18.155 LECTURE 6
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Abstract. Notes before and then after lecture.

Read: Chapter 3, end of Sect 1.

Before lecture

Last lecture I promised to discuss 'indefinite integration' of tempered distributions, for the moment in one dimension. By the fundamental theorem of calculus this amounts to discussing the invertibility of the differential operator d/dx.

Lemma 1. *The linear map*

$$\frac{d}{dx} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) \tag{1}$$

is injective with closed range of codimension one and (hence)

$$\frac{d}{dx} : \mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R}) \tag{2}$$

is surjective with one-dimensional null space.

Exercise 1. Work out what this says about the operator $-d^2/dx^2$ which is the Laplacian in one dimension.

Hint after lecture. If you want to do this 'properly', define a generalized inverse for $-d^2/dx^2$ by generalizing (4) below to show that

$$\phi = \frac{\int_{\mathbb{R}} \phi}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) + c(\int_{\mathbb{R}} x\phi(x))x\exp(-\frac{x^2}{2})\frac{d^2}{dx^2}\eta,\ \eta \in \mathcal{S}(\mathbb{R}) \tag{3}$$

where the constant c should be chosen carefully. Since $\int x\exp(-x^2/2) = 0$ and $x\exp(-x^2/2) = -d/dx\exp(-x^2/2)$ this means that $\eta = I\psi$ for the correct choice of constant and the fact that d^2/dx^2 has closed range of codimension two on $\mathcal{S}(\mathbb{R})$ follows. $\square$

Proof. The injectivity of d/dx on test functions is clear enough since no constant function other than 0 can be in $\mathcal{S}(\mathbb{R})$. To characterize the range we really just need to integrate but I will write down the answer.

$$\phi \in \mathcal{S}(\mathbb{R}) \Longrightarrow \phi = \frac{\int_{\mathbb{R}} \phi}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) + \frac{d}{dx}\psi,\ \psi \in \mathcal{S}(\mathbb{R}). \tag{4}$$

In fact ψ is unique and the map $I : \phi \longmapsto \psi$ is a 'left inverse' to d/dx so let me write down the identities you should check

$$I \circ \frac{d}{dx} = \mathrm{Id},\ \frac{d}{dx} \circ I = \mathrm{Id} - \Pi,\ \Pi(\phi) = \frac{\int_{\mathbb{R}} \phi}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}). \tag{5}$$

To prove (4) just observe that the difference of ϕ and $\Pi\phi$, which is the first term on the right, has integral 0. So if we define $I\phi = \psi$ by the second equality

$$I\phi = \psi(x) = \int_{-\infty}^{x} (\phi - \Pi\phi) = -\int_{x}^{\infty} (\phi - \Pi\phi) \tag{6}$$

the third follows by the vanishing of the integral. We certainly get a smooth function which is rapidly vanishing with all derivatives as $x \to -\infty$ from the first equality and as $x \to \infty$ from the second. So indeed $I : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ is a continuous linear map. Estimates on $d\psi/dx$ are immediate and the rapid decay estimates on ϕ follow from those of ϕ so that for instance

$$\|\psi\|_{(k)} \le C_k \|\phi\|_{k+n+1} \tag{7}$$

where need to estimate the integral of ϕ. This proves (4).

Check that (1) and (5) follow.

Now to get (2) observe that if $u \in \mathcal{S}'(\mathbb{R})$ then by the continuity of I we can define

$$v(\phi) = -u(I\phi) \Longrightarrow \frac{dv}{dx}(\mu) = -v(\frac{d\mu}{dx}) = u(I\frac{d\mu}{dx}) = u(\mu),\ \forall\ \mu \in \mathcal{S}(\mathbb{R}). \tag{8}$$

This shows the surjectivity on tempered distributions. Similarly if $\frac{du}{dx} = 0$ then

$$u(\phi) = \frac{\int \phi}{\sqrt{2\pi}} u(\exp(-\frac{x^2}{2}) = c \int \phi \Longrightarrow u = c \text{ is constant} \tag{9}$$

using (4). So the null space of d/dx is indeed one dimensional, spanned by the constant functions. □

Exercise 2. Write down a *left* inverse of d/dx on $\mathcal{S}'(\mathbb{R})$ (it is implicit in (8)) and the identities corresponding to (5) (they are the other way around). Are these two 'generalized inverses' unique?

