18.155 LECTURE 8
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Abstract. Notes before and then after lecture.

Read:
This week: Ellipticity of polynomials and elliptic regularity. This pre-lecture discussion is for both L8 and L9.

Before lecture

- First I want to do a little more localization. For any open set $\Omega \subset \mathbb{R}^n$ we have defined $\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^{-\infty}(\Omega)$. There are many space in between these two, but clearly they need to admit lots of growth near the boundary. We define

$$H^s_{\mathrm{loc}}(\Omega) = \{u \in \mathcal{C}^{-\infty}(\Omega); \phi u \in H^s(\mathbb{R}^n)\ \forall\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)\}. \tag{1}$$

  I add a little discussion on these spaces and where we are going with them for you to check how much you have absorbed about localization and the identifications and results implicit in this definition. If you can't follow this and are brave enough, let me know so I can try to fix things.
  - First recall we *can* multiply distributions on Ω by smooth functions and then $\phi u \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega) = (\mathcal{C}^\infty(\Omega))' \subset \mathcal{S}'(\mathbb{R}^n)$ by 'extension as zero' corresponding by duality to the restriction map $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\Omega)$. So it is meaningful to say $\phi u \in H^s(\mathbb{R}^n)$ and hence the definition makes sense.
  - Note that $H^s_{\mathrm{loc}}(\mathbb{R}^n)$ is *not* contained in $H^s(\mathbb{R}^n)$. (Why?)
  - Then you can think briefly about the topology. We can make $H^s_{\mathrm{loc}}(\Omega)$ into a metric space (it is a Fréchet space) by taking an exhaustion by compact sets K_j of Ω and corresponding elements $\phi_j \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ such that $\phi_j = 1$ in a neighbourhood of K_j. From the requirement that $\phi_j u \in H^s(\mathbb{R}^n)$ for all j it follows that $u \in H^s_{\mathrm{loc}}(\Omega)$? (Why?)
  - Then we can define

$$d(u,v) = \sum_j 2^{-j} \frac{\|\phi_j(u-v)\|_{H^s}}{1 + \|\phi_j(u-v)\|_{H^s}} \tag{2}$$

    in terms of H^s norms. Why is this a metric on $H^s_{\mathrm{loc}}(\Omega)$ and why is the space complete?
  - Then you might like to check that any differential operator of order m with smooth coefficients, $P = \sum\limits_{|\alpha| \le m} a_\alpha(x) D\alpha$, $a_\alpha \in \mathcal{C}^\infty(\Omega)$ defines a continuous linear map $P : H^{s+m}_{\mathrm{loc}}(\Omega) \longrightarrow H^s_{\mathrm{loc}}(\Omega)$ for every s.

  - To orient you a little let me continue. Such a differential operator is said to be elliptic if the homogeneous polynomials depending on $x \in \Omega$ as parameter,

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \tag{3}$$

    do not vanish on $\Omega \times (\mathbb{R}^n_\xi \setminus \{0\})$.
  - Elliptic regularity says that for such an elliptic operator, if $u \in \mathcal{C}^{-\infty}(\Omega)$ and $Pu \in H^s_{\text{loc}}(\Omega)$ then $u \in H^{s+m}_{\text{loc}}(\Omega)$.
  - This week we will prove this for constant coefficient differential operators.
- Another concept I want to introduce now – because it is rather useful and important – is that of a symbol (technically of type 1 for the moment). These form a linear space for each order $m \in \mathbb{R}$, $S^m(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ and are modelled on two case we already understand. Namely polynomials of degree (less than or equal to) m and our 'weight functions' $b = \langle x\rangle^m = (1+|x|^2)^{m/2}$. The latter satisfy estimates

$$|\partial_x^\alpha b(x)| \le C_\alpha \langle x\rangle^{m-|\alpha|} \ \forall\ \alpha. \tag{4}$$

  In words, each derivative lowers the order by one – polynomials do this too of course.

