3. Lecture 3

13 February, 2007. Parametrices for elliptic operators.

Photo1, Photo2, Photo3, Photo4, Photo5, Photo6, Photo7, Photo8, Photo9

Today I will extend to a variable coefficient elliptic operator the results we had last time in the constant coefficient case. If you recall, last time I worked out *a priori* estimates for solutions of the elliptic operator in terms of Sobolev norms. To use these we need to show the regularity of solutions and I will do this by constructing parameterices just like last time in the constant coefficient case.

Theorem 2. *If $P(z,D)$ is an elliptic differential operator of order m with smooth coefficients in $\Omega \subset \mathbb{R}^n$ then there is a continuous linear operator*

$$Q : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega) \tag{3.1}$$

such that

$$P(z,D)Q = \operatorname{Id} - R_R,\ \ QP(z,D) = \operatorname{Id} - R_L \tag{3.2}$$

where R_R, R_L are properly-supported smoothing operators.

That is, both R_R and R_L have kernels in $\mathcal{C}^\infty(\Omega^2)$ with proper supports. We will in fact conclude that

$$Q : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^{s+m}_{\mathrm{loc}}(\Omega),\ \forall\ s \in \mathbb{R} \tag{3.3}$$

using the *a priori* estimates.

To construct at least a first approximation to Q we will use essentially the same formula as in the constant coefficient case. Thus consider

$$Q_0 f(z) = \int_\Omega q(z, z - z')\chi(z,z')f(z')dz'. \tag{3.4}$$

Here q is defined as last time, except it now depends on both variables, rather than just the difference, and is defined by inverse Fourier transform

$$q_0(z,Z) = \mathcal{F}^{-1}_{\zeta \longmapsto Z}\hat{q}_0(z,\zeta),\ \ \hat{q}_0 = \frac{1-\chi(z,\zeta)}{P(z,\zeta} \tag{3.5}$$

where $\chi \in \mathcal{C}^\infty(\Omega \times \mathbb{R})$ is chosen to have compact support in the second variable, so $\operatorname{supp}(\chi) \cap (K \times \mathbb{R}^n)$ is compact for each $K \Subset \Omega$, and to be equal to 1 on such a large set that $P(z,\zeta) \neq 0$ on the support of $1 - \chi(z,\zeta)$. Thus the right side makes sense and the inverse Fourier transform exists.

Last time I showed how to bound the ζ derivatives of such a quotient, using the ellipticity of P. The same argument works for derivatives with respect to z except no decay results. That is, for any compact set $K \Subset \Omega$

$$|D^\beta_z D^\alpha_\zeta \hat{q}_0(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{-m-|\alpha|}. \tag{3.6}$$

Now the argument concerning the singularities of q_0 works with z derivatives as well. It shows that

$$(z_j - z'_j)^{N+k} q_0(z, z-z') \in C^N(\Omega \times \mathbb{R}^n) \text{ if } k + m > n/2. \tag{3.7}$$

So as before

$$\operatorname{sing\,supp} q_0 \subset \operatorname{Diag} = \{(z,z) \in \Omega^2\}. \tag{3.8}$$

So, as in the earlier case, changing the cutoff function in (3.4) changes Q_0 by a properly supported smoothing operator and this will not affect the validity of (3.2) one way or the other! For the moment not worrying too much about how to make sense of (3.4) consider (formally)

$$P(z,D)Q_0f = \int_\Omega P(z,D_Z)q_0(z,Z)_{Z=z-z'}\chi(z,z')f(z')dz' + E_1 + R_1. \tag{3.9}$$

