1. Lecture 1

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What do I want to do:-

(1) Variable coefficient, elliptic PDE.
(2) Monopoles
(3) Something else if time permits

I have pretty much decided not to use one book but to take bits and pieces from several. For one thing there does not seem to be an approptriate treatment of monopoles. The elliptic part of the course is intended to serve as an introduction to many parts of geometic pde and differential geometry. The part on monopoles is basically an extended application of the first part.

First let me recall some things from 18.155 – or at least try to stir some memory of what it is about. Namely, using the Fourier transform we can say a lot about *constant coefficient linear* differential operators. In this course we want to develop methods to handle variable coefficient, non-linear and (because of gauge-invariance) non-elliptic operators. However in all three directions of generalization we will still ultimaltely use constant coefficient elliptic methods as a starting point.

A linear, constant coefficient differential operator can be thought of as a map

$$P : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ of the form } Pu(z) = \sum_{|\alpha|\le m} c_\alpha(z) D^\alpha u(z),$$
$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n},\ D_j = \frac{1}{i}\frac{\partial}{\partial z_j}, \tag{1.1}$$

but it also acts on various other spaces. So, really it is just a polynomial $P(\zeta)$ in n variables. This 'characteristic polynomial' has the property that

$$\mathcal{F}(P(D)u)(\zeta) = P(\zeta)\mathcal{F}u(\zeta) \tag{1.2}$$

and this is why the Fourier tranform is especially useful. Still it does not solve the important questions directly.

Question 1. $P(D)$ is always injective as a map (1.1) but usually not surjective. When is it surjective? If $\Omega \subset \mathbb{R}^n$ is a non-empty open set then

$$P(D) : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega) \tag{1.3}$$

is never injective, for which polynomials is it surjective?

The fist three points are relatively easy. As a map (1.1) $P(D)$ is injective since if $P(D)u = 0$ then by (1.2), $P(\zeta)\mathcal{F}u(\zeta) = 0$ on $\mathbb{R}^n$. However, a non-trivial polynomial cannot vanish on an open set, i.e. the set where it is non-zero is dense, so $\mathcal{F}u(\zeta) = 0$ on $\mathbb{R}^n$ (by continuity) and hence $u = 0$ by the invertibility of the Fourier transform. So (1.1) is injective (of course excepting the case that P is the zero polynomial). When is it surjective? That is, when can every $f \in \mathcal{S}(\mathbb{R}^n)$ be written as $P(D)u$ with $u \in \mathcal{S}(\mathbb{R}^n)$? Taking the Fourier transform again, this is the same as asking when every $g \in \mathcal{S}(\mathbb{R}^n)$ can be written in the form $P(\zeta)v(\zeta)$ with $v \in \mathcal{S}(\mathbb{R}^n)$. If $P(\zeta)$ has a zero in $\mathbb{R}^n$ then this is not possible, since $P(\zeta)v(\zeta)$ always vanishes at such a point. It is a little trickier to see the converse, that $P(\zeta) \neq 0$ on $\mathbb{R}^n$ implies that $P(D)$ in (1.1) is surjective. Why is this not obvious? Because we need to show that $v(\zeta) = g(\zeta)/P(\zeta) \in \mathcal{S}(\mathbb{R}^n)$ whenever $g \in \mathcal{S}(\mathbb{R}^n)$. Certainly, $v \in \mathcal{C}^\infty(\mathbb{R}^n)$ but we need to show that the derivatives decay rapidly at infinity. To do this we really need to show that

$$P(\zeta) \neq 0 \text{ on } \mathbb{R}^n \Longrightarrow C|P(\zeta)| \geq (1+|\zeta|)^a \tag{1.4}$$

for some C and $a \in \mathbb{R}$ (possibly negative) This is true! Thirdly the non-injectivity in (1.3) is obvious for the opposite reason. Namely for any polynomial there exists $\zeta \in \mathbb{C}^n$ such that $P(\zeta) = 0$. Since

$$P(D)e^{i\zeta\cdot z} = P(\zeta)e^{i\zeta\cdot z} \tag{1.5}$$

such a zero gives rise to a non-trivial element of the null space of (1.3). You can find an extensive discussion of the density of these sort of 'exponential' solutions (with polynomial factors) in all solutions in Hörmander's book [1].

What about the surjectivity of (1.3)? It is not always surjective unless Ω is *convex* but there are decent answers, to find them you should look under P-convexity in [1]. If $P(\zeta)$ is elliptic then (1.3) is surjective for every open Ω.

