18.155 LECTURE 9
6 OCTOBER, 2016

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Abstract. Notes before and then after lecture.

Read:
This week: Ellipticity of polynomials and elliptic regularity. This pre-lecture discussion is for both L8 and L9.

Before lecture

- First I want to do a little more localization. For any open set $\Omega \subset \mathbb{R}^n$ we have defined $\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^{-\infty}(\Omega)$. There are many space in between these two, but clearly they need to admit lots of growth near the boundary. We define

$$H^s_{\mathrm{loc}}(\Omega) = \{u \in \mathcal{C}^{-\infty}(\Omega); \phi u \in H^s(\mathbb{R}^n)\ \forall\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)\}. \tag{1}$$

  I add a little discussion on these spaces and where we are going with them for you to check how much you have absorbed about localization and the identifications and results implicit in this definition. If you can't follow this and are brave enough, let me know so I can try to fix things.
  - First recall we *can* multiply distributions on Ω by smooth functions and then $\phi u \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega) = (\mathcal{C}^\infty(\Omega))' \subset \mathcal{S}'(\mathbb{R}^n)$ by 'extension as zero' corresponding by duality to the restriction map $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\Omega)$. So it is meaningful to say $\phi u \in H^s(\mathbb{R}^n)$ and hence the definition makes sense.
  - Note that $H^s_{\mathrm{loc}}(\mathbb{R}^n)$ is *not* contained in $H^s(\mathbb{R}^n)$. (Why?)
  - Then you can think briefly about the topology. We can make $H^s_{\mathrm{loc}}(\Omega)$ into a metric space (it is a Fréchet space) by taking an exhaustion by compact sets K_j of Ω and corresponding elements $\phi_j \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ such that $\phi_j = 1$ in a neighbourhood of K_j. From the requirement that $\phi_j u \in H^s(\mathbb{R}^n)$ for all j it follows that $u \in H^s_{\mathrm{loc}}(\Omega)$? (Why?)
  - Then we can define

$$d(u, v) = \sum_j 2^{-j} \frac{\|\phi_j(u-v)\|_{H^s}}{1 + \|\phi_j(u-v)\|_{H^s}} \tag{2}$$

    in terms of H^s norms. Why is this a metric on $H^s_{\mathrm{loc}}(\Omega)$ and why is the space complete?
  - Then you might like to check that any differential operator of order m with smooth coefficients, $P = \sum\limits_{|\alpha| \le m} a_\alpha(x) D\alpha$, $a_\alpha \in \mathcal{C}^\infty(\Omega)$ defines a continuous linear map $P : H^{s+m}_{\mathrm{loc}}(\Omega) \longrightarrow H^s_{\mathrm{loc}}(\Omega)$ for every s.

  - To orient you a little let me continue. Such a differential operator is said to be elliptic if the homogeneous polynomials depending on $x \in \Omega$ as parameter,

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \tag{3}$$

    do not vanish on $\Omega \times (\mathbb{R}^n_\xi \setminus \{0\})$.
  - Elliptic regularity says that for such an elliptic operator, if $u \in \mathcal{C}^{-\infty}(\Omega)$ and $Pu \in H^s_{\mathrm{loc}}(\Omega)$ then $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$.
  - This week we will prove this for constant coefficient differential operators.
- Another concept I want to introduce now – because it is rather useful and important – is that of a symbol (technically of type 1 for the moment). These form a linear space for each order $m \in \mathbb{R}$, $S^m(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ and are modelled on two case we already understand. Namely polynomials of degree (less than or equal to) m and our 'weight functions' $b = \langle x\rangle^m = (1+|x|^2)^{m/2}$. The latter satisfy estimates

$$|\partial^\alpha_x b(x)| \le C_\alpha \langle x\rangle^{m-|\alpha|} \ \forall\ \alpha. \tag{4}$$

  In words, each derivative lowers the order by one – polynomials do this too of course.

