2. Lecture 2

8 February, 2007: Elliptic regularity, continued.

Photographs by Chris Kottke. Photo1, Photo2, Photo3, Photo4, Photo5, Photo6, Photo7, Photo8, Photo9, Photo10, Photo11, Photo12, Photo13, Photo14, Photo15.

Last time we showed that if $P(D)$ is elliptic of order m and $u \in \mathcal{C}^{-\infty}(\Omega)$, for $\Omega \subset \mathbb{R}^n$ open, satisfies $P(D)u \in H^s_{\text{loc}}(\Omega)$ then $u \in H^{m+s}_{\text{loc}}(\mathbb{R}^n)$ and for any ϕ, $\psi \in \mathcal{C}^\infty_c(\Omega)$ with $\phi = 1$ in a neighbourhood of $\text{supp}(\phi)$,

$$\|\psi u\|_{s+m} \le C\|\psi P(D)u\|_s + C'\|\phi u\|_{s+m-1} \tag{2.1}$$

for any $M \in \mathbb{R}$, with C' depending only on ψ, ϕ, M and $P(D)$ and C only on $P(D)$ (so neither depends on u). We proceed to try to do the same thing in the variable coefficient case, so for

$$P(z, D) = \sum_{|\alpha|\le m} p_\alpha(z)D^\alpha,\ p_\alpha \in \mathcal{C}^\infty(\Omega). \tag{2.2}$$

We now assume ellipticity for the polynomial $P(z, \zeta)$ for each $z \in \Omega$. This is the same thing as ellipticity for the principal part, i.e. the condition for each compact subset of Ω

$$|\sum_{|\alpha|=m} p_\alpha(z)\zeta^\alpha| \ge C(K)|\zeta|^m,\ z \in K \Subset \Omega, C(K) > 0. \tag{2.3}$$

Now, we got the estimate (2.1) by iteration from the case $M = s + m - 1$ (by nesting cutoff functions). Pick a point $\bar{z} \in \Omega$. In a small ball around $\bar{z}$ the coefficients are almost constant. In fact by Taylor's theorem

$$P(z, \zeta) = P(\bar{z}, \zeta) + Q(z, \zeta),\ Q(z, \zeta) = \sum_j (z - \bar{z})_j P_j(z, \bar{z}, \zeta) \tag{2.4}$$

where the P_j are also polynomials of degree m in ζ and smooth in z in the ball (and in $\bar{z}$.) We can apply the estimate (2.1) for $P(\bar{z}, D)$ and $s = 0$ to find

$$\|\psi u\|_m \le C\|\psi\left(P(z, D)u - Q(z, D)\right)u\|_0 + C'\|\phi u\|_{m-1}. \tag{2.5}$$

Because the coefficients are small we then find

$$\begin{aligned} \|\psi Q(z, D)u\|_0 &\le \sum_{j,|\alpha|\le m} \|(z - \bar{z})_j r_{j,\alpha} D^\alpha \psi u\|_0 + C'\|\phi u\|_{m-1} \\ &\le \delta C\|\psi u\|_m + C'\|\phi u\|_{m-1}. \end{aligned} \tag{2.6}$$

What we would like to say next is that we can choose δ so small that $\delta C < \frac{1}{2}$ and so inserting (2.6) into (2.5) we would get

$$\begin{aligned} \|\psi u\|_m &\le C\|\psi P(z, D)u\|_0 + C\|\psi Q(z, D)u\|_0 + C'\|\phi u\|_{m-1} \\ &\le C\|\psi P(z, D)u\|_0 + \frac{1}{2}\|\psi u\|_m + C'\|\phi u\|_{m-1} \\ &\Longrightarrow \frac{1}{2}\|\psi u\|_m \le C\|\psi P(z, D)u\|_0 + C'\|\phi u\|_{m-1}. \end{aligned} \tag{2.7}$$

However, there is a problem here. Namely this is an *a priori* estimate – to move the norm term from right to left we need to know that it is *finite*. Really, that is what we are trying to prove! So more work is required. Nevertheless we will eventually get essentially the same estimate as in the constant coefficient case.