  Really we normally think of symbols as being 'on the Fourier transform side'. Clearly $S^m(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and we can state the important result we want as follows

  Theorem 1. *If $b \in S^m(\mathbb{R}^n)$ then $v = \mathcal{G}b \in \mathcal{S}'(\mathbb{R}^n)$ is such that*

$$\begin{gathered} v* : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^n)\ \forall\ s \in \mathbb{R} \\ \textit{if } \chi \in \mathcal{C}^\infty_c(\mathbb{R}^n),\ \chi = 1 \textit{ near } 0 \textit{ then } (1-\chi)v \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow \operatorname{singsupp} v \subset \{0\}. \end{gathered} \tag{5}$$

- Suppose $P(\xi) = \sum_{|\alpha|\le m} c_\alpha \xi^\alpha$ is a polynomial of degree m then the following conditions are equivalent
  (1) P is elliptic (of order m)

$$P_m(\xi) = \sum_{|\alpha|=m} c_\alpha\xi^\alpha \ne 0 \ \text{ for } \xi \in \mathbb{R}^n. \tag{6}$$

  (2) There exists $c > 0$ such that

$$|P(\xi)| \ge c|\xi|^n \text{ in } |\xi| > 1/c. \tag{7}$$

  (3) There exists $b \in S^{-m}(\mathbb{R}^n)$ such that

$$Pb = 1 - \phi,\ \phi \in \mathcal{C}^\infty_c(\mathbb{R}^n). \tag{8}$$

- Probably not until Thursday. If P is an elliptic polynomial, which by definition is the same as saying $P(D) = \sum_{|\alpha|\le m} c_\alpha D^\alpha$, $D_j = -i\partial_j$, is an elliptic operator, and $u \in \mathcal{C}^{-\infty}(\Omega)$ is such that $P(D)u \in H^s_{\text{loc}}(\Omega)$ then $u \in H^{s+m}_{\text{loc}}(\Omega)$.
- First we will prove that if $P(D)u \in \mathcal{C}^\infty(\Omega)$ then $u \in \mathcal{C}^\infty(\Omega)$ assuming of course that P is elliptic. This follows by taking the b in the last part of the

result immediately above and setting $E = \chi \mathcal{G} b$ where $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is equal to 1 near 0. This gives us a 'convolution parameterix'

$$P(D)E = \delta + \psi, \ \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n). \tag{9}$$

The convolution operator defined by E satisfies

$$\begin{gathered} E* : \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n), \\ \operatorname{singsupp}(E * f) \subset \operatorname{singsupp}(f) \\ P(D)(E * f) = (P(D)) * f = f + \psi * f. \end{gathered} \tag{10}$$

Now apply this to $f_j = P(D)(\phi_j u)$ where $\phi_j \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ and $\phi_j = 1$ in a neighbourhood of K_j for an exhaustion K_j and it follows that

$$\begin{aligned} P(D)(E * (\phi_j u)) &= E * (P(D)(\phi_j u)) = E * f_j \\ &= (P(D)E) * (\phi_j u) = (\delta + \psi) * (\phi_j u) = \phi_j u + \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n). \end{aligned} \tag{11}$$

However, $\operatorname{singsupp}(f_j) \cap K_j = \emptyset$ so $\operatorname{singsupp}(\phi_j u) \cap K_j = \emptyset$ from the second part of (10) but from this, for all j it follows that $u \in \mathcal{C}^\infty(\Omega)$.

- The Sobolev version is similar (see L9).

After lecture

I discussed symbols on $\mathbb{R}^n$ and the properties of their inverse Fourier transforms. Most of this is in the notes but let me go through it quickly here to make sure you caught it.

First we define $S^m(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ by requiring that an element a have bounds

$$|\partial_\xi^\alpha a(\xi)| \le C_\alpha \langle \xi \rangle^{m-|\alpha|}. \tag{12}$$

For $m = N$ this includes all polynomials of degree at most N. Moreover

$$\langle \xi \rangle^m \in S^m(\mathbb{R}^n). \tag{13}$$

By applying the Leibniz rule for differentiation of a product we see that

$$S^m(\mathbb{R}^n) \cdot S^{m'}(\mathbb{R}^n) \subset S^{m+m'}(\mathbb{R}^n) \tag{14}$$

where in fact this is always an equality because of (13).

In fact using these facts we can see that

$$a \in S^m(\mathbb{R}^n) \Longleftrightarrow |\partial_\xi^\alpha \xi^\beta a| \le C_{\alpha,\beta} \langle \xi \rangle^{M-|\alpha|+|\beta|} \ \forall \ \alpha, \beta. \tag{15}$$

Proposition 1. *If $a \in S^m(\mathbb{R}^n)$ then the inverse Fourier transform*

1. $\mathcal{G}a = b + e$, $b \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$, $e \in \mathcal{S}(\mathbb{R}^n)$
2. $b* : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^n)$ *for all* s.
3. $\operatorname{singsupp}(b) \subset \{0\}$.