To apply $P(z,D)$ we just need to apply D^α to E_0f, multiply the result by $p_\alpha(z)$ and add. Applying D_z^α (formally) under the integral sign in (3.4) each derivative may fall on either the 'parameter' z in $q_0(z,z-z')$, the variable $Z=z-z'$ or else on the cutoff $\chi(z,z')$. Now, if χ is every differentiated the result vanishes near the diagonal and as a consequence of (3.8) this gives a smooth kernel. So any such term is included in R_1 in (3.9) which is a smoothing operator and we only have to consider derivatives falling on the first or second variables of q_0. The first term in (3.9) corresponds to *all* derivatives falling on the second variable. Thus

$$E_1f = \int_\Omega P(z,D_Z)e_1(z,z-z')\chi(z,z')f(z')dz' \tag{3.10}$$

is the new term which arises from at least one derivative of the coefficients of q_0 (which is to say ultimately the coefficient of $P(z,\zeta)$. We need to examine this in detail. First however notice that we may rewrite (3.9) as

$$P(z,D)Q_0f = \operatorname{Id} + E_1 + R_1' \tag{3.11}$$

where E_1 is unchanged and R_1' is a new properly supported smoothing operator which comes from the fact that

$$P(z,\zeta)q_0(z,\zeta) = 1-\zeta(z,\zeta) \Longrightarrow P(z,D_Z)q_0(z,Z) = \delta(Z)+r(z,Z),\ r\in\mathcal{C}^\infty(\Omega\times\mathbb{R}^n) \tag{3.12}$$

from the choice of q_0. This part is just as in the constant coefficient case.

So, it is the new error term, E_1 which we must examine more carefully. This arises, as already noted, directly from the fact that the coefficients of $P(z,D)$ are not assumed to be constant, hence $q_0(z,Z)$ depends parameterically on z and this is differentiated in (3.9). So, using Leibniz' formula to get an explicit representation of e_1 in (3.10) we see that

$$e_1(z,Z) = \sum_{|\alpha|\le m,\ |\gamma|<m} p_\alpha(z)\binom{\alpha}{\gamma}D_z^{\alpha-\gamma}D_Z^\gamma q_0(z,Z). \tag{3.13}$$

The precise form of this expansion is not really significant. What *is* important is that at most $m-1$ derivatives are acting on the second variable of $q_0(z,Z)$ since all the terms where all m act here have already been treated. Taking the Fourier transform in the second variable, as before, we find that

$$\hat{e}_1(z,\zeta) = \sum_{|\alpha|\le m,\ |\gamma|<m} p_\alpha(z)\binom{\alpha}{\gamma}D_z^{\alpha-\gamma}\zeta^\gamma\hat{q}_0(z,\zeta) \in \mathcal{C}^\infty(\Omega\times\mathbb{R}^n). \tag{3.14}$$

So $\hat{e}_1$ is the product of $q_0(z,\zeta)$ and a polynomial in ζ of degree at most $m-1$ (with smooth dependence on z. We may therefore transfer the estimates (3.6) to e_1 and conclude that

$$|D_z^\beta D_\zeta^\alpha\hat{e}_1(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{-1-|\alpha|}. \tag{3.15}$$

Let us denote by $S^m(\Omega\times\mathbb{R}^n)\subset\mathcal{C}^\infty(\Omega\times\mathbb{R}^n)$ the linear space of functions satisfyign (3.6) when $-m$ is replaced by m, i.e.

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{m-|\alpha|} \Longleftrightarrow a\in S^m(\Omega\times\mathbb{R}^n). \tag{3.16}$$

This allows (3.15) to be written succinctly as $\hat{e}_1 \in S^{-1}(\Omega \times \mathbb{R}^n)$.

To summarize so far, we have chosen $\hat{q}_0 \in S^{-m}(\Omega \times \mathbb{R}^n)$ such that with Q_0 given by (3.4),

$$P(z,D)Q_0 = \operatorname{Id} + E_1 + R_1' \tag{3.17}$$

where E_1 is given by the same formula (3.4), as (3.10), where now $\hat{e}_1 \in S^{-1}(\Omega \times \mathbb{R}^n)$. In fact we can easily generalize this discussion, to do so let me use the notation

$$\operatorname{Op}(a)f(z) = \int_\Omega A(z, z-z')\chi(z,z')f(z')dz', \text{ if } \hat{A}(z,\zeta) = a(z,\zeta) \in S^m(\Omega \times \mathbb{R}^n). \tag{3.18}$$

Proposition 4. *If $a \in S^{m'}(\Omega \times \mathbb{R}^n)$ then*

$$P(z,D)\operatorname{Op}(a) = \operatorname{Op}(pa) + \operatorname{Op}(b) + R \tag{3.19}$$

where R is a (properly supported) smoothing operator and $b \in S^{m'+m-1}(\Omega \times \mathbb{R}^n)$.