Theorem 1. *If $P(z, D)$ is an elliptic differential operator of order m with smooth coefficients in $\Omega \subset \mathbb{R}^n$ and $u \in \mathcal{C}^{-\infty}(\Omega)$ is such that $P(z, D)u \in H^s_{\mathrm{loc}}(\Omega)$ for some $s \in \mathbb{R}$ then $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$ and for any ϕ, $\psi \in \mathcal{C}^\infty_c(\Omega)$ with $\phi = 1$ in a neighbourhood of* $\operatorname{supp}(\psi)$ *and $M \in \mathbb{R}$, there exist constants C (depending only on P and ψ) and C' (independent of u) such that*

$$\|\psi u\|_{m+s} \leq C\|\phi P(z, D)u\|_s + C'\|\phi u\|_M. \tag{2.8}$$

Let me add here to what I did in the lecture to observe how to get the *a priori* estimate first for general s, rather than $s = 0$ and then for general ψ (since up to this point it is only for ψ with sufficiently small support). In the estimates in (2.6) the L^2 norm of a product is estimated by the L^∞ norm of one factor and the L^2 norm of the other. For general Sobolev norms such an estimate does not hold, but something similar does.

Lemma 1. *If $u \in H^s(\mathbb{R}^n)$ and $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ then*

$$\|\psi u\|_s \leq \|\psi\|_{L^\infty}\|u\|_s + C\|u\|_{s-1} \tag{2.9}$$

where the constant depends on s and ψ but not u.

Proposition 2. *Under the hypotheses of Theorem 1 if in addition $u \in \mathcal{C}^\infty(\Omega)$ then* (2.8) *follows.*

See the addenda for proofs.

At this point let me return to the discussion for the constant coefficient case. I will construct an operator Q_Ω which will turn out to be an inverse, modulo smoothing errors, for $P(D)$ acting functions on Ω. The idea is to set

$$Q_\Omega f(z) = \int_\Omega q(z - z')\chi(z, z')f(z')dz' \tag{2.10}$$

where q is from (1.19). For the moment let us not worry about the precise meaning of the integral, only that it should not be ruled out by support difficulties. For this we want $\chi \in \mathcal{C}^\infty(\Omega^2)$ to have proper support in the following sense:

$$\text{If } K \subset \Omega \text{ then } \pi_R\left((\Omega \times K) \cap \operatorname{supp}(\chi)\right) \cup \pi_L\left((L \times \Omega) \cap \operatorname{supp}(\chi)\right) \Subset \Omega. \tag{2.11}$$

Here π_L, $\pi_R : \Omega^2 \longrightarrow \Omega$ are the two projections, onto left and right factors. This condition means that if we multiply the integral in (2.10) on the left by $\phi(z)$, $\phi \in \mathcal{C}^\infty_c(\Omega)$ then the integrand has compact support in z' as well – and so should exist at least as a distributional pairing. The second property we want of χ is that it should not change the properties of q as a convolution operator too much. This reduces to

$$\chi = 1 \text{ in a neighbourhood of } \operatorname{Diag} = \{(z, z); z \in \Omega\} \subset \Omega^2. \tag{2.12}$$

Before discussing why these conditions help us, let me just check that it is possible to find such a ψ. This follows easily from the existence of a partition of unity in Ω as follows. I claim that it is possible to find functions $\phi_i \in \mathcal{C}^\infty_c(\Omega)$, $i \in \mathbb{N}$, which have locally finite supports (i.e. any compact subset of Ω only meets the supports of a finite number of the ϕ_i,) that $\sum\limits_i \phi_i(z) = 1$ in Ω and also that there exist functions $\phi'_i \in \mathcal{C}^\infty_c(\Omega)$, also with locally finite supports in the same sense and such that $\phi'_i = 1$ on a neighborhood of the support of ϕ_i. I leave the existence of such functions as an exercise.

Accepting that such functions exists, consider

$$\chi(z,z')=\sum_i \phi_i(z)\phi_i'(z'). \tag{2.13}$$

Any compact subset of Ω^2 is contained in a compact set of the form $K\times K$ and hence meets the supports of only a finite number of terms in (2.13). Thus the sum is locally finite and hence $\chi\in\mathcal{C}^\infty(\Omega^2)$. Moreover, its support has the property (2.11). Clearly, by the assumption that $\phi_i'=1$ on the support of ϕ_i and that the latter form a partition of unity, $\psi(z,z)=1$. In fact $\chi(z,z')=1$ in a neighborhood of the diagonal since each z has a neighborhood N such that $z'\in N$, $\psi_i(z)\neq 0$ implies $\psi_i'(z')=1$. Thus we have shown that such a cutoff function ψ exists.