- Support of a continuous function in two ways, $\mathcal{C}^\infty_{\text{c}}(\Omega)$.
- Compact exhaustion of an open set.
- The topology of $\mathcal{C}^\infty_{\text{c}}(\Omega)$.
- The space $\mathcal{C}^{-\infty}(\Omega)$ of distributions on an open set
- Restriction including $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}(\Omega)$.
- Sheaves (the definition only)
- If $K \subset \Omega$ is a compact subset of an open set then there exists $\phi \in \mathcal{C}^\infty_{\text{c}}(\Omega)$ such that $\phi = 1$ in a neighbourhood of K.
- Vanishing of a distribution on an open set.
- The sheaf properties.
- Support of a distribution
- I actually got o here.
- Singular support of a distribution
- Support and singular support of convolutions
- I hope to get to around here.

$$\mathcal{C}^{-\infty}(\mathbb{R}^n) * \mathcal{C}^{-\infty}_{\text{c}}(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n). \tag{10}$$

- Convolution and supports.
- Fundamental solutions of constant coefficient differential operators
- Examples.
- Ellipticity

- Parametrices

After lecture

I did go through the proof that if $K \Subset \Omega$ is a compact subset of an open set then there exists $\mu \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ such that $0 \le \mu \le 1$ and $\mu = 1$ in an open set containing K. If the proof isn't in the notes I will add one.

I mentioned at the beginning of the Lecture that the Laplacian $\varDelta = \partial_1^2 - \dots - \partial_n^2$ on $\mathbb{R}^n$ has similar properties, which we could prove at the moment and will prove later. Namely

$$\begin{aligned} \varDelta : \mathcal{S}(\mathbb{R}^n) &\longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ is injective and} \\ \varDelta : \mathcal{S}'(\mathbb{R}^n) &\longrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ is surjective} \end{aligned} \tag{11}$$

where the null space on $\mathcal{S}(\mathbb{R}^n)$ consists of the infinite dimensional space of harmonic polynomials. The range on $\mathcal{S}(\mathbb{R}^n)$ is closed, and consists precisely of those $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int p(x)\phi(x) = 0$ for every harmonic polynomial p.

I said I would add the proof that the distibution spaces $\mathcal{C}^{-\infty}(\Omega)$ form a sheaf over $\mathbb{R}^n$ (or similarly over any open subset of $\mathbb{R}^n$). We really do not need this ...

The presheaf axioms follow by duality from the inclusions

$$U \subset V \subset \mathbb{R}^n \text{ open } \Longrightarrow \mathcal{C}^\infty_{\mathrm{c}}(U) \subset \mathcal{C}^\infty_{\mathrm{c}}(V) \subset \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \tag{12}$$

where these 'inclusions' involve extending functions as zero outside there initial domains. Thus the restriction of $u \in \mathcal{C}^{-\infty}(V)$ to $u\big|^V_U \in \mathcal{C}^{-\infty}(U)$ is just obtained by restricting the domain of the linear functional from $\mathcal{C}^\infty_{\mathrm{c}}(V)$ to $\mathcal{C}^\infty_{\mathrm{c}}(U)$. It follows immediately that $\big|^U_U = \mathrm{Id}$ and that if $U \subset V \subset W$ are all open then

$$\big|^V_U \circ \big|^W_V = \big|^W_U, \tag{13}$$

the categorical property.