Proof. Follow through the discussion above with $\hat{q}_0$ replaced by a. □

So, we wish to get rid of the error term E_1 in (3.10) to as great an extend as possible. To do so we add to Q_0 a second term $Q_1 = \operatorname{Op}(a_1)$ where

$$a_1 = -\frac{1-\chi}{P(z,\zeta)}\hat{e}_1(z,\zeta) \in S^{-m-1}(\Omega \times \mathbb{R}^n). \tag{3.20}$$

Indeed

$$S^{m'}(\Omega \times \mathbb{R}^n)S^{m''}(\Omega \times \mathbb{R}^n) \subset S^{m'+m''}(\Omega \times \mathbb{R}^n) \tag{3.21}$$

(pretty much as though we are multiplying polynomials). With this choice of Q_1 the identity (3.19) becomes

$$P(z,D)Q_1 = -E_1 + \operatorname{Op}(b_2) + R_2,\ b_2 \in S^{-2}(\Omega \times \mathbb{R}^n) \tag{3.22}$$

since $p(z,\zeta)a_1 = -\hat{e}_1 + r'(z,\zeta)$ where $\operatorname{supp}(r')$ is compact in the second variable and so contributes a smoothing operator and by definition $E_1 = \operatorname{Op}(\hat{e}_1)$.

Now we can proceed by induction, let me formalize it a little.

Lemma 3. *If $P(z,D)$ is elliptic with smooth coefficients on Ω then we may choose a sequence of elements $a_i \in S^{-m-i}(\Omega \times \mathbb{R}^n)$ $i = 0, 1, \ldots,$ such that if $Q_i = \operatorname{Op}(a_i)$ then*

$$P(z,D)(Q_0 + Q_1 + \cdots + Q_j) = \operatorname{Id} + E_{j+1} + R_j',\ E_{j+1} = \operatorname{Op}(b_{j+1}) \tag{3.23}$$

with R_j a smoothing operator and $b_j \in S^{-j}$, $j = 1, 2, \ldots$.

Proof. We have already taken the first two steps! Namely with $a_0 = \hat{q}_0$, given by (3.5), (3.17) is just (3.23) for $j = 0$. Then, with a_1 given by (3.20), adding (3.22) to (3.20) gives (3.23) for $j = 1$. Proceeding by induction we may assume that we have obtained (3.23) for some j. Then we simply set

$$a_{j+1} = -\frac{1-\chi(z,\zeta)}{P(z,\zeta)}b_{j+1}(z,\zeta) \in S^{-j-1-m}(\Omega \times \mathbb{R}^n)$$

where we have used (3.21). Setting $Q_{j+1} = \operatorname{Op}(a_{j+1})$ the identity (3.19) becomes

$$P(z,D)Q_{j+1} = -E_{j+1} + E_{j+2} + R_{j+1}'',\ E_{j+2} = \operatorname{Op}(b_{j+2}) \tag{3.24}$$

for some $b_{j+2} \in S^{-j-2}(\Omega \times \mathbb{R}^n)$. Adding (3.24) to (3.23) gives the next step in the inductive argument. □

Consider the error term in (3.23) for large j. From the estimates on an element $a \in S^m(\Omega \times \mathbb{R}^n)$

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)| \leq C_{\alpha,\beta}(K)(1+|\zeta|)^{m-|\alpha|} \tag{3.25}$$

it follows that if $m < -n-k$ then $\zeta^\gamma a$ is integrable in ζ with all its z derivatives for $|\zeta| \leq k$. Thus the inverse Fourier transform has derivatives in all variables up to order k. Applied to the error term in (3.23) we conclude that

$$E_j = \mathrm{Op}(b_j) \text{ has kernel in } \mathcal{C}^{j-n-1}(\Omega^2) \text{ for large } j. \tag{3.26}$$

Thus as j increases the error terms in (3.23) have increasingly smooth kernels.