Now, why do we want (2.12)? This is important because

$$\operatorname{sing\,supp}(q)\subset\{0\} \tag{2.14}$$

as follows from (1.14). Indeed these estimates on the Fourier transform show that

$$z^\alpha q(z)\in C^N(\mathbb{R}^n) \text{ if } |\alpha|>n+N \tag{2.15}$$

since this is enough to show that the Fourier transform is L^1. At every point of $\mathbb{R}^n$ other than 0 one of the z_j is non-zero and so, taking $z^\alpha=z_j^k$, (2.15) shows that $q(z)$ is in C^N in $\mathbb{R}^n\setminus\{0\}$ for all N, i.e. (2.14) holds.

Thus the distribution $q(z-z')$ in (2.10) is only singular at the diagonal. It follows that different choices of χ with the properties listed above lead to kernels in (2.10) which differ by smooth functions in Ω^2 with proper supports.

Definition 2. An properly supported smoothing operator, which is by defninition given by an integral operator

$$Ef(z)=\int_\Omega E(z,z')f(z')dz' \tag{2.16}$$

where $E\in\mathcal{C}^\infty(\Omega^2)$ has proper support (so both maps

$$\pi_L,\ \pi_R:\operatorname{supp}(E)\longrightarrow\Omega \tag{2.17}$$

are proper) gives continuous operators

$$E:\mathcal{C}^{-\infty}(\Omega)\longrightarrow\mathcal{C}^\infty(\Omega),\ CmIc(\Omega)\longrightarrow\mathcal{C}_c^\infty(\Omega) \tag{2.18}$$

and has an adjoint of the same type.

Proposition 3. *If $P(D)$ is an elliptic operator with constant coefficients then the kernel in* (2.10) *defines an operator $Q_\Omega:\mathcal{C}^{-\infty}(\Omega)\longrightarrow\mathcal{C}^{-\infty}(\Omega)$ which maps $H^s_{\mathrm{loc}}(\Omega)$ to $H^{s+m}_{\mathrm{loc}}(\Omega)$ for each $s\in\mathbb{R}$ and gives a 2-sided parametrix for $P(D)$ in Ω :*

$$P(D)Q_\Omega=\operatorname{Id}-R,\ Q_\Omega P(D)=\operatorname{Id}-R' \tag{2.19}$$

where R and R' are smoothing operators.

Proof. Since we have already seen that changing χ in (2.10) changes Q_Ω by a smoothing operator – and that such a change will just change R and R' in (2.19), we can use the explicit choice for χ made above in terms of a partition unity. Thus, multiplying on the left by some $\mu\in\mathcal{C}_c^\infty(\Omega)$ the sum becomes finite and

$$\mu Q_\Omega f=\sum_j \mu\psi_j q*(\psi_j' f). \tag{2.20}$$

It follows that Q_Ω acts on $\mathcal{C}^{-\infty}(\Omega)$. To check (2.19) we may apply $P(D)$ to (2.20) and consider a region where $\mu = 1$. Since $P(D)q* = \mathrm{Id} - \tilde{R}*$ where $\tilde{R} \in \mathcal{S}(\mathbb{R}^n)$, $P(D)Q_\Omega f = \mathrm{Id} - R$ where additional 'error terms' arise from any differentiation of ψ_j, but all such terms have smooth kernels (since $\psi'_j = 1$ on the support of ψ_j and $q(z - z')$ is smooth outside the diagonal. The second identity in (2.19) comes from the same computation for the adjoint of $P(D)$ and E_Ω. □

Addenda to Lecture 2

Proof of Lemma 1. This is really a standard estimate for Sobolev spaces. Recall that the Sobolev norm is related to the L^2 norm by

$$\|u\|_s = \|\langle D\rangle^s u\|_{L^2}. \tag{2.21}$$

Here $\langle D\rangle^s$ is the convolution operator with kernel defined by its Fourier transform

$$\langle D\rangle^s u = R_s * u,\ \widehat{R_s}(\zeta) = (1 + |\zeta)^2)^{\frac{s}{2}}. \tag{2.22}$$

To get (2.9) use the following standard bound on the commutator.

Lemma 2. *If $\psi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$M_s = [\psi, R_s*] : H^t(\mathbb{R}^n) \longrightarrow H^{t-s+1}(\mathbb{R}^n) \tag{2.23}$$

is bounded for each t.