Proof. Using the product properties above it follows that

$$|\partial_\xi^\alpha \xi^\beta a| \le (1 + |\xi|)^{-n-1} \text{ if } |\alpha| \ge |\beta| - m + n + 1. \tag{16}$$

The right side here is in $L^1(\mathbb{R}^n)$ so from the standard properties of the Fourier transform

$$x^\alpha \partial_x^\beta \mathcal{G} a \text{ is continuous and bounded if } |\alpha| \ge |\beta| - m + n + 1. \tag{17}$$

Taking $2N > |\beta| - m + n + 1$ it follows that

$$|x|^{2N} \partial_x^\beta \mathcal{G} a \text{ is continuous and bounded }. \tag{18}$$

If we choose a cutoff $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ which is equal to 1 nea 0 (which is the only point where $|x|^{2N}$ is not invertible) it follows that

(19)
$$(1+|x|^2)^N \partial_x^\beta (1-\chi)\mathcal{G}a \text{ is continuous and bounded } \Longrightarrow e = (1-\chi)\mathcal{G}a \in \mathcal{S}(\mathbb{R}^n).$$

Now, convolution of $\mathcal{S}$ and $\mathcal{S}'$ is well-defined with values in $\mathcal{S}'$ as is convolution by $\mathcal{C}^{-\infty}_{\mathrm{c}}$ and $\mathcal{S}'$ and in both cases we know that

$$\widehat{b * u} = \hat{b}\hat{u},\ \widehat{e * u} = \hat{e}\hat{u}. \tag{20}$$

In fact we know that $e* : H^s(\mathbb{R}^n) \longrightarrow H^\infty(\mathbb{R}^n)$ and hence we deduce the second property in the statement from the fact that multiplication by a maps $\langle\xi\rangle^{-s}L^2(\mathbb{R}^n)$ to $\langle\xi\rangle^{-s+m}L^2(\mathbb{R}^n)$.

The singular support condition on $\mathcal{G}a$, or equivalent b, follows from (18). $\square$

I then proceeded to apply this to

$$a = \frac{1-\chi}{P(\xi)} \in S^{-m}(\mathbb{R}^n) \tag{21}$$

where P is an elliptic polynomial and $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ vanishes on a sufficiently large ball (to contain all the real zeros of P in its interior).

Another thing that I talked about, which is a little out of the main line of the course, is that the singular support is in fact the support for sections of a sheaf. The basic claim here is not quite obvious; it amounts to:

Lemma 1. *The linear spaces $\mathcal{C}^{-\infty}(\Omega)/\mathcal{C}^\infty(\Omega)$, defined for any open set $\Omega \subset \mathbb{R}^n$, form a sheaf.*

There is no real necessity for you to understand this, however enquiring minds like to know these things. You might like to go through the arguments below to check that the $H^s_{\mathrm{loc}}(\Omega)$ form a sheaf as do the quotients $\mathcal{C}^{-\infty}(\Omega)/H^s_{\mathrm{loc}}(\Omega)$. An element $u \in \mathcal{C}^{-\infty}(\Omega)$ projects to an element of the quotient space and the suuport in this sense is $\operatorname{singsupp}_{H^s}(u)$ – this is the complement of the largest set on which u is eual to an element of H^s_{loc}.

In fact showing that the quotients form a sheaf is a cohomological problem, in sheaf or Čech cohomology. You might want to work out this relationsip youself, or ask.

The main ingredient allows one to show that these are fine sheaves.

Proposition 2 (Partitions of unity)**.** *Let $\Omega_\alpha \subset \mathbb{R}^n$ be a collection of open sets indexed by $\alpha \in A$, setting $\Omega = \bigcup_{\alpha\in A} \Omega_\alpha \subset \mathbb{R}^n$, there exist elements $\psi_\alpha \in \mathcal{C}^\infty(\Omega)$ indexed by A with the following properties*

(1) *For each α,* $\operatorname{supp}(u_\alpha) \subset \Omega_\alpha$.
(2) $0 \le \psi_\alpha \le a$.
(3) *For each $K \Subset \Omega$ $K \cap \operatorname{supp}(\psi_\alpha) \neq \emptyset$ for at most a finite set of α.*
(4) $\sum\limits_\alpha \psi_\alpha = 1$.

