PROBLEM SET 8, 18.155
DUE 18 NOVEMBER, 2016

One thing that I have not been able to describe is the *wavefront set* of a distribution, so I ask you to assimilate the definition and deduce some basic properties. This notion involves cones in $\mathbb{R}^n \setminus \{0\}$ so let me define 'the open cone of aperture $\epsilon > 0$ around a point' to be

$$\Gamma(\bar{\xi}, \epsilon) = \left\{ \xi \in \mathbb{R}^n \setminus \{0\}; |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}| < \epsilon \right\}. \tag{1}$$

Make sure you see that this is just a ball around the point in the sphere $\bar{\xi}/|\bar{\xi}| \in \mathbb{S}^{n-1}$ extended radially.

If $u \in \mathcal{C}^{-\infty}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, the wave front set of u is the subset

$$\mathrm{WF}(u) \subset \Omega \times (\mathbb{R}^n \setminus \{0\}) \tag{2}$$

defined in terms of its complement

$$\begin{aligned} &\Omega \times (\mathbb{R}^n \setminus \{0\}) \ni (\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u) \Longleftrightarrow \\ &\qquad \exists\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega),\ \phi(\bar{x}) \neq 0 \text{ and } \epsilon > 0 \text{ such that} \\ &\qquad\qquad \sup_{\Gamma} |\xi|^N |\mathcal{F}(\phi u)(\xi)| < \infty\ \forall\ N,\ \Gamma = \Gamma(\bar{\xi}, \epsilon). \end{aligned} \tag{3}$$

The idea is that the wavefront set gives information about the (co-)direction of singularities, not just their position.

Q1. For $u \in \mathcal{C}^{-\infty}(\Omega)$ show that

(a) $\mathrm{WF}(u) \subset \Omega \times (\mathbb{R}^n \setminus \{0\})$ is closed (as a subset of course)

(b) $\mathrm{WF}(u)$ is 'conic' i.e.

$$(x, \xi) \in \mathrm{WF}(u) \Longrightarrow (x, t\xi) \in \mathrm{WF}(u),\ (x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\}),\ t > 0. \tag{4}$$

(c)

$$\mathrm{WF}(u) \subset \mathrm{singsupp}(u) \times (\mathbb{R}^n \setminus \{0\}). \tag{5}$$

Q2. Given $\bar{\xi} \in \mathbb{R}^n \setminus \{0\}$ and $\epsilon_1 > \epsilon_2 > 0$ construct a(n almost) conic cut-off $0 \leq \psi \in S^0(\mathbb{R}^n)$ (the symbol space) such that

$$\operatorname{supp} \psi \subset \Gamma(\bar{\xi}, \epsilon_1),\ \psi = 1 \text{ on } \Gamma(\bar{\xi}, \epsilon_2) \cap \{|\xi| > 2\}. \tag{6}$$

Show that $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$ is equivalent to

$$\psi \mathcal{F}(\phi u) \in \mathcal{S}(\mathbb{R}^n) \Longleftrightarrow b_\psi * (\phi u) \in \mathcal{S}(\mathbb{R}^n),\ \hat{b}_\psi = \psi, \tag{7}$$

for some $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$, $\phi(\bar{x}) \neq 0$, $\epsilon_1 > \epsilon_2 > 0$.

Hint:- One way is easy here. The other way the issue is that the definition of $\mathrm{WF}(u)$ only gives directly the condition that $b_\psi * \phi u \in H^\infty(\mathbb{R}^n)$ (the intersection of the Sobolev spaces). You should recall that b_ψ is the sum of a compactly supported distribution and an element of $\mathcal{S}(\mathbb{R}^n)$.

Q3. Now show that $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$ implies that for some $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$, $\phi(\bar{x}) \neq 0$ and some cone $\Gamma(\bar{x}, \epsilon)$, $\epsilon > 0$

$$b * (\phi u) \in \mathcal{S}(\mathbb{R}^n)\ \forall\ \hat{b} \in S^m(\mathbb{R}^n),\ \operatorname{supp}(\hat{b}) \subset \Gamma(\bar{\xi}, \epsilon). \tag{8}$$

Hint: This is not hard.

Q4. (a) Recall (you do not have to prove this, I did it in class and it should be in the notes by now – see L16) that if $b \in S^m(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ then there exist $\phi_\alpha \in \mathcal{S}(\mathbb{R}^n)$ and $b_\alpha \in S^{m-j}$ such that given k there exists $N = N_k$ such that the operator

$$E_N : u \longmapsto b * (\phi u) - \sum_{|\alpha| \leq N} \phi_j (b_j * u) \tag{9}$$

has Schwartz kernel in $\mathcal{C}^k(\mathbb{R}^{2n})$.

(b) Conclude that if (8) holds then for any $\mu \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$

$$b * (\mu\phi u) \in \mathcal{S}(\mathbb{R}^n)\ \forall\ \hat{b} \in S^m(\mathbb{R}^n),\ \operatorname{supp}(\hat{b}) \subset \Gamma(\bar{\xi}, \epsilon).$$

Hint: A kernel in $\mathcal{C}^k(\mathbb{R}^{2n})$ defines a map from $H^{-k}_{\mathrm{c}}(\mathbb{R}^n)$ to $H^k_{\mathrm{loc}}(\mathbb{R}^n)$ so as k increases this becomes 'increasingly a smoothing operator'. If you know something about the support properties as well (from its definition) you get more.

(c) Hence deduce that $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$ is equivalent to the apparently stronger statement that for some $\epsilon > 0$

$$b * (\phi u) \in \mathcal{S}(\mathbb{R}^n)\ \forall\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega),\ \operatorname{supp}\phi \subset B(\bar{x}, \epsilon),\quad \hat{b} \in S^m(\mathbb{R}^n),\ \operatorname{supp}(\hat{b}) \subset \Gamma(\bar{\xi}, \epsilon). \tag{10}$$

Q5. Show that for any $u \in \mathcal{C}^{-\infty}(\Omega)$ the wavefront set is a refinement of the singular support in the sense that

$$\pi(\mathrm{WF}(u)) = \operatorname{singsupp}(u),\ \pi(x, \xi) = x \tag{11}$$

Q6-Opt. Show the 'microellipticity of elliptic operators': If $P(x, D) = \sum\limits_{|\alpha| \leq m} p_\alpha(x) D^\alpha$ has coefficients $p_\alpha \in \mathcal{C}^\infty(\Omega)$ and is elliptic in Ω then

$$\mathrm{WF}(P(x, D)u) = \mathrm{WF}(u)\ \forall\ u \in \mathcal{C}^{-\infty}(\Omega). \tag{12}$$

Q7-opt. Show that if $u, v \in \mathcal{C}^{-\infty}(\Omega)$ and there is no point $(x, \xi) \in \mathrm{WF}(u)$ such that $(x, -\xi) \in \mathrm{WF}(v)$ then it is possible to define the product $uv \in \mathcal{C}^{-\infty}(\Omega)$ consistently with multiplcation when one element is smooth.

Hint: First think about the corresponding result for singular supports, which is just that $\mathrm{singsupp}(u) \cap \mathrm{singsupp}(v)$ allows you to define uv and try to do something similar.