# 4. Lecture 4

15 February, 2007: Coordinate invariance.

[Currently very succinct to say the least!]

Photographs as usual by Chris Kottke.

Photo1, Photo2, Photo3, Photo4, Photo5, Photo6, Photo7, Photo8, Photo9, Photo10, Photo11, Photo12, Photo13

Let $\Omega_i \subset \mathbb{R}^n$ be open and $f : \Omega_1 \longrightarrow \Omega_2$ a diffeomorphism, so it is a $\mathcal{C}^\infty$ map, which is equivalent to the condition

$$f^*u \in \mathcal{C}^\infty(\Omega_1)\ \forall\ u \in \mathcal{C}^\infty(\Omega_2),\ f^*u = u \circ f,\ f^*u(z) = u(f(z)), \tag{4.1}$$

and has a $\mathcal{C}^\infty$ inverse $f^{-1} : \Omega_2 \longrightarrow \Omega_1$.

Such a map induces an isomorphism $f^* : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}_c^\infty(\Omega_1)$.

Recall also that, as a homeomorphism, $f^*$ identifies the measurable functions on $\Omega_2$ with those on $\Omega_1$. Since it is continuously differentiable it also identifies $L^1_{\text{loc}}(\Omega_2)$ with $L^1_{\text{loc}}(\Omega_1)$ and

$$u \in L^1_{\text{c}}(\Omega_2) \Longrightarrow \int_{\Omega_1} f^*u(z)|J_f(z)|dz = \int_{\Omega_1} u(z')dz',\ J_f(z) = \det \frac{\partial f_i(z)}{\partial z_j}. \tag{4.2}$$

The absolute value appears because the definition of the Lebesgue integral is through the Lebesgue measure.

It follows that $f^* : L^2_{\text{loc}}(\Omega_2) \longrightarrow L^2_{\text{loc}}(\Omega_1)$ is also an isomorphism. If $u \in L^2(\Omega_2)$ has support in some compact subset $K \Subset \Omega_2$ then $f^*u$ has support in the compact subset $f^{-1}(K) \Subset \Omega_1$ and

$$\|f^*u\|^2_{L^2} = \int_{\Omega_1} |f^*u|^2 dz \le C(K) \int_{\Omega_1} |f^*u|^2 |J_f(z)|dz = C(K)\|u\|^2_{L^2}. \tag{4.3}$$

Distributions are defined by duality, as the continuous linear functionals:-

$$u \in \mathcal{C}^{-\infty}(\Omega) \Longrightarrow u : \mathcal{C}_c^\infty(\Omega) \longrightarrow \mathbb{C}. \tag{4.4}$$

We always embed the smooth functions in the distributions using integration. This presents a small problem here, namely it is not consistent under pull-back. Indeed if $u \in \mathcal{C}^\infty(\Omega_2)$ and $\mu \in \mathcal{C}_c^\infty(\Omega_1)$ then

$$\begin{gathered}\int_{\Omega_1} f^*u(z)\mu(z)|J_f(z)|dz = \int_{\Omega_2} u(z')(f^{-1})^*\mu(z')dz' \text{ or} \\ \int_{\Omega_1} f^*u(z)\mu(z)dz = \int_{\Omega_2} u(z')(f^{-1})^*\mu(z')|J_{f^{-1}}(z')|dz', \\ \text{since } f^*J_{f^{-1}} = (J_f)^{-1}.\end{gathered} \tag{4.5}$$

So, if we want distributions to be 'generalized functions', so that the identification of $u \in \mathcal{C}^\infty(\Omega_2)$ as an element of $\mathcal{C}^{-\infty}(\Omega_2)$ is consistent with the identification of $f^*u \in \mathcal{C}^\infty(\Omega_1)$ as an element of $\mathcal{C}^{-\infty}(\Omega_1)$ we need to use (4.5). Thus we *define*

$$f^* : \mathcal{C}^{-\infty}(\Omega_2) \longrightarrow \mathcal{C}^{-\infty}(\Omega_1) \text{ by } f^*u(\mu) = u((f^{-1})^*\mu|J_{f^{-1}}|). \tag{4.6}$$

There are better ways to think about this, namely in terms of densities, but not at the moment. Of course one should check that $f^*$ is a map as indicated.