  Really we normally think of symbols as being 'on the Fourier transform side'. Clearly $S^m(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and we can state the important result we want as follows

  Theorem 1. *If $b \in S^m(\mathbb{R}^n)$ then $v = \mathcal{G}b \in \mathcal{S}'(\mathbb{R}^n)$ is such that*

$$\begin{gathered} v* : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^n)\ \forall\ s \in \mathbb{R} \\ \textit{if } \chi \in \mathcal{C}^\infty_c(\mathbb{R}^n),\ \chi = 1 \textit{ near } 0 \textit{ then } (1-\chi)v \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow \operatorname{singsupp} v \subset \{0\}. \end{gathered} \tag{5}$$

- Suppose $P(\xi) = \sum\limits_{|\alpha|\le m} c_\alpha \xi^\alpha$ is a polynomial of degree m then the following conditions are equivalent
  (1) P is elliptic (of order m)

$$P_m(\xi) = \sum_{|\alpha|=m} c_\alpha\xi^\alpha \ne 0 \ \text{ for } \xi \in \mathbb{R}^n. \tag{6}$$

  (2) There exists $c > 0$ such that

$$|P(\xi)| \ge c|\xi|^n \text{ in } |\xi| > 1/c. \tag{7}$$

  (3) There exists $b \in S^{-m}(\mathbb{R}^n)$ such that

$$Pb = 1 - \phi,\ \phi \in \mathcal{C}^\infty_c(\mathbb{R}^n). \tag{8}$$

- Probably not until Thursday. If P is an elliptic polynomial, which by definition is the same as saying $P(D) = \sum\limits_{|\alpha|\le m} c_\alpha D^\alpha$, $D_j = -i\partial_j$, is an elliptic operator, and $u \in \mathcal{C}^{-\infty}(\Omega)$ is such that $P(D)u \in H^s_{\mathrm{loc}}(\Omega)$ then $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$.
- First we will prove that if $P(D)u \in \mathcal{C}^\infty(\Omega)$ then $u \in \mathcal{C}^\infty(\Omega)$ assuming of course that P is elliptic. This follows by taking the b in the last part of the

result immediately above and setting $E = \chi \mathcal{G}b$ where $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is equal to 1 near 0. This gives us a 'convolution parameterix'

$$P(D)E = \delta + \psi,\ \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n). \tag{9}$$

The convolution operator defined by E satisfies

$$\begin{gathered} E* : \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n), \\ \operatorname{singsupp}(E * f) \subset \operatorname{singsupp}(f) \\ P(D)(E * f) = (P(D)) * f = f + \psi * f. \end{gathered} \tag{10}$$

Now apply this to $f_j = P(D)(\phi_j u)$ where $\phi_j \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ and $\phi_j = 1$ in a neighbourhood of K_j for an exhaustion K_j and it follows that

$$\begin{aligned} P(D)(E * (\phi_j u)) &= E * (P(D)(\phi_j u)) = E * f_j \\ &= (P(D)E) * (\phi_j u) = (\delta + \psi) * (\phi_j u) = \phi_j u + \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n). \end{aligned} \tag{11}$$

However, $\operatorname{singsupp}(f_j) \cap K_j = \emptyset$ so $\operatorname{singsupp}(\phi_j u) \cap K_j = \emptyset$ from the second part of (10) but from this, for all j it follows that $u \in \mathcal{C}^\infty(\Omega)$.

- The Sobolev version is similar.

After lecture

I started off talking about the space $H^s_{\mathrm{loc}}(\Omega)$ for an open set $\Omega \subset \mathbb{R}^n$ mostly as a reminder about how to do things. First the definition

$$H^s_{\mathrm{loc}}(\Omega) = \{u \in \mathcal{C}^{-\infty}(\Omega) = (\mathcal{C}^\infty_{\mathrm{c}}(\Omega))'; \phi u \in H^s(\mathbb{R}^n),\ \forall\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)\}. \tag{12}$$

This involves a few steps. First that $\phi u \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega)$ is defined (via the weak definition $\phi u(\psi) = u(\phi\psi)$ which makes sense for all $\psi \in \mathcal{C}^\infty(\Omega)$ because $\phi\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$). Then that $\mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega) \hookrightarrow \mathcal{S}'(\mathbb{R}^m)$ by 'extension as zero outside Ω' which is dual to the restriction map $\big|_\Omega \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\Omega)$ so that $\phi u \in \mathcal{S}'(\mathbb{R}^n)$ makes sense and then that $H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is a well-defined subspace.

