}(z,\zeta) = a(z,\zeta) \in S^m(\Omega \times \mathbb{R}^n). \tag{3.18}$$

Proposition 4. *If $a \in S^{m'}(\Omega \times \mathbb{R}^n)$ then*

$$P(z,D)\,\mathrm{Op}(a) = \mathrm{Op}(pa) + \mathrm{Op}(b) + R \tag{3.19}$$

where R is a (properly supported) smoothing operator and $b \in S^{m'+m-1}(\Omega \times \mathbb{R}^n)$.

Proof. Follow through the discussion above with $\hat{q}_0$ replaced by a. □

So, we wish to get rid of the error term E_1 in (3.10) to as great an extend as possible. To do so we add to Q_0 a second term $Q_1 = \mathrm{Op}(a_1)$ where

$$a_1 = -\frac{1-\chi}{P(z,\zeta)}\hat{e}_1(z,\zeta) \in S^{-m-1}(\Omega \times \mathbb{R}^n). \tag{3.20}$$

Indeed

$$S^{m'}(\Omega \times \mathbb{R}^n)S^{m''}(\Omega \times \mathbb{R}^n) \subset S^{m'+m''}(\Omega \times \mathbb{R}^n) \tag{3.21}$$

(pretty much as though we are multiplying polynomials). With this choice of Q_1 the identity (3.19) becomes

$$P(z,D)Q_1 = -E_1 + \mathrm{Op}(b_2) + R_2,\ b_2 \in S^{-2}(\Omega \times \mathbb{R}^n) \tag{3.22}$$

since $p(z,\zeta)a_1 = -\hat{e}_1 + r'(z,\zeta)$ where $\mathrm{supp}(r')$ is compact in the second variable and so contributes a smoothing operator and by definition $E_1 = \mathrm{Op}(\hat{e}_1)$.

Now we can proceed by induction, let me formalize it a little.

Lemma 3. *If $P(z,D)$ is elliptic with smooth coefficients on Ω then we may choose a sequence of elements $a_i \in S^{-m-i}(\Omega \times \mathbb{R}^n)$ $i = 0, 1, \dots,$ such that if $Q_i = \mathrm{Op}(a_i)$ then*

$$P(z,D)(Q_0 + Q_1 + \cdots + Q_j) = \mathrm{Id} + E_{j+1} + R_j',\ E_{j+1} = \mathrm{Op}(b_{j+1}) \tag{3.23}$$

with R_j a smoothing operator and $b_j \in S^{-j}$, $j = 1, 2, \dots$.

Proof. We have already taken the first two steps! Namely with $a_0 = \hat{q}_0$, given by (3.5), (3.17) is just (3.23) for $j = 0$. Then, with a_1 given by (3.20), adding (3.22) to (3.20) gives (3.23) for $j = 1$. Proceeding by induction we may assume that we have obtained (3.23) for some j. Then we simply set

$$a_{j+1} = -\frac{1-\chi(z,\zeta)}{P(z,\zeta)}b_{j+1}(z,\zeta) \in S^{-j-1-m}(\Omega \times \mathbb{R}^n)$$

where we have used (3.21). Setting $Q_{j+1} = \mathrm{Op}(a_{j+1})$ the identity (3.19) becomes

$$P(z,D)Q_{j+1} = -E_{j+1} + E_{j+2} + R_{j+1}'',\ E_{j+2} = \mathrm{Op}(b_{j+2}) \tag{3.24}$$

for some $b_{j+2} \in S^{-j-2}(\Omega \times \mathbb{R}^n)$. Adding (3.24) to (3.23) gives the next step in the inductive argument. □

Consider the error term in (3.23) for large j. From the estimates on an element $a \in S^m(\Omega \times \mathbb{R}^n)$

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{m-|\alpha|} \tag{3.25}$$

it follows that if $m < -n-k$ then $\zeta^\gamma a$ is integrable in ζ with all its z derivatives for $|\zeta| \le k$. Thus the inverse Fourier transform has derivatives in all variables up to order k. Applied to the error term in (3.23) we conclude that

$$E_j = \mathrm{Op}(b_j) \text{ has kernel in } \mathcal{C}^{j-n-1}(\Omega^2) \text{ for large } j. \tag{3.26}$$

Thus as j increases the error terms in (3.23) have increasingly smooth kernels.