For the sheaf property suppose $U = \bigcup_\alpha U_\alpha$ is an open cover and for each α we are given $u_\alpha \in \mathcal{C}^{-\infty}(U_\alpha)$ such that

$$u_\alpha\big|^{U_\alpha}_{U_\alpha \cap U_\beta} = u_\beta\big|^{U_\beta}_{U_\alpha \cap U_\beta}. \tag{14}$$

then we wish to show that there is a unique $u \in \mathcal{C}^{-\infty}(U)$ such that

$$u\big|^U_{U_\alpha} = u_\alpha. \tag{15}$$

Lemma 2. *If $K \Subset U$ and $U = \bigcup_\alpha U_\alpha$ is an open cover then there exist a finite collection $\mu_j \in \mathcal{C}^\infty_{\mathrm{c}}(U_{\alpha_j})$ $j = 1, \dots, N$ such that*

$$\sum_{j=1}^N = 1 \textit{ in a neighbourhood of } K. \tag{16}$$

Proof. Since K is compact it is covered by a finite number of the U_α. So we can proceed by induction, showing that if K is covered by N open sets U_j then there are corresponding $\mu_j \in \mathcal{C}^\infty_{\mathrm{c}}(U_j)$ summing to 1 in an open set containing K. We did this explicitly when there is one open set. Suppose there are N and consider $K' = K \setminus U_1 \Subset \bigcup_{j>1}$. We can apply the inductive hypothesis and find functions $\mu'_j \in \mathcal{C}^\infty_{\mathrm{c}}(U_j)$, $j > 1$, summing to 1 in an open set $V \supset K'$. Then we can find

$\mu_1' \in \mathcal{C}^\infty_c(U_1)$ such that $\mu_1' = 1$ in an open set containing $K \setminus V \subset U_1$. It follows that together with the μ_j for $j > 1$,

$$\mu_1 = \mu_1'(1 - \sum_{j>1} \mu_j') \in \mathcal{C}^\infty_c(U_1),\ \mu_j = (1-\mu_1')\mu_j',\ j > 1. \tag{17}$$

Then on an open set contaning $K \setminus V$ $\mu_1' = 1$ so $\mu_1 = (1 - \sum\limits_{j>1} \mu_j')$ and $\mu_j = \mu_j'$ for $j > 1$ and on an open set containing K' $mu_1 = \mu_1'$ and $\sum\limits_{j>1} \mu_j = (1-\mu_1')$ so these functions fulfil the inductive hypothesis for N sets. □

Now returning to the sheaf property, given the covering U_α and $\phi \in \mathcal{C}^\infty_c(U)$ then we can apply the Lemma for $K = \operatorname{supp}(\phi)$ and set

$$u(\phi) = \sum_j u_{\alpha_j}(\mu_j \phi) \tag{18}$$

for some finite subcover. Since the μ_j depend only on the choice of some compact set K, u is continuous on $\mathcal{C}^\infty_c(K)$.

In fact u is independent of the choice of the μ_j, since if ν_l, which may correspond to a different finite open subcover U_{β_l} of K, we can use the fact that $\sum\limits_l \nu_l = 1$ in an open set containing K and hence all the supports of the $\mu_j\phi)$ to write

$$\sum_j u_{\alpha_j}(\mu_j\phi) = \sum_l \sum_j u_{\alpha_j}(\nu_l \mu_j \phi) = \sum_l \sum_j u_{\beta_l}(\nu_l \mu_j \phi) = \sum_l u_{\beta_l}(\nu_l \phi) \tag{19}$$

where we use the consistency condition (14).

To see that u satisfies (15), suppose that $\phi \in \mathcal{C}^\infty(U_\alpha)$ for some α. The definition (18) then corresponds to an open cover of a compact subset of U_α, so each $\mu_j\phi$ in (18) has support in $U_\alpha \cap u_{\alpha_j}$, so indeed

$$u_{\alpha_j}(\mu_j\phi) = u_\alpha(\mu_j\phi) \Longrightarrow u(\phi) = u_\alpha(\phi). \tag{20}$$

That u it is unique follows from a similar argument.

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