Lemma 4. *If $J \in \mathcal{C}^k(\Omega^2)$ is properly supported then the operator with kernel J (also denoted J) is a map*

$$J : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^k_{\mathrm{loc}}(\Omega) \ \forall\ s \geq -k. \tag{3.27}$$

As a result of this, and (3.23), the operator $Q_{(k)} = \sum\limits_{j=0}^{k} Q_j$ comes increasingly close to satisfying the first identity in (3.2), except that the error term is only finitely (but arbitrarily) smoothing. Since this is enough for what we want here I will banish the actual solution of (3.2) to the appended notes.

Lemma 5. *For k sufficiently large to left parametrix $Q_{(k)}$ is a continuous operator on $\mathcal{C}^\infty(\Omega)$ and*

$$Q_{(k)} : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^{s+m}_{\mathrm{loc}}(\Omega) \ \forall\ s \in \mathbb{R}. \tag{3.28}$$

Proof. So far I have been rather cavalier in treating $\mathrm{Op}(a)$ for $a \in S^m(\Omega \times \mathbb{R}^n)$ as an operator without showing that this is really the case, however this is a rather easy exercise in distribution theory. Namely, from the basic properties of the Fourier transform and Sobolev spaces

$$A(z, z-z') \in \mathcal{C}^k(\Omega; H^{-n-1+m-k}_{\mathrm{loc}}(\Omega)) \ \forall\ k \in \mathbb{N}. \tag{3.29}$$

It follows that $\mathrm{Op}(a) : H^{n+1-m+k}_{\mathrm{c}}(\Omega)$ into $\mathcal{C}^k(\Omega)$ and in fact into $\mathcal{C}^k_{\mathrm{c}}(\Omega)$ by the properness of the support. In particular it does define an operator on $\mathcal{C}^\infty(\Omega)$ as we have been pretending and the steps above are easily justified.

A similar argument (which I will not give since it is better to do it by duality, as I will describe in the addenda) shows that for any fixed s

$$A : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^S_{\mathrm{loc}}(\Omega) \tag{3.30}$$

for some S.

Now, if $f \in H^s_{\mathrm{loc}}(\Omega)$ then it may be approximated by a sequence $f_j \in \mathcal{C}^\infty(\Omega)$ in the topology of $H^s_{\mathrm{loc}}(\Omega)$, so $\mu f_j \to \mu f$ in $H^s(\mathbb{R}^n)$ for each $\mu \in \mathcal{C}^\infty_c(\Omega)$. Set $u_j = Q_{(k)} f_j \in \mathcal{C}^\infty(\Omega)$ as we have just seen, where k is fixed but will be chosen to be large. Then from our identity $P(z,D)Q_{(k)} = \mathrm{Id} + R_{(k)}$ it follows that

$$P(z,D)u_j = f_j + g_j,\ g_j = R_{(k)} f_j \to R_{(k)} f \in H^N_{\mathrm{loc}}(\Omega) \tag{3.31}$$

for k large enough depending on s. Thus the right side converges in $H^s_{\mathrm{loc}}(\Omega)$ and by (3.30), $u_j \to u$ in some $H^S_{\mathrm{loc}}(\Omega)$. But now we can use the *a priori* estimates (2.8) on $u_j \in \mathcal{C}^\infty(\Omega)$ to conclude that

$$\|\psi u_j\|_{s+m} \leq C\|\phi(f_j + g_j)\|_s + C''\|\phi u_j\|_S \tag{3.32}$$

to see that ψu_j is bounded in $H^{s+m}(\mathbb{R}^n)$ for any $\psi \in \mathcal{C}^\infty_c(\Omega)$. In fact, applied to the difference $u_j - u_l$ it shows the sequence to be Cauchy. Hence in fact $u \in H^{s+m}_{\text{loc}}(\Omega)$ and the estimates (2.8) hold for this u. That is, $Q_{(k)}$ has the mapping property (3.28) for large k. □

In fact the continuity property (3.28) holds for all $\mathrm{Op}(a)$ where $a \in S^m(\Omega \times \mathbb{R}^n)$, not just those which are parametrices for elliptic differential operators. I will comment on this below.