Lemma 4. *If $J \in \mathcal{C}^k(\Omega^2)$ is properly supported then the operator with kernel J (also denoted J) is a map*

$$J : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^k_{\mathrm{loc}}(\Omega) \ \forall\ s \ge -k. \tag{3.27}$$

As a result of this, and (3.23), the operator $Q_{(k)} = \sum_{j=0}^{k} Q_j$ comes increasingly close to satisfying the first identity in (3.2), except that the error term is only finitely (but arbitrarily) smoothing. Since this is enough for what we want here I will banish the actual solution of (3.2) to the appended notes.

Lemma 5. *For k sufficiently large to left parametrix $Q_{(k)}$ is a continuous operator on $\mathcal{C}^\infty(\Omega)$ and*

$$Q_{(k)} : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^{s+m}_{\mathrm{loc}}(\Omega) \ \forall\ s \in \mathbb{R}. \tag{3.28}$$

Proof. So far I have been rather cavalier in treating $\mathrm{Op}(a)$ for $a \in S^m(\Omega\times\mathbb{R}^n)$ as an operator without showing that this is really the case, however this is a rather easy exercise in distribution theory. Namely, from the basic properties of the Fourier transform and Sobolev spaces

$$A(z, z-z') \in \mathcal{C}^k(\Omega; H^{-n-1+m-k}_{\mathrm{loc}}(\Omega)) \ \forall\ k \in \mathbb{N}. \tag{3.29}$$

It follows that $\mathrm{Op}(a) : H^{n+1-m+k}_{\mathrm{c}}(\Omega)$ into $\mathcal{C}^k(\Omega)$ and in fact into $\mathcal{C}^k_{\mathrm{c}}(\Omega)$ by the properness of the support. In particular it does define an operator on $\mathcal{C}^\infty(\Omega)$ as we have been pretending and the steps above are easily justified.

A similar argument (which I will not give since it is better to do it by duality, as I will describe in the addenda) shows that for any fixed s

$$A : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^S_{\mathrm{loc}}(\Omega) \tag{3.30}$$

for some S.

Now, if $f \in H^s_{\mathrm{loc}}(\Omega)$ then it may be approximated by a sequence $f_j \in \mathcal{C}^\infty(\Omega)$ in the topology of $H^s_{\mathrm{loc}}(\Omega)$, so $\mu f_j \to \mu f$ in $H^s(\mathbb{R}^n)$ for each $\mu \in \mathcal{C}^\infty_c(\Omega)$. Set $u_j = Q_{(k)} f_j \in \mathcal{C}^\infty(\Omega)$ as we have just seen, where k is fixed but will be chosen to be large. Then from our identity $P(z,D)Q_{(k)} = \mathrm{Id} + R_{(k)}$ it follows that

$$P(z,D)u_j = f_j + g_j,\ g_j = R_{(k)} f_j \to R_{(k)} f \in H^N_{\mathrm{loc}}(\Omega) \tag{3.31}$$

for k large enough depending on s. Thus the right side converges in $H^s_{\mathrm{loc}}(\Omega)$ and by (3.30), $u_j \to u$ in some $H^S_{\mathrm{loc}}(\Omega)$. But now we can use the *a priori* estimates (2.8) on $u_j \in \mathcal{C}^\infty(\Omega)$ to conclude that

$$\|\psi u_j\|_{s+m} \le C\|\phi(f_j+g_j)\|_s + C''\|\phi u_j\|_S \tag{3.32}$$

to see that ψu_j is bounded in $H^{s+m}(\mathbb{R}^n)$ for any $\psi \in \mathcal{C}_c^\infty(\Omega)$. In fact, applied to the difference $u_j - u_l$ it shows the sequence to be Cauchy. Hence in fact $u \in H^{s+m}_{\text{loc}}(\Omega)$ and the estimates (2.8) hold for this u. That is, $Q_{(k)}$ has the mapping property (3.28) for large k. □

In fact the continuity property (3.28) holds for all $\operatorname{Op}(a)$ where $a \in S^m(\Omega \times \mathbb{R}^n)$, not just those which are parametrices for elliptic differential operators. I will comment on this below.

