PROBLEM SET 4, 18.155
DUE 14 OCTOBER, 2016

Q1. Show that if $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ there exists a sequence $u_n \in \mathcal{C}_c^{\infty}(\mathbb{R}^n)$ such that $u_n \to u$ weakly, which just means that

$$u_n(f) \longrightarrow u(f) \;\forall\; f \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{1}$$

Hint: You only have to 'put things together' to get this. We know that there is a sequence $\phi_n \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\phi_n * f \to f$ in $\mathcal{S}(\mathbb{R}^n)$ for any $f \in \mathcal{S}(\mathbb{R}^n)$ where the support of ϕ_n approaches 0. We defined $u_n = \phi_n * u$ (even when u does not have compact support) by $(\phi_n * u)(g) = u(\check{\phi}_n * g))$ where $\check{\phi}_n(x) = \phi_n(-x)$ (you can actually choose ϕ_n to be even). Then $\check{\phi}_n * g \to g$ uniformly with all derivatives on any compact set and this is enough to prove (1).

Q2. Using this, or otherwise, show that the Fourier transform of a compactly supported distribution is a smooth function of slow growth (it is actually an entire function on $\mathbb{C}^n$ with certain exponential bounds as we shall see later).

Hint: The weak density above allows you so see that the Fourier transform of $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ is the function $u(\chi \exp(-i\xi \cdot -))$ where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ has $\chi = 1$ in a neighbourhood of $\operatorname{supp}(u)$. Check that this makes sense and is true when $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and so holds in general. Then you need to think about the resulting function of ξ.

Q3. Show that if $u_n \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $u_n \to u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ in the sense of Q1 above then there is a fixed compact set K containing $\operatorname{supp}(u_n)$ for all n.

Hint: Show that if the conclusion did not hold then there would exist a subsequence n_k and a sequence $x_k \in \mathbb{R}^n$ with $|x_k| \to \infty$, $x_k \notin \bigcup_{j<k} \operatorname{supp}(u_{n_j})$ and such that $u_{n_k}(x_k) \neq 0$. Construct a sequence $\psi_k \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ supported very near x_k such that $\int u_{n_k} \sum\limits_{j \le k} \psi_j = k$, $u_{n_k}(\psi_j) = 0$ if $j > k$ and such that $\sum\limits_j \psi_j = v$ converges in $\mathcal{C}^\infty(\mathbb{R}^n)$. So conclude that $u_n(v)$ does not converge contrary to the assumed weak convergence.

Q4. Show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $P(D)u \in \mathcal{S}(\mathbb{R}^n)$ where $P(D)$ is an elliptic operator, then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a function of slow growth.

Hint: Take the Fourier transform and see that $P(\xi)\hat{u}(\xi) \in \mathcal{S}(\mathbb{R}^n)$. Use a cut-off near zero to conclude from this that $\hat{u}$ is the sum of a term in $\mathcal{S}(\mathbb{R}^n)$ (supported away from 0) and an element of $\mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n)$; now take the inverse FT and use a result above.

Q5. Assuming that ϕ, $\psi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ have disjoint supports and that $E \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n)$ has $\operatorname{singsupp}(E) \subset \{0\}$ show that

$$\mathcal{C}^{-\infty}(\mathbb{R}^n) \ni u \longmapsto \phi(E * (\psi u)) \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n).$$

Q6. (Optional) Given any (relatively of course) closed subset of an open set $\Omega \subset \mathbb{R}^n$ show that there is a distribution $u \in \mathcal{C}^{-\infty}(\Omega)$ with this as singular support. The usual argument is called 'condensation of singularities' if you want to look it up.

Q7. (Optinal) A differential operator $P(D)$ with constant coefficients is said to be *hypoelliptic* if for every (equivalently any one non-empty) open set Ω

$$\operatorname{singsupp}(P(D)u) = \operatorname{singsupp}(u) \ \forall \ u \in \mathcal{C}^{-\infty}(\Omega).$$

Show that this condition is equivalent to the existence of a function of slow growith v such that $P(\xi)v(\xi) = 1 + e(\xi)$, $e \in \mathcal{S}(\mathbb{R}^n)$. [It is actually equivalent to the existence of any smooth function with this property but that involves more work.]