There is also the question of the second identity in (3.2), at least in the same finite-order-error sense. To solve this we may use the transpose identity. Thus taking formal transposes this second identity should be equivalent to

$$P^t Q^t = \mathrm{Id} - R^t_L. \tag{3.33}$$

The transpose of $P(z, D)$ is the differential operotor

$$P^t(z, D) = \sum_{|\alpha| \le m} (-D)^\alpha_z p_\alpha(z). \tag{3.34}$$

This is again of order m and after a lot of differenttiation to move the coefficients back to the left we see that its leading part is just $P_m(z, -D)$ where $P_m(z, D)$ is the leading part of $P(z, D)$. Thus it is also elliptic in Ω. Thus to construct a solution to (3.34), up to finite order errors, we need just apply Lemma 3 to the transpose differential operator. This $Q'_{(N)} = \mathrm{Op}(a'_{(N)}$ exists with the property

$$P^t(z, D) Q'_{(N)} = \mathrm{Id} - R'_{(N)} \tag{3.35}$$

where the kernel of $R'_{(N)}$ is in $C^N(\Omega^2)$. Since this property is preserved under transpose we have indeed solved the second identity in (3.2) up to an arbitrarily smooth error.

Of course the claim in Theorem 2 is that the one operator satisfies both identities, whereas we have constructed two operator which each satisfy one of them, up to finite smoothing error terms

$$P(z, D) Q_R = \mathrm{Id} - R_R, \ \ Q_L P(z, D) = \mathrm{Id} - R_L. \tag{3.36}$$

However these operators must themselves be equal up to finite smoothing error terms since composing the first identity on the left with Q_L and the second on the right with Q_R shows that

$$Q_L - Q_L R_R = Q_L P(z, D) Q_R = Q_R - R_L Q_R \tag{3.37}$$

where the associativity of operator composition has been used. We have already checked the mapping property(3.28) for both Q_L and Q_R, assuming the error terms are sufficiently smoothing. It follows that the composite error terms here map $H^{-p}_{\text{loc}}(\Omega)$ into $H^p_{\text{loc}}(\Omega)$ where $p \to \infty$ with k with the same also true of the transposes of these operators. Such an operator has kernel in $C^{p'}(\Omega^2)$ where again $p' \to \infty$ with k. Thus the difference of Q_L and Q_R itself becomes arbitrarily smoothing as $k \to \infty$.

Finally then we have proved most of Theorem 2 except with arbitrarily finitely smoothing errors. In fact we have not quite proved the regularity statement that $P(z, D)u \in H^s_{\text{loc}}(\Omega)$ implies $u \in H^{s+m}_{\text{loc}}(\Omega)$ although we can very close in the proof of Lemma 5. In fact, now that we know that $Q_{(k)}$ is also a right parametrix, i.e. satisfies the second identity in (2.8) up to arbitrarily smoothing errors, this too follows. Namely it also follows from the discussion above that $Q_{(k)}$ is an operator on $\mathcal{C}^{-\infty}(\Omega)$. Then

$$Q_P(k)P(z,D)u = u + v_k,\ \ \psi v_k \in H^{s+m}(\Omega)$$

for large enough k so (3.28) implies $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$ and the *a priori* estimates become real estimates on all solutions.

Addenda to Lecture 3

Asymptotic completeness to show that really can get smoothing errors.
Some discussion of pseudodifferential operators – adjoints, composition and boundedness
Some more reassurance as regards operators, kernels and mapping properties – since I have treated these fairly shabbily!

References

[1] L. Hörmander, *The analysis of linear partial differential operators*, vol. 2, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.