The second last condition means that the sum in the last one if finite over any compact subset, so this makes sense. The ψ_α are said to form a *partition of unity subordinate to the open cover Ω_α of Ω.*

Proof. We have done most of the work for this. We showed that there is a sequence $\phi_j \in \mathcal{C}^\infty_c(\Omega)$ such that $0 \le \phi_j \le 1$, $\phi_{j+1} = 1$ in a neighbourhood of $\operatorname{supp}(\phi_j)$ and that any compact set $K \subset \{\phi_j = 1\}$ for some j.

We also showed that if $K \Subset \Omega$ and Ω_α is an open cover of K then there exist $\chi_\alpha \in \mathcal{C}^\infty_c(\Omega_\alpha)$, with $0 \le \chi_\alpha \le 1$, only a finite number not identically zero and such that

$$\sum_\alpha \chi_\alpha = 1 \text{ in a neighbourhood of } K. \tag{22}$$

We can combine these to constructins by setting $\mu_1 = \phi_1$ and $\mu_j = \phi_j - \phi_{j-1}$ for $j \ge 2$. Then $0 \le \mu_j \le 1$, each compact set only meets a finite number of the $\operatorname{supp}(\mu_j)$ and

$$\sum_j \mu_j = 1. \tag{23}$$

Now for each j choose $\chi_{\alpha,j}$ as above such that (22) holds for each j, with $K_j = \operatorname{supp} \mu_j$. Then set

$$\psi_\alpha = \sum_j \chi_{\alpha,j}. \tag{24}$$

The sum is locally finite, so makes sense and $\psi_\alpha \in \mathcal{C}^\infty(\Omega)$. Each $\psi_{\alpha,j}$ has support in Ω_α, hence so does the sum and the final condition in the proposition follows from (23) and the choice of the $\psi_{\alpha,j}$. □

I did not really show that the spaces $\mathcal{C}^{-\infty}(\Omega)$ form a sheaf. The restriction maps are clear enough, if $\Omega_1 \subset \Omega_2$ are open then

(25)
$$\mathcal{C}^{-\infty}(\Omega_2) \ni u \longmapsto u\big|_{\Omega_1} \in \mathcal{C}^{-\infty}(\Omega_1),\ u\big|_{\Omega_1}(\phi) = u(\phi)\ \forall\ \phi \in \mathcal{C}^\infty_c(\Omega_1) \subset \mathcal{C}^\infty_c(\Omega_2).$$

I leave it to you to check the presheaf properties.

For the sheaf properties we are given $u_\alpha \in \mathcal{C}^{-\infty}(\Omega_\alpha)$ for a collection of open sets, with

$$u_\alpha\big|_{\Omega_\alpha\cap\Omega_\beta} = u_\beta\big|_{\Omega_\alpha\cap\Omega_\beta}. \tag{26}$$

Given $\phi \in \mathcal{C}^\infty_c(\Omega)$, $\Omega = \bigcup_\alpha \Omega_\alpha$, we can define

$$u(\phi) = \sum_\alpha u_\alpha(\psi_\alpha\phi) \tag{27}$$

using a partition of unity as above. This makes sense since $\psi_\alpha\phi \in \mathcal{C}^\infty_c(\Omega_\alpha)$. Notice that this sum is finite, with the same number of terms, for all ϕ with $\operatorname{supp}(\phi) \subset K \Subset \Omega$. From this it follows readily that $u \in \mathcal{C}^{-\infty}(\Omega)$.

You can check that u does not depend on the choice of the partition of unity, because of (26), but we actually do not need to do so. Just check that $u \in \mathcal{C}^{-\infty}(\Omega)$ has the property

$$u\big|_{\Omega_\alpha} = u_\alpha, \text{ i.e. } u(\phi) = u_\alpha(\phi)\ \forall\ \phi \in \mathcal{C}^\infty_c(\Omega_\alpha). \tag{28}$$

Indeed, if $\operatorname{supp}(\phi) \Subset \Omega_\alpha$ then for every $\beta \ne \alpha$, $\operatorname{supp}(\phi\psi_\beta) \Subset \Omega_\alpha \cap \Omega_\beta$, so in the (finite) sum (27) in all the terms with $\beta \ne \alpha$ we can use (26) to replace u_β by u_α, so indeed

$$u(\phi) = \sum_\beta u_\alpha(\phi\psi_\beta) = u_\alpha(\phi). \tag{29}$$

To see the uniqueness, just note that (27) follows from (28).