There is also the question of the second identity in (3.2), at least in the same finite-order-error sense. To solve this we may use the transpose identity. Thus taking formal transposes this second identity should be equivalent to

$$P^t Q^t = \operatorname{Id} - R_L^t. \tag{3.33}$$

The transpose of $P(z, D)$ is the differential operotor

$$P^t(z, D) = \sum_{|\alpha| \le m} (-D)_z^\alpha p_\alpha(z). \tag{3.34}$$

This is again of order m and after a lot of differenttiation to move the coefficients back to the left we see that its leading part is just $P_m(z, -D)$ where $P_m(z, D)$ is the leading part of $P(z, D)$. Thus it is also elliptic in Ω. Thus to construct a solution to (3.34), up to finite order errors, we need just apply Lemma 3 to the transpose differential operator. This $Q'_{(N)} = \operatorname{Op}(a'_{(N)}$ exists with the property

$$P^t(z, D) Q'_{(N)} = \operatorname{Id} - R'_{(N)} \tag{3.35}$$

where the kernel of $R'_{(N)}$ is in $C^N(\Omega^2)$. Since this property is preserved under transpose we have indeed solved the second identity in (3.2) up to an arbitrarily smooth error.

Of course the claim in Theorem 2 is that the one operator satisfies both identities, whereas we have constructed two operator which each satisfy one of them, up to finite smoothing error terms

$$P(z, D) Q_R = \operatorname{Id} - R_R,\ Q_L P(z, D) = \operatorname{Id} - R_L. \tag{3.36}$$

However these operators must themselves be equal up to finite smoothing error terms since composing the first identity on the left with Q_L and the second on the right with Q_R shows that

$$Q_L - Q_L R_R = Q_L P(z, D) Q_R = Q_R - R_L Q_R \tag{3.37}$$

where the associativity of operator composition has been used. We have already checked the mapping property(3.28) for both Q_L and Q_R, assuming the error terms are sufficiently smoothing. It follows that the composite error terms here map $H^{-p}_{\text{loc}}(\Omega)$ into $H^p_{\text{loc}}(\Omega)$ where $p \to \infty$ with k with the same also true of the transposes of these operators. Such an operator has kernel in $C^{p'}(\Omega^2)$ where again $p' \to \infty$ with k. Thus the difference of Q_L and Q_R itself becomes arbitrarily smoothing as $k \to \infty$.

Finally then we have proved most of Theorem 2 except with arbitrarily finitely smoothing errors. In fact we have not quite proved the regularity statement that $P(z, D)u \in H^s_{\text{loc}}(\Omega)$ implies $u \in H^{s+m}_{\text{loc}}(\Omega)$ although we can very close in the proof of Lemma 5. In fact, now that we know that $Q_{(k)}$ is also a right parametrix, i.e. satisfies the second identity in (2.8) up to arbitrarily smoothing errors, this too follows. Namely it also follows from the discussion above that $Q_{(k)}$ is an operator on $\mathcal{C}^{-\infty}(\Omega)$. Then

$$Q_P(k)P(z,D)u = u + v_k,\ \ \psi v_k \in H^{s+m}(\Omega)$$

for large enough k so (3.28) implies $u \in H^{s+m}_{\text{loc}}(\Omega)$ and the *a priori* estimates become real estimates on all solutions.

Addenda to Lecture 3

Asymptotic completeness to show that really can get smoothing errors.
Some discussion of pseudodifferential operators – adjoints, composition and boundedness
Some more reassurance as regards operators, kernels and mapping properties – since I have treated these fairly shabbily!

4. Lecture 4

15 February, 2007: Coordinate invariance.

[Currently very succinct to say the least!]

Photographs as usual by Chris Kottke.

Photo1, Photo2, Photo3, Photo4, Photo5, Photo6, Photo7, Photo8, Photo9, Photo10, Photo11, Photo12, Photo13

Let $\Omega_i \subset \mathbb{R}^n$ be open and $f : \Omega_1 \longrightarrow \Omega_2$ a diffeomorphism, so it is a $\mathcal{C}^\infty$ map, which is equivalent to the condition

$$f^*u \in \mathcal{C}^\infty(\Omega_1)\ \forall\ u \in \mathcal{C}^\infty(\Omega_2),\ f^*u = u \circ f,\ f^*u(z) = u(f(z)), \tag{4.1}$$

and has a $\mathcal{C}^\infty$ inverse $f^{-1} : \Omega_2 \longrightarrow \Omega_1$.

Such a map induces an isomorphism $f^* : \mathcal{C}^\infty_c(\Omega_2) \longrightarrow \mathcal{C}^\infty_c(\Omega_1)$.