To discuss elliptic regularity, let me recall that Sobolev spaces were extensively discussed in 18.155. The basic global Sobolev spaces are defined in terms of the Fourier transform

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); (1+|\zeta|^2)^{-\frac{s}{2}}\mathcal{F}u(\zeta) \in L^2(\mathbb{R}^n)\}. \tag{1.6}$$

It turns out that these spaces are not really very useful as global spaces on $\mathbb{R}^\cdot$ so we will later look at modified versions of them. Even this definition depends on knowing a few things. First that $\mathcal{F}$, the Fourier transform, acts as an isomorphism on $\mathcal{S}'(\mathbb{R}^n)$, the dual space of $\mathcal{S}(\mathbb{R}^n)$. Secondly that multiplication by $(1+|\zeta|^2)^{\frac{t}{2}}$ on $\mathcal{S}'(\mathbb{R}^n)$ is defined for any $t \in \mathbb{R}$ and thirdly that $L^2(\mathbb{R}^n)$ is embedded as a linear subspace of $\mathcal{S}'(\mathbb{R}^n)$. I invite you to remember these things – if they need clarification now would be a good time to ask. With these caveats, (1.6) is a good defintion of the Sobolev spaces and any constant coefficient differential operator of order m defines a continuous linear map

$$P(D) : H^{s+m}(\mathbb{R}^n) \longmapsto H^s(\mathbb{R}^n). \tag{1.7}$$

This map is *always* injective (of course I assume that P is not the zero polynomial). Why? It is seldom surjective. Recall that P is said to be elliptic (either as a polynomial or as a differential operator) if it is of order m and there is a constant $c > 0$ such that

$$|P(\zeta)| \geq c(1+|\zeta|)^m \text{ in } \{\zeta \in \mathbb{R}^n; |\zeta| > 1/c\}. \tag{1.8}$$

Exercise 1. Show that $P(D)$ is surjective as a map (1.7) if and only if P is elliptic and $P(\zeta) \neq 0$ on $\mathbb{R}^n$.

So, this is not frequently the case. In fact one of the questions I want to get to early on in this course – even though we are interested in variable coefficient operators – is improving (1.7) to get an isomorphism at least for homogeneous elliptic operators. One reason for this is that we need it for monopoles.

Let me recall *elliptic regularity* for constant coefficient operators. Since this is a local issue, I first want to define local versions of the Sobolev spaces. Maybe you already saw all this.

Definition 1. If $\Omega \subset \mathbb{R}^n$ is an open set then

$$H^s_{\text{loc}}(\Omega) = \left\{u \in \mathcal{C}^{-\infty}(\Omega); \phi u \in H^s(\mathbb{R}^n) \ \forall \ \phi \in \mathcal{C}^\infty_c(\Omega)\right\}. \tag{1.9}$$

Again you need to know what $\mathcal{C}^{-\infty}(\Omega)$ is (it is the dual of $\mathcal{C}^{\infty}_c(\Omega)$) and that multiplication by $\phi \in \mathcal{C}^{\infty}_c(\Omega)$ defines a linear continuous map from $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ to $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and gives a bounded operator on $H^m(\mathbb{R}^n)$ for all m.

Proposition 1. *If $P(D)$ is elliptic, $u \in \mathcal{C}^{-\infty}(\Omega)$ is a distribution on an open set and $P(D)u \in H^s_{\mathrm{loc}}(\Omega)$ then $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$.*

Let me discuss this in two slightly different ways. The first, older, approach is direct regularity estimates. The second is via the use of a parametrix.

First the regularity estimates. An easy case of Proposition 1 is if $u \in \mathcal{C}^{-\infty}_c(\Omega)$ has compact support to start with. Then $P(D)u$ also has compact support so in this case

$$u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n) \text{ and } P(D)u \in H^s(\mathbb{R}^n). \tag{1.10}$$

Then of course the Fourier transform works like a charm. Namely

$$(1+|\zeta|^2)^{\frac{s}{2}} P(\zeta)\hat{u}(\zeta) \in L^2 \Longrightarrow (1+|\zeta|^2)^{\frac{s+m}{2}} F(\zeta)\hat{u}(\zeta),\ F(\zeta) = (1+|\zeta|^2)^{-\frac{m}{2}} P(\zeta). \tag{1.11}$$

Ellipticity implies that $F(\zeta)$ is bounded above and below on $|\zeta| > 1/c$ and hence can be inverted there by a bounded function. Since $\hat{u}$ is smooth everywhere it follows from (1.11) that $(1+|\zeta|^2)^{\frac{s+m}{2}}\hat{u} \in L^2(\mathbb{R}^n)$ and hence that $u \in H^{s+m}(\mathbb{R}^n)$ as expected.

To do the general case of an open set we need to use cutoffs. We want to show that $\psi u \in H^{s+m}(\mathbb{R}^n)$ where $\psi \in \mathcal{C}^{\infty}_c(\Omega)$ is some fixed but arbitrary element. We can always choose some function $\phi \in \mathcal{C}^{\infty}_c(\Omega)$ which is equal to one in a neighbourhood of the support of ψ. Then $\phi u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ so $\phi u \in H^t(\mathbb{R}^n)$ for some (unknown) $t \in \mathbb{R}$. We will show that $\psi u \in H^T(\mathbb{R}^n)$ where T is the smaller of $s+m$ and $t+1$. To see this, compute

$$P(D)(\psi u) = \psi P(D)u + \sum_{|\beta|\le m-1, |\gamma|\ge 1} c_{\beta,\gamma} D^{\gamma}\psi D^{\beta}\phi u. \tag{1.12}$$