Proof. Since the Sobolev spaces are defined in terms of the Fourier transform, first conjugate and observe that (2.23) is equivalent to the boundeness of the integral operator with kernel

$$K_{s,t}(\zeta,\zeta') = (1+|\zeta|^2)^{\frac{t-s+1}{2}}\hat{\psi}(\zeta-\zeta')\left((1+|\zeta'|^2)^{\frac{s}{2}} - (1+|\zeta|^2)^{\frac{s}{2}}\right)(1+|\zeta'|^2)^{-\frac{t}{2}} \tag{2.24}$$

on $L^2(\mathbb{R}^n)$. If we insert the characteristic function for the region near the diagonal

$$|\zeta - \zeta'| \le \frac{1}{4}(|\zeta| + |\zeta'|) \Longrightarrow |\zeta| \le 2|\zeta'|,\ |\zeta'| \le 2|\zeta| \tag{2.25}$$

then $|\zeta|$ and $|\zeta'|$ are of comparable size. Using Taylor's formula

$$\begin{aligned}(1+|\zeta'|^2)^{\frac{s}{2}} - (1+|\zeta|^2)^{\frac{s}{2}} &= s(\zeta-\zeta')\cdot\int_0^1 (t\zeta + (1-t\zeta')\left(1 + |t\zeta + (1-t)\zeta'|^2\right)^{\frac{s}{2}-1} dt \\ &\Longrightarrow \left|(1+|\zeta'|^2)^{\frac{s}{2}} - (1+|\zeta|^2)^{\frac{s}{2}}\right| \le C_s|\zeta-\zeta'|(1+|\zeta|)^{s-1}.\end{aligned} \tag{2.26}$$

It follows that in the region (2.25) the the kernel in (2.24) is bounded by

$$C|\zeta - \zeta'||\hat{\psi}(\zeta - \zeta')|. \tag{2.27}$$

In the complement to (2.25) the kernel is rapidly decreasing in ζ and ζ' in view of the rapid decrease of $\hat{\psi}$. Both terms give bounded operators on L^2, in the first case using the same estimates that show convolution by an element of $\mathcal{S}$ to be bounded. □

From (2.23), (writing 0 for the L^2 norm)

$$\begin{aligned}\|\psi u\|_s = \|R_s * (\psi u)\|_0 &\le \|\psi(R_s * u)\|_0 + \|M_s u\|_0 \\ &\le \|\psi\|_{L^\infty}\|R_s u\|_0 + \|u\|_{s-1} \le \|\psi\|_{L^\infty}\|u\|_s + \|u\|_{s-1}.\end{aligned} \tag{2.28}$$

This completes the proof of (2.9) and so of Lemma 1. □

Proof of Proposition 2. First we can generalize (2.5), now using Lemma 1. Thus, if ψ has support near the point $\bar{z}$

$$\begin{aligned}(2.29)\quad \|\psi u\|_{s+m} &\le C\|\psi P(\bar{z},D)u\|_s + \|\phi Q(z,D)\psi u\|_s + C'\|\phi u\|_{s+m-1} \\ &\le C\|\psi P(\bar{z},D)u\|_s + \delta C\|\psi uu\|_{s+m} + C'\|\phi u\|_{s+m-1}.\end{aligned}$$

This gives the extension of (2.7) to general s (where now we are assuming that u is indeed smooth:

$$(2.30)\qquad \|\psi u\|_{s+m} \le C_s\|\psi P(z,D)u\|_s + C'\|\phi u\|_{s+m-1}.$$

Now, given a general element $\psi \in \mathcal{C}_c^\infty(\Omega)$ and $\phi \in \mathcal{C}_c^\infty(\Omega)$ with $\phi = 1$ in a neighbourhood of $\operatorname{supp}(\psi)$ we may choose a partition of unity psi_j for $\operatorname{supp}(\psi)$ for each element of which (2.30) holds for some $\phi_j \in \mathcal{C}_c^\infty(\Omega)$ where in addition $\phi = 1$ in a neighbourhood of $\operatorname{supp}(\phi_j)$. Then, with various constants

$$\begin{aligned}(2.31)\quad \|\psi u\|_{s+m} \le \sum_j \|\psi_j\|_{s+m} &\le C_s\sum_j \|\psi_j\phi P(z,D)u\|_s + C'\sum_j \|\phi_j\phi u\|_{s+m-1} \\ &\le C_s(K)\|\phi P(z,D)u\|_s + C''\|\phi u\|_{s+m-1},\end{aligned}$$

where K is the support of ψ and Lemma 1 has been used again. This removes the restriction on supports.

Now, to get the full (a priori) estimate (2.8), where the error term on the right has been replaced by one with arbitrarily negative Sobolev order, it is only necessary to iterate (2.31) on a nested sequence of cutoff functions as we did earlier in the constant coefficient case.

This completes the proof of Proposition 2. □

References

[1] L. Hörmander, *The analysis of linear partial differential operators*, vol. 2, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.