Recall also that, as a homeomorphism, f^* identifies the measurable functions on Ω_2 with those on Ω_1. Since it is continuously differentiable it also identifies $L^1_{\text{loc}}(\Omega_2)$ with $L^1_{\text{loc}}(\Omega_1)$ and

$$u \in L^1_{\text{c}}(\Omega_2) \Longrightarrow \int_{\Omega_1} f^*u(z)|J_f(z)|dz = \int_{\Omega_1} u(z')dz',\ J_f(z) = \det \frac{\partial f_i(z)}{\partial z_j}. \tag{4.2}$$

The absolute value appears because the definition of the Lebesgue integral is through the Lebesgue measure.

It follows that $f^* : L^2_{\text{loc}}(\Omega_2) \longrightarrow L^2_{\text{loc}}(\Omega_1)$ is also an isomorphism. If $u \in L^2(\Omega_2)$ has support in some compact subset $K \Subset \Omega_2$ then f^*u has support in the compact subset $f^{-1}(K) \Subset \Omega_1$ and

$$\|f^*u\|^2_{L^2} = \int_{\Omega_1} |f^*u|^2 dz \le C(K) \int_{\Omega_1} |f^*u|^2 |J_f(z)|dz = C(K)\|u\|^2_{L^2}. \tag{4.3}$$

Distributions are defined by duality, as the continuous linear functionals:-

$$u \in \mathcal{C}^{-\infty}(\Omega) \Longrightarrow u : \mathcal{C}^\infty_c(\Omega) \longrightarrow \mathbb{C}. \tag{4.4}$$

We always embed the smooth functions in the distributions using integration. This presents a small problem here, namely it is not consistent under pull-back. Indeed if $u \in \mathcal{C}^\infty(\Omega_2)$ and $\mu \in \mathcal{C}^\infty_c(\Omega_1)$ then

$$\begin{gathered}\int_{\Omega_1} f^*u(z)\mu(z)|J_f(z)|dz = \int_{\Omega_2} u(z')(f^{-1})^*\mu(z')dz' \text{ or}\\ \int_{\Omega_1} f^*u(z)\mu(z)dz = \int_{\Omega_2} u(z')(f^{-1})^*\mu(z')|J_{f^{-1}}(z')|dz',\\ \text{since } f^*J_{f^{-1}} = (J_f)^{-1}.\end{gathered} \tag{4.5}$$

So, if we want distributions to be 'generalized functions', so that the identification of $u \in \mathcal{C}^\infty(\Omega_2)$ as an element of $\mathcal{C}^{-\infty}(\Omega_2)$ is consistent with the identification of $f^*u \in \mathcal{C}^\infty(\Omega_1)$ as an element of $\mathcal{C}^{-\infty}(\Omega_1)$ we need to use (4.5). Thus we *define*

$$f^* : \mathcal{C}^{-\infty}(\Omega_2) \longrightarrow \mathcal{C}^{-\infty}(\Omega_1) \text{ by } f^*u(\mu) = u((f^{-1})^*\mu|J_{f^{-1}}|). \tag{4.6}$$

There are better ways to think about this, namely in terms of densities, but not at the moment. Of course one should check that f^* is a map as indicated.

Now, with these definitions we have

Theorem 3. *For every $s \in \mathbb{R}$, any diffeomorphism $f : \Omega_1 \longrightarrow \Omega_2$ induces an isomorphism*

$$f^* : H^s_{\mathrm{loc}}(\Omega_2) \longleftrightarrow H^s_{\mathrm{loc}}(\Omega_1). \tag{4.7}$$

Proof. We know this already for $s = 0$. To prove it for $0 < s < 1$ we use the norm equivalent to the standard Fourier transform norm:-

$$\|u\|_s^2 = \|u\|_{L^2}^2 + \int_{\mathbb{R}^{2n}} \frac{|u(z) - u(\zeta)|^2}{|z-\zeta|^{2s+n}} dz dz'. \tag{4.8}$$

See Sect 7.9 of [1]. Then if $u \in H^s_{\mathrm{c}}(\Omega_2)$ has support in $K \Subset \Omega_2$ with $0 < s < 1$, certainly $u \in L^2$ so $f^*u \in L^2$ and we can bound the second part of the norm in (4.8) on f^*u :