With the final ϕu replaced by u this is just the effect of expanding out the derivatives on the product. Namely, the $\psi P(D)u$ term is when no derivative hits ψ and the other terms come from at least one derivative hitting ψ. Since $\phi = 1$ on the support of ψ we may then insert ϕ without changing the result. This the first term on the right in (1.12) is in $H^s(\mathbb{R}^n)$ and all terms in the sum are in $H^{t-m+1}(\mathbb{R}^n)$. Applying the simple case discussed above it follows that $\psi u \in H^r(\mathbb{R}^n)$ with r the minimum of $s+m$ and $t+1$. This result holds for any ψ with support in the interior of the set where $\phi = 1$. We can insert a chain of functions, of any finite length k, each contained in the interior of the set where the next is equal to 1, with ϕ having the largest support and ψ the smallest. The argument above then shows that r is the minimum of $t+k$ and $s+m$ for any k, proving the desired regularity.

The second method is actually quite similar, but we avoid the iteration technique, by doing it all at once. Namely, going back to the easy case of a function on $\mathbb{R}^n$ let's give the map a name:-

$$Q(D) : f \in \mathcal{S}'(\mathbb{R}^n) \longmapsto \mathcal{F}^{-1}\left(\frac{1-\chi(\zeta)}{P(\zeta)}\hat{f}(\zeta)\right) \in \mathcal{S}'(\mathbb{R}^n). \tag{1.13}$$

Here $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is chosen to be equal to one on the set $|\zeta| \le \frac{1}{c} + 1$ corresponding to the ellipticity estimate (1.8). Thus $\frac{1-\chi(\zeta)}{P(\zeta)} \in \mathcal{C}^\infty(\mathbb{R}^n)$ is bounded and in fact

$$|D_\zeta^\alpha \frac{1-\chi(\zeta)}{P(\zeta)}| \le C_\alpha (1+|\zeta|)^{-m-|\alpha|} \ \forall \ \alpha. \tag{1.14}$$

This has a straightforward proof by induction. Namely, these estimates are trivial on any compact set, where the function is smooth, so we need only consider the region where $\chi(\zeta) = 1$. The inductive statement is that for polynomials H_α,

$$D_\zeta^\alpha \frac{1}{P(\zeta)} = \frac{H_\alpha(\zeta)}{(P(\zeta))^{|\alpha|+1}}, \ \deg(H_\alpha) \le (m-1)|\alpha|. \tag{1.15}$$

From this (1.14) follows.

So

$$Q(D) : H^s(\mathbb{R}^n) \longrightarrow H^{s+m}(\mathbb{R}^n) \tag{1.16}$$

is continuous for each s and it is also an essential inverse of $P(D)$ in the sense that as operators on $\mathcal{S}'(\mathbb{R}^n)$

$$Q(D)P(D) = P(D)Q(D) = \mathrm{Id} - E, \ E : H^s(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \ \forall \ s \in \mathbb{R}. \tag{1.17}$$

So, in the global case of $\mathbb{R}^n$, we get elliptic regularity by applying $Q(D)$ to both sides of the eqution $P(D)u = f$ to find

$$f \in H^s(\mathbb{R}^n) \Longrightarrow u = Eu + Qf \in H^{s+m}(\mathbb{R}^n). \tag{1.18}$$

The idea then, is to do the same thing for $P(D)$ acting on functions on the open set Ω and then for a variable coefficient operator. The problem of course is that $Q(D)$ does not act on functions (or distributions) defined just on Ω, then need to be defined on the whole of $\mathbb{R}^n$ and to be tempered. Now, $Q(D)$ is a convolution operator. Namely

$$Qf = q * f, \ q \in \mathcal{S}'(\mathbb{R}^n), \ \hat{q} = \frac{1-\chi(\zeta)}{P(\zeta)}. \tag{1.19}$$

This in fact is exactly what (1.13) means, since

$$\mathcal{F}(q * f) = \hat{q}\hat{f}. \tag{1.20}$$

We can write out convolution by a smooth function (which q is not, but let's not worry about that yet) as an integral

$$q * f(\zeta) = \int_{\mathbb{R}^n} q(z-z')f(z')dz'. \tag{1.21}$$

Restating the problem, (1.21) is an integral (really a distributional pairing) over the whole of $\mathbb{R}^n$ not the subset Ω. In essence the cutoff argument above inserts a cutoff ϕ in front of f (really of course in front of u but not to worry). So, let's think about inserting a cutoff into (1.21), replacing it by

$$Qf(\zeta) = \int_{\mathbb{R}^n} q(z-z')\psi(z,z')f(z')dz'. \tag{1.22}$$

Here we will take $\psi \in \mathcal{C}^\infty(\Omega^2)$. To get the integrand to have compact support in Ω for each $z \in \Omega$ we want to arrange that the projection onto the second variable, z'

$$\Omega \times \Omega \supset \mathrm{supp}(\psi) \longrightarrow \Omega \tag{1.23}$$

should be proper – the inverse image of a compact subset $K \subset \Omega$, namely $(\Omega \times K)\,\mathrm{supp}(\psi)$ should be compact in Ω^2.

References

[1] L. Hörmander, *The analysis of linear partial differential operators*, vol. 2, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.