PROBLEM SET 3, 18.155
DUE SEPTEMBER 30, 2016

In Lecture 6 I will talk about distributions on open sets and supports. I want you to go through the basics of distribution theory 'yet again' to find the distributions of compact support another way. As usual the questions starting at Q6 are mostly for you amusement. The notation L6/7 indicates that I am not sure when I will prove the result you need here, which is that

> The only distributions with support contained in $\{0\}$ are finite sums of derivatives of the delta 'function' at the origin.

Q1 (L6) Suppose $\Omega \subset \mathbb{R}^n$ is open, consider

$$\mathcal{C}^\infty(\Omega) = \{\phi : \Omega \longrightarrow \mathbb{C}; \phi \text{ infinitely differentiable}\}. \tag{1}$$

Take an exhaustion of Ω by an increasing sequence K_j of compact subsets (meaning every compact subset of Ω is contained in one of the K_j) and show that $\mathcal{C}^\infty(\Omega)$ is complete as a metric space where

$$\begin{gathered} d(\phi,\psi) = \sum_k 2^{-k} \frac{\|\phi-\psi\|_{(k)}}{1+\|\phi-\psi\|_{(k)}}, \\ \|\phi\|_{(k)} = \sup_{|\alpha|\le k,\ x\in K_k} |D^\alpha \phi(x)|. \end{gathered} \tag{2}$$

[Note that these are only seminorms, so make sure you understand why the vanishing of the distance between two points implies that they are equal.]

Q2 Conclude that an element of the dual space, denoted $\mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega)$, (this is an apparently weird notation for the continuous linear maps $u : \mathcal{C}^\infty(\Omega) \longrightarrow \mathbb{C}$ but there is a method here) is precisely a linear map such that for some C and k

$$|u(\phi)| \le C\|\phi\|_{(k)}\ \forall\ \phi \in \mathcal{C}^\infty(\Omega). \tag{3}$$

Q3 (L6) Observe that for every $\Omega \subset \mathbb{R}^n$ open there is a restriction map $\big|_\Omega : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\Omega)$ and use this to show that

$$u'(f) = u(f\big|_\Omega)\ \forall\ f \in \mathcal{S}(\mathbb{R}^n) \text{ defines } u' \in \mathcal{S}'(\mathbb{R}^n). \tag{4}$$

and gives an injection identifying $\mathcal{C}_{\mathrm{c}}^{-\infty}(\Omega) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ with the tempered distributions with compact support contained in Ω as defined in L6.

Q4 Show that for any non-trivial polynomial in one variable $P = P(-i\frac{d}{dx})$, defines a surjective operator

$$P : \mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R}) \tag{5}$$

and find the null space of this map.

Hint: Factorize the polynomial and treat each of the factors separately. For the ones with non-real roots use the Fourier transform. I did the case of $\frac{d}{dx}$ in class, the case of a general real root follows from this.

Remark: Use of the Fourier transform shows that you have proved Łojasieciwz division Theorem for $n = 1$.

What Łojasieciwz division Theorem says is that every non-trivial (i.e. not identically zero) polynomail P in n variables, has a tempered inverse, $U \in \mathcal{S}'(\mathbb{R}^n)$ such that $P(\xi)U(\xi) = \delta_0$. Show that to do this it suffices to show there is $F \in \mathcal{S}'(\mathbb{R}^n)$ such that $|P(\xi)|^2 f = 1$ on $\mathbb{R}^n$. As far as I am aware there are three proofs of this and none are easy. There are estimates by Hörmander, simplifying earlier estimates by Łojasieciwz. There is a clever proof by Atiyah using the Hironaka resolution theorem from algebraic geoemtry and there is a proof showing the meromorphy of $|P(\xi)|^z$. Fortunately it is not important for us but you might like to look at one of them.

Q5 (L6/7) Let $\varDelta = D_1^2 + \cdots + D_n^2$ $D_j = -i\partial_j$, be the (positive) Laplacian on $\mathbb{R}^n$. Show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varDelta u = 0$ then u is a polynomial.

Q6 (Optional) Prove a variant of the Schwartz Representation theorem which says that $u \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n)$ can be written in the form

$$u = \sum_{|\alpha|+|\beta|\le k} x^\alpha D_x^\beta u_{\alpha,\beta},\ u_{\alpha,\beta} \in \mathcal{C}_{\mathrm{c}}^0(\mathbb{R}^n). \tag{6}$$

Q7 (Optional) A compact set K is said to be 'regular' if any distribution $u \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n)$ with support in K can be written in the form (6) with the $u_{\alpha,\beta}$ continuous and supported in K. Show that the unit ball is regular.