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \frac{|u(f(z)) - u(f(z'))|^2}{|z-z'|^{2s+n}} dz d\zeta \\ &\quad = \int_{\mathbb{R}^{2n}} \frac{|u(z') - u(\zeta')|^2}{|g(z') - g(\zeta')|^{2s+n}} |J_g(z')||J_g(\zeta')| dz' d\zeta' \\ &\qquad\qquad C \le \int_{\mathbb{R}^{2n}} \frac{|u(z) - u(z')|^2}{|z-z'|^{2s+n}} dz dz' \end{aligned} \tag{4.9}$$

since $C|g(z') - g(\zeta')| \ge |z' - \zeta'|$ where $g = f^{-1}$. For the spaces of order $m + s$, $0 \le s < 1$ and $m \in \mathbb{N}$ we can proceed by estimating the norms of the derivatives and for negative orders we proceed by duality. □

Consider the issue of differential operators more carefully. If $P : \mathcal{C}^\infty(\Omega_1) \longrightarrow \mathcal{C}^\infty(\Omega_1)$ is a differential operator of order m with smooth coefficients, then so is

$$P_f : \mathcal{C}^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_2),\ P_f v = (f^{-1})^*(Pf^*v). \tag{4.10}$$

However, the formula for the coefficients, i.e. the explicit formula for P_f, is rather complicated:-

$$P = \sum_{|\alpha| \le m} \Longrightarrow P_f = \sum_{|\alpha| \le m} p_\alpha(g(z'))(J_f(z')D_{z'})^\alpha \tag{4.11}$$

since we have to do some serious differentiation to move all the Jacobian terms to the left.

Even though the formula (4.11) is complicated, the leading part of it is rather simple. Observe that we can compute the leading part of a differential operator by 'oscillatory testing'. Thus, on an open set Ω consider

$$P(z,D)(e^{it\psi}u) = e^{it\psi} \sum_{k=0}^m t^k P_k(z,D)u,\ u \in \mathcal{C}^\infty(\Omega),\ \psi \in \mathcal{C}^\infty(\Omega),\ t \in \mathbb{R}. \tag{4.12}$$

Here the $P_k(z, D)$ are differential operators of order $m-k$ acting on u (they involve derivatives of ψ of course). Indeed, the only way a factor of t can occur is from a derivative acting on $e^{it\psi}$ through

$$D_{z_j} e^{it\psi} = e^{it\psi} t \frac{\partial \psi}{\partial z_j}. \tag{4.13}$$

Thus, the coefficient of t^m involves no differentiation of u at all and takes the simple form

$$P_m = \sum_{|\alpha|=m} p_\alpha(z)(d\psi)^\alpha \in \mathcal{C}^\infty(\Omega). \tag{4.14}$$

In particular, the value of this function at any point $z \in \Omega$ is determined once we know $d\psi$, the differential of ψ at that point. Using this observation, we can easily compute the leading part of P_f given that of P in (4.10). Namely if $\psi \in \mathcal{C}^\infty(\Omega_2)$ and $(P_f)(z')$ is the leading part of P_f for

$$\begin{aligned}\sigma_m(P_f)_m(z')v &= \lim_{t\to\infty} t^{-m}e^{-it\psi}P_f(z', D_{z'})(e^{it\psi}v)\\ &= \lim_{t\to\infty} t^{-m}e^{-it\psi}g^*(P(z,D_z)(e^{itf^*\psi}f^*v)\\ &= g^*(\lim_{t\to\infty} t^{-m}e^{-itf^*\psi}g^*(P(z,D_z)(e^{itf^*\psi}f^*v) = g^*P_m(z,f^*\psi)f^*v.\end{aligned} \tag{4.15}$$

Thus

$$\sigma_m(P_f(z',\zeta')) = g^*\sigma_m(P(z,\zeta). \tag{4.16}$$

5. Lecture 5

22 February, 2007.

(1) Matrices of differential operators
(2) Operators on vector bundles
(3) Global elliptic estimates
(4) Elliptic operators are Fredholm on Sobolev spaces and $\mathcal{C}^\infty$.
(5) Index theorem.
(6) Self-adjointness, resolvent family, compactness.
(7) Discreteness of spectrum; completeness.
(8) Homogeneous first order operators with constant coefficients on $\mathbb{R}^n$.
(9) Compact manifolds with boundary
(10) The b-Sobolev spaces
    Integral order
    Mellin transform
    General real order
(11) Radial compactification of $\mathbb{R}^n$.
(12) Isomorphism properties of homogeneos elliptic operators (of first order).

References

[1] L. Hörmander, *The analysis of linear partial differential operators*, vol. 2, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.