Now, with these definitions we have

**Theorem 3.** *For every $s \in \mathbb{R}$, any diffeomorphism $f : \Omega_1 \longrightarrow \Omega_2$ induces an isomorphism*

$$f^* : H^s_{\mathrm{loc}}(\Omega_2) \longleftrightarrow H^s_{\mathrm{loc}}(\Omega_1). \tag{4.7}$$

*Proof.* We know this already for $s = 0$. To prove it for $0 < s < 1$ we use the norm equivalent to the standard Fourier transform norm:-

$$\|u\|_s^2 = \|u\|_{L^2}^2 + \int_{\mathbb{R}^{2n}} \frac{|u(z) - u(\zeta)|^2}{|z-\zeta|^{2s+n}} dz dz'. \tag{4.8}$$

See Sect 7.9 of [1]. Then if $u \in H^s_{\mathrm{c}}(\Omega_2)$ has support in $K \Subset \Omega_2$ with $0 < s < 1$, certainly $u \in L^2$ so $f^*u \in L^2$ and we can bound the second part of the norm in (4.8) on $f^*u$ :

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \frac{|u(f(z)) - u(f(z'))|^2}{|z-z'|^{2s+n}} dz d\zeta \\ &\qquad = \int_{\mathbb{R}^{2n}} \frac{|u(z') - u(\zeta')|^2}{|g(z') - g(\zeta')|^{2s+n}} |J_g(z')||J_g(\zeta')| dz' d\zeta' \\ &\qquad\qquad C \leq \int_{\mathbb{R}^{2n}} \frac{|u(z) - u(z')|^2}{|z-z'|^{2s+n}} dz dz' \end{aligned} \tag{4.9}$$

since $C|g(z') - g(\zeta')| \geq |z' - \zeta'|$ where $g = f^{-1}$. For the spaces of order $m + s$, $0 \leq s < 1$ and $m \in \mathbb{N}$ we can proceed by estimating the norms of the derivatives and for negative orders we proceed by duality. □

Consider the issue of differential operators more carefully. If $P : \mathcal{C}^\infty(\Omega_1) \longrightarrow \mathcal{C}^\infty(\Omega_1)$ is a differential operator of order $m$ with smooth coefficients, then so is

$$P_f : \mathcal{C}^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_2),\ P_f v = (f^{-1})^*(Pf^*v). \tag{4.10}$$

However, the formula for the coefficients, i.e. the explicit formula for $P_f$, is rather complicated:-

$$P = \sum_{|\alpha| \leq m} \Longrightarrow P_f = \sum_{|\alpha| \leq m} p_\alpha(g(z'))(J_f(z')D_{z'})^\alpha \tag{4.11}$$

since we have to do some serious differentiation to move all the Jacobian terms to the left.

Even though the formula (4.11) is complicated, the leading part of it is rather simple. Observe that we can compute the leading part of a differential operator by 'oscillatory testing'. Thus, on an open set $\Omega$ consider

$$P(z, D)(e^{it\psi}u) = e^{it\psi} \sum_{k=0}^m t^k P_k(z, D)u,\ u \in \mathcal{C}^\infty(\Omega),\ \psi \in \mathcal{C}^\infty(\Omega),\ t \in \mathbb{R}. \tag{4.12}$$

Here the $P_k(z, D)$ are differential operators of order $m - k$ acting on $u$ (they involve derivatives of $\psi$ of course). Indeed, the only way a factor of $t$ can occur is from a derivative acting on $e^{it\psi}$ through

$$D_{z_j} e^{it\psi} = e^{it\psi} t \frac{\partial \psi}{\partial z_j}. \tag{4.13}$$

Thus, the coefficient of $t^m$ involves no differentiation of $u$ at all and takes the simple form

$$P_m = \sum_{|\alpha|=m} p_\alpha(z)(d\psi)^\alpha \in \mathcal{C}^\infty(\Omega). \tag{4.14}$$

In particular, the value of this function at any point $z \in \Omega$ is determined once we know $d\psi$, the differential of $\psi$ at that point. Using this observation, we can easily compute the leading part of $P_f$ given that of $P$ in (4.10). Namely if $\psi \in \mathcal{C}^\infty(\Omega_2)$ and $(P_f)(z')$ is the leading part of $P_f$ for

$$\begin{aligned}(4.15)\quad \sigma_m(P_f)_m(z')v &= \lim_{t\to\infty} t^{-m}e^{-it\psi}P_f(z', D_{z'})(e^{it\psi}v)\\ &= \lim_{t\to\infty} t^{-m}e^{-it\psi}g^*(P(z, D_z)(e^{itf^*\psi}f^*v)\\ = g^*(\lim_{t\to\infty} t^{-m}&e^{-itf^*\psi}g^*(P(z, D_z)(e^{itf^*\psi}f^*v) = g^*P_m(z, f^*\psi)f^*v.\end{aligned}$$

Thus

$$\sigma_m(P_f(z', \zeta')) = g^*\sigma_m(P(z, \zeta). \tag{4.16}$$

## References

[1] L. Hörmander, *The analysis of linear partial differential operators*, vol. 2, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.