Proof of Lemma 1. I leave it to you to show that the quotients

$$F(\Omega) = \mathcal{C}^{-\infty}(\Omega)/\mathcal{C}^{\infty}(\Omega)$$

form a presheaf – this is always the case for the quotient of a sheaf of linear spaces by a subsheaf.

It is the sheaf condition on the quotient that is not so obvious. What we need to show is that if $u_\alpha \in F(\Omega_\alpha)$ for a collection of open sets and $u_\alpha\big|_{\Omega_\alpha\cap\Omega_\beta} = u_\beta\big|_{\Omega_\alpha\cap\Omega_\beta}$ then we need to find $u \in F(\Omega)$, $\Omega = \bigcup_\alpha \Omega_\alpha$, such that $u\big|_{\Omega_\alpha} = u_\alpha$. We can choose $v_\alpha \in \mathcal{C}^{-\infty}(\Omega_\alpha)$ representing $u_\alpha = [v_\alpha]$, but then the compatibility condition becomes

$$v_{\alpha,\beta} = v_\alpha - v_\beta \in \mathcal{C}^{\infty}(\Omega_\alpha \cap \Omega_\beta) \tag{30}$$

where I have dropped the notation for the restriction maps. The claim is that there exist $w_\alpha \in \mathcal{C}^{\infty}(\Omega_\alpha)$ such that

$$w_\alpha\big|_{\Omega_\alpha\cap\Omega_\beta} - w_\beta\big|_{\Omega_\alpha\cap\Omega_\beta} = v_{\alpha,\beta} \in \mathcal{C}^{\infty}(\Omega_\alpha \cap \Omega_\beta) \tag{31}$$

for all α, β. This is the vanishing of a Čech obstruction.

To arrange (31) we define

$$w_\alpha = \sum_{\beta\neq\alpha} v_{\alpha,\beta}\psi_\beta \in \mathcal{C}^{\infty}(\Omega_\alpha) \tag{32}$$

where ψ_α is a partition of unity as discussed above. This makes sense, even though $v_{\alpha,\beta}$ is only defined on $\Omega_\alpha \cap \Omega_\beta$, since $\psi_\beta \in \mathcal{C}^{\infty}(\Omega)$ has support in Ω_β. This means that ψ_β vanishes outside a relatively closed subset of Ω_β which means it vanishes in a *neighbourhood* of $\Omega_\alpha \setminus \Omega_\beta$. So each of the terms in (32) is initially defined on $\Omega_\alpha \cap \Omega_\beta$ but vanishes near $\Omega_\alpha \setminus \Omega_\beta$ so the extension as zero outside this set is smooth on Ω_α.

Now observe that (31) does indeed hold for this choice of the w_α's. Restricted to the intersection of Ω_α and Ω_β the difference can be written

$$\begin{aligned} &\sum_{\gamma\neq\alpha} v_{\alpha,\gamma}\psi_\gamma - \sum_{\gamma\neq\beta} v_{\beta,\gamma}\psi_\gamma \\ &= v_{\alpha,\beta}\psi_\beta - v_{\beta,\alpha}\psi_\alpha + \sum_{\gamma\neq\alpha,\beta} v_{\alpha,\beta}\psi_\gamma \\ &= v_{\alpha,\beta} \end{aligned} \tag{33}$$

where we use the fact that $v_{\beta,\gamma}$ is antisymmetric and $v_{\beta,\gamma} + v_{\gamma,\alpha} + v_{\alpha,\beta} = 0$ on the triple intersections $\Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma$.

Now, replace v_α by the new representative $v_\alpha - w_\alpha$ and observe that the sheaf property for $\mathcal{C}^{-\infty}(\Omega)$ gives a grobal representative which solves the problem. $\square$

Make sure you understand that if $v \in \mathcal{C}^{\infty}(\Omega_1 \cap \Omega_2)$ for two open sets and $\chi \in \mathcal{C}^{\infty}(\Omega_1 \cup \Omega_2)$ has support in Ω_2 then $\chi u \in \mathcal{C}^{\infty}(\Omega_1)$ is well-defined by extending it as zero outside the intersection.

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