Then recall that we can find an exhaustion of Ω by compact sets and a corresponding sequence of cut-offs, $\phi_j \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$, $0 \le \phi_j \le 1$ (which we don't use here) such that for each compact set $K \Subset \Omega$ there exists j such that $\phi_j = 1$ on K (and so by increasing K a little, maybe for a different j, $\phi_j = 1$ in a neighbourhood of K). We can also arrange that $\phi_{j+1} = 1$ in a neighoubrhood of $\operatorname{supp}(\phi_j)$ just by thinning out the sequence a bit.

The point of this sequence is that it allows us to replace the apparently uncountable number of conditions in (12) by the countable collection

$$u \in H^s_{\mathrm{loc}}(\Omega) \Longleftrightarrow \phi_j u \in H^s(\mathbb{R}^n)\ \forall\ j. \tag{13}$$

Of course this is a subset of the conditions in (12) so holds if $u \in H^s_{\mathrm{loc}}(\Omega)$. Conversely, the support properties mean that for any $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ there is a j such that $\phi_j = 1$ on a neighbourhood of the support of ϕ and hence $\phi_j \phi = \phi$. So from the right side of (13) it follows that

$$\phi u = \phi\phi_j u = \phi(\phi_j u) \in H^s(\mathbb{R}^n)$$

giving the equivalence in (13).

Now, this allows us to see that $H^s(\Omega)$ is a Fréchet space. There is a countable collection of semi-norms defined on it, in this case

$$\|u\|_j = \|\phi_j u\|_{H^s} \tag{14}$$

which collectively capture everything – if $\|u\|_j = 0$ for all j then $\phi_j u = 0$ for all j and hence $\phi u = 0$ for $\phi \in \mathcal{C}^\infty_c(\Omega)$. However this implies that $u(\phi) = 0$, i.e. $u = 0$ in $\mathcal{C}^{-\infty}(\Omega)$.

The condition that a space be Fréchet with respect to such a sequence of seminorms is that it be complete with respect to the metric

$$u(u, v) = \sum_j 2^{-j} \frac{\|u - v\|_j}{1 + \|u - v\|_j}. \tag{15}$$

We know that a sequence u_n which is Cauchy with respect to this distance is Cauchy with respect to each seminorm (check that you DO know this) and hence that $\phi_j u_n$ is Cauchy in $H^s(\mathbb{R}^n)$ of each j. By the completeness, this implies convergence, $\phi_j u_n \to v_j$ in $H^s(\mathbb{R}^n)$. So we need to check that there exists $u \in H^s_{\mathrm{loc}}(\Omega)$ such that $\phi_j u = v_j$ for all j, since this implies $\phi_j u_n \to \phi u$. Observe from the properties of the ϕ_j that

$$\phi_{j+1}\phi_j u_n = \phi_j u_n \longrightarrow v_j \Longrightarrow \phi_{j+1}\phi_j u_n \to \phi_j v_{j+1},\ v_j = \phi_j v_{j+1}.$$

This means we can define

$$u(\psi) = \lim_{j\to\infty} v_j(\psi)\ \forall\ \psi \in \mathcal{C}^\infty_c(\Omega) \tag{16}$$

where in fact the sequence is constant for each ψ as soon as $\psi\phi_j = \psi$.

I (meaning you) should check that u so defined is an element of $\mathcal{C}^{-\infty}(\Omega)$ and then $u_n \to u$ in terms of the distance.

So $H^s_{\mathrm{loc}}(\Omega)$ is a Fréchet space – and I am assuming you can pretty much 'see' this by now.

What does $H^s(\Omega)$ being a Fréchet space actually buy us? I did not talk about this in lecture but I will get to it eventually. Many of the standard results from Banach spaces carry over, for instance Hahn-Banach. So if you have a linear function defined on a subspace and continuous there (with respect to the metric) then it can (provided you believe in the Axiom of Choice or something a little weaker) be extended to a continous linear functional on the whole space. This is actually just the usual Hahn-Banach since the original continuity means the absolute value is bounded by a multiple and this is what Hahn-Banach assumes.

Now, I introduced these spaces mainly to give the corresponding version of elliptic regularity. For any polynomial of order m

$$P(D) : H^{s+m}_{\mathrm{loc}}(\Omega) \longrightarrow H^s_{\mathrm{loc}}(\Omega). \tag{17}$$

Theorem 2. *If $P(\xi)$ is an elliptic polynomial of order m (really a polynomial elliptic of order m, so it* really *is of order m and no lower) then for any open set Ω*

$$u \in \mathcal{C}^{-\infty}(\Omega),\ P(D)u \in H^s_{\mathrm{loc}}(\Omega) \Longrightarrow u \in H^{s+m}_{\mathrm{loc}}(\Omega). \tag{18}$$

In fact (less usefully) the 'converse' is also true – if $P(D)$ is such that this holds for some non-trivial (non-empty) open set and some s then $P(D)$ must be elliptic of order m.

Proof. I did this in the case of $s = \infty$ last time. The key ingredient is the same, our 'parametrix' for $P(D)$ which was constructed last time. Namely we found a distribution $b \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ with the following properties

$$(19)\qquad \begin{gathered} P(D)b = \delta + v,\ v \in \mathcal{C}_c^\infty(\mathbb{R}^n) \\ \operatorname{singsupp}(b) \subset \{0\} \\ b* : H^s(\mathbb{R}^n) \longrightarrow H^{s+m}(\mathbb{R}^n)\ \forall\ s. \end{gathered}$$

You can usefully recall how this was constructed, but for the moment we want to use it.

The obstruction is that convolution only works on $\mathbb{R}^n$. So we need to cut things off and this introduces errors. Let's use our ϕ_j's to do the cut-offs. The basic ideas is that $b*$, convolution with b, is 'almost' an inverse to $P(D)$, specifically

$$(20)\qquad b * (P(D)w) = P(D)(b * w) = (P(D)b) * w = \delta * w + v * w, \text{ for any } w \in \mathcal{C}^{-\infty}(\mathbb{R}^n).$$

Now, $v * w \in \mathcal{C}^\infty(\mathbb{R}^n)$. If we apply (20) for $w = \phi_{j+1}u$ and reorganize

$$(21)\qquad \phi_{j+1}u = -v * (\phi_{j+1}u) + b * (P(D)(\phi_{j+1}u).$$

We need to use the locality of $P(D)$ to analyze the last term. A cheap way is to muliply (21) by ϕ_j :

$$(22)\qquad \phi_j u = -\phi_j(v * (\phi_{j+1}u)) + \phi_j(b * (P(D)(\phi_{j+1}u)).$$

Recall the support properties of convolution:

$$(23)\qquad \operatorname{supp}(b * w) \subset \operatorname{supp}(b) + \operatorname{supp}(w) \Longrightarrow \\ \operatorname{singsupp}(b * w) \subset \operatorname{singsupp}(b) + \operatorname{supp}(w),\ b \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n).$$

The second part follows from the first, since if $b \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then $b * w \in \mathcal{C}^\infty(\mathbb{R}^n)$. To prove the second part it suffices to show that for any compact set K containing $\operatorname{singsupp}(b)$ (which is compact) in its interior the weaker version holds

$$\operatorname{singsupp}(b * w) \subset K + \operatorname{supp}(w).$$

To see this we can use a cut-off $\psi \in \mathcal{C}_c^\infty(K)$ which is equal to 1 in a neighbourhood of $\operatorname{singsupp}(b)$ – and we know that this exists. This writes $b = \psi b + (1 - \psi)b$ where the second term is smooth so does not contribute to the singular support. So we can use the first part of (23) to prove the second.

Now, let's think about the last term in (22). We can expand out the action of the differential operator on the product

$$(24)\qquad P(D)(\phi_{j+1}u) = \phi_{j+1}P(D)u + \sum_{|\alpha|<m} \psi_\alpha D^\alpha u.$$

Here the sum comes from the terms where at least one derivative lands on the smooth factor ϕ_{j+1} – which is why the sum is over $|\alpha| < m$. This allows an inductive approach to regularity (which is in the notes somewhere). Instead we can observe that

$$(25)\qquad \operatorname{supp}(\psi_\alpha) \cap \operatorname{supp}(\phi_j) = \emptyset.$$

This follows from the fact that $\phi_{j+1} = 1$ in a neighbourhood of $\operatorname{supp}\phi_j$. So, since we have differentiated it at least once to get the ψ_α, (25) follows. Writing out (22) in this form gives

$$(26)\qquad \phi_j u = \phi_j(b * (\phi_{j+1}P(D)u)) + \sum_{|\alpha|<m} \phi_j(b * \psi_\alpha D^\alpha u) - \phi_j(v * (\phi_{j+1}u))$$

where I have written the important term first. The last term is smooth and by assumption

$$
\begin{gathered}
P(D)u \in H^s_{\mathrm{loc}}(\mathbb{R}^n) \Longrightarrow \phi_j(b * (\phi_{j+1}P(D)u)) \in H^{s+m}(\mathbb{R}^n) \\
\operatorname{singsupp}(b * \psi_\alpha D^\alpha u) \subset \operatorname{supp}(\psi_\alpha) \Longrightarrow \phi_j(b * \psi_\alpha D^\alpha u) \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n).
\end{gathered} \tag{27}
$$

The second line here comes from (23), the fact that $\operatorname{singsupp}(b) = \{0\}$ (why can't it be smaller? – we only really need inclusion) and then (25) which shows that $\phi_j(b * \psi_\alpha D^\alpha u)$ has empty singular support, i.e. is smooth.

So we have proved the Sobolev form of elliptic regularity □

I then discussed the efficacy of having a fundamental solution. This is a distribution $E \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ such that

$$P(D)E = \delta. \tag{28}$$

The most direct use of this comes from setting

$$u = E * f,\ f \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \tag{29}$$

where we need compactness of the support to make sure that the convolution is well-defined. Then

$$P(D)u = (P(D)E) * f = \delta * f = f. \tag{30}$$

So, if we have a fundamental solution then for any $f \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)$ we can find $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ such that $P(D)u = f$. Of course the converse is also true since we can get such an E by setting $f = \delta$.

Question 1. If $P(D)$ is elliptic and E_1, E_2 are two fundamental solutions what can be said about the regularity of $E_1 - E_2$?

In fact any $P(D)$, other than the zero polynomial, has such a fundamental solution. I will show this later after talking about Paley-Wiener.

This suggests we think about the mapping properties of $P(D)$:

$$
\begin{gathered}
P(D) : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \\
P(D) : \mathcal{C}^{-\infty}(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^n) \\
P(D) : \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \\
P(D) : \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)
\end{gathered} \tag{31}
$$

The third map is injective. This follows from the current homework since if $P(D)u = 0$ where u has compact support we can take the Fourier transform and see that

$$P(\xi)\hat{u} = 0. \tag{32}$$

You are supposed to check that $\hat{u}$ is a smooth function, and since a polynomial is non-zero on an open dense subset of the reals, it follows from (32) that $\hat{u} = 0$ and hence $u = 0$. It follows that the fourth map is injective too.

A similar argument shows that this third map can *never* be surjective except in the case that $P(\xi)$ s a non-zero constant. This will follow from the Paley-Wiener theorem later. So let me exclude this case of $P(\xi)$ constant. What is then true is that the first and second maps are never injective but are surjective. Whereas the third and fourth maps are always injective but never surjective.

Proposition 1. *The third and fourth maps in* (31) *have continuous inverses (for non-trivial